# HW8 - Solutions

Saturday, December 7, 2019 7:30 PM

1. (Section 4.4 - Exercise 2)

Verify that $x(t) = \ln(1+t)$

$y(t) = e^t$

is a solution of the system $\dot{x} = e^{-x}$, $\dot{y} = e^{e^x - 1}$,

and find its orbits.

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For $x = \ln(1+t)$ we obtain $e^x = 1+t$, $t = e^x - 1$, and

$$\dot{x} = \frac{dx}{dt} = \frac{1}{1+t} = \frac{1}{e^x} = e^x,$$

while for $y = e^t$ we derive $\dot{y} = \frac{dy}{dt} = e^t = e^{e^x - 1}$.

Notice that this system has no equilibrium solutions.

For finding orbits, we start with

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{e^{e^x - 1}}{e^{-x}} = e^x\left(e^{e^x - 1}\right)$$

$$y = \int e^x\left(e^{e^x - 1}\right) dx = \int e^u du = e^u + c = e^{e^x - 1} + c$$

$e^x - 1 = u$, $e^x dx = du$

The orbits of the given system are curves

$y = e^{e^x - 1} + c$, $c$ - arbitrary real constant.

2. (Section 4.4 - Exercise 8)

Find orbits of the system $\dot{x} = y + x^2 y$
$\dot{y} = 3x + xy^2.$

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Solving equations
$$y + x^2 y = y(1+x^2) = 0$$
$$3x + xy^2 = x(3+y^2) = 0$$
we find the only equilibrium point of the system is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
( notice that $1+x^2>0$, for all $x$, and $3+y^2>0$, for all $y$)

For finding orbits, consider the differential equation

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{3x+xy^2}{y+x^2y} = \frac{x(3+y^2)}{y(1+x^2)}.$$

It is separable and we can solve it in the following way:

$$\int \frac{y}{3+y^2}\,dy = \int \frac{x}{1+x^2}\,dx$$

$r = 3+y^2$, $dr = 2y\,dy$

$s = 1+x^2$, $ds = 2x\,dx$

$$\frac{1}{2}\int \frac{dr}{r} = \frac{1}{2}\int \frac{ds}{s}$$

$$\ln|3+y^2| = \ln|1+x^2| + c_1$$

$$|3+y^2| = c_2\,|1+x^2|, \quad c_2 = e^{c_1}$$
$$3+y^2 = c(1+x^2)$$
$$y^2 = c(1+x^2) - 3$$

Orbits of the given sistem are :

1) equilibrium point $(0,0)$
2) the curves $y^2 = c(1+x^2) - 3$, $c \neq 3$
3) the half-lines $y = \sqrt{3}\,x$, $x > 0$
   $y = \sqrt{3}\,x$, $x < 0$
   $y = -\sqrt{3}\,x$, $x > 0$, and
   $y = -\sqrt{3}\,x$, $x < 0$.

(remark: if in the solution curves $y^2 = c(1+x^2) - 3$ we formally take $c = 3$, we obtain $y^2 = 3(1+x^2) - 3 = 3x^2$, and $y = \pm\sqrt{3}\cdot x$ – here we need to exclude $(0,0)$, thus orbits are 4 half-lines)

3. (Section 4.7 - Exercise 3)

Draw the phase portrait of the system

$$\dot{x} = \begin{bmatrix} 4 & -1 \\ -2 & 5 \end{bmatrix} x.$$

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For system matrix $A = \begin{bmatrix} 4 & -1 \\ -2 & 5 \end{bmatrix}$ we find

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & -1 \\ -2 & 5-\lambda \end{vmatrix} = (4-\lambda)(5-\lambda) - 2 = 20 - 4\lambda - 5\lambda + \lambda^2 - 2$$

$$= \lambda^2 - 9\lambda + 18 = 0 , \qquad \lambda_{1,2} = \frac{9 \pm \sqrt{81-72}}{2} = \frac{9 \pm 3}{2}$$

Eigenvalues of $A$ are $\lambda_1 = 3$, $\lambda_2 = 6$.

Equilibrium solution $(0,0)$ is nodal source.

$\lambda_1 = 3$: $\quad (A - \lambda_1 I)v = 0$

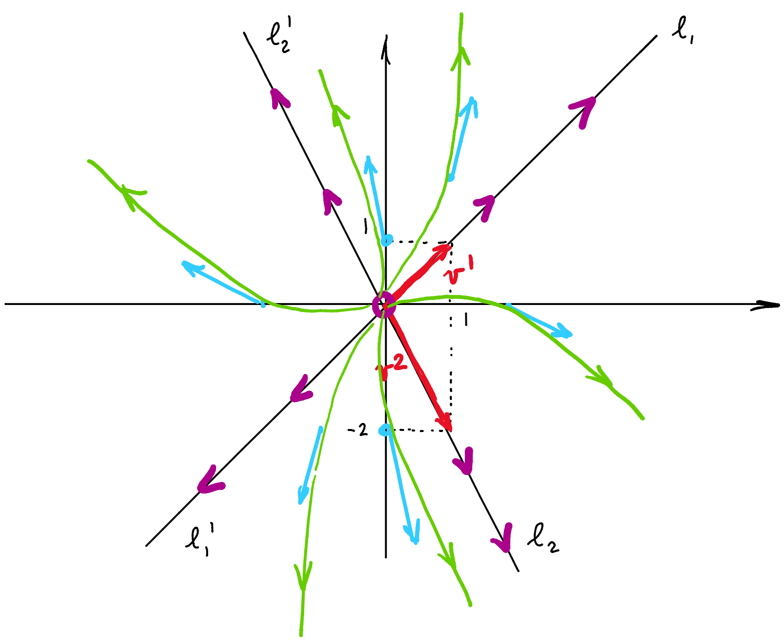
$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_1 = v_2$$

$\Rightarrow$ eigenvector for $\lambda_1 = 3$ is $v^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 6$: $\quad (A - \lambda_2 I) = 0$

$$\begin{bmatrix} -2 & -1 \\ -2 & -1 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -2v_1 - v_2 = 0 \,, \quad v_2 = -2v_1$$

$\Rightarrow$ eigenvector for $\lambda_2 = 6$ is $v^2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$




| | $\dot{x}$ | $\dot{y}$ |
|---|---|---|
| (0,1) | -1 | 5 |
| (0,-2) | 2 | -10 |
| (1,2) | 2 | 8 |
| (-2,0) | -8 | 4 |
| (2,0) | 8 | -4 |
| (-1,-2) | -2 | -8 |

4. (Section 4.7 - Exercise 6)

Draw the phase portrait of the system

$$\dot{x} = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} x.$$

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For the system matrix $A = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix}$ we find

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -1 \\ 5 & -3-\lambda \end{vmatrix} = (3-\lambda)(-3-\lambda) + 5 = -9 - 3\lambda + 3\lambda + \lambda^2 + 5$$

$$= \lambda^2 - 4 = 0$$

Eigenvalues of $A$ are $\lambda_1 = -2$, $\lambda_2 = 2$, and equilibrium point $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is saddle point.

$\lambda_1 = -2$ : $(A - \lambda_1 I) v = 0$

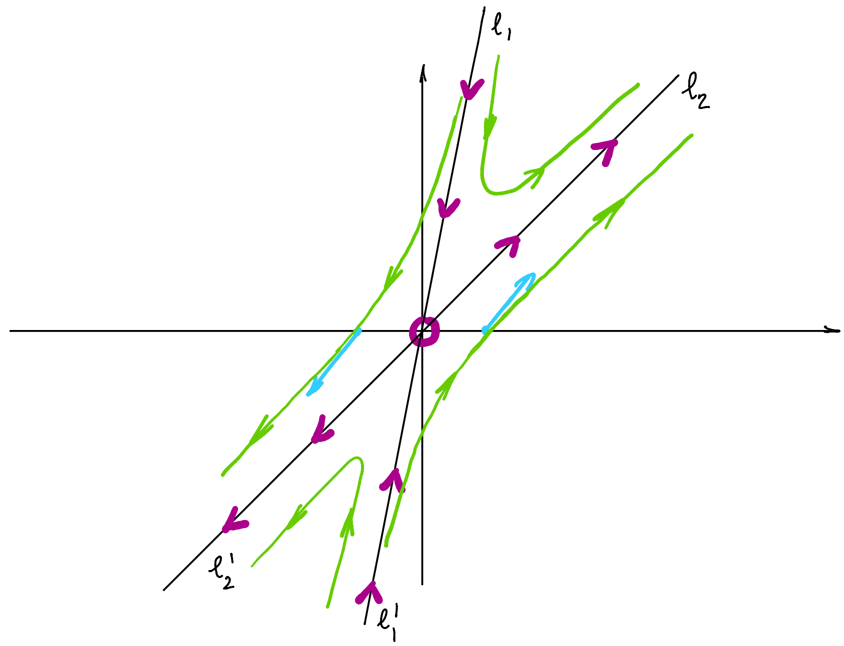
$$\begin{bmatrix} 5 & -1 \\ 5 & -1 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 5v_1 - v_2 = 0, \quad v_2 = 5v_1$$

$\Rightarrow$ eigenvector for $\lambda_1 = -2$ is $v^1 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

$\lambda_2 = 2$ : $(A - \lambda_2 I) v = 0$

$$\begin{bmatrix} 1 & -1 \\ 5 & -5 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_1 = v_2$$

$\Rightarrow$ eigenvector for $\lambda_2 = 2$ is $v^2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$




| | $\dot{x}$ | $\dot{y}$ |
|---|---|---|
| $(-1,0)$ | $-3$ | $-5$ |
| $(1,0)$ | $3$ | $5$ |

5. (Section 4.7 - Exercise 9)

Draw the phase portrait of the system

$$\dot{x} = \begin{bmatrix} 2 & 1 \\ -5 & -2 \end{bmatrix} x.$$

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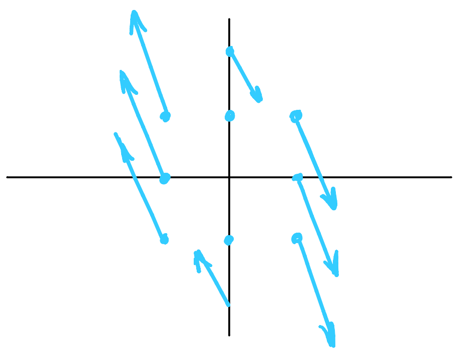
For the system matrix $A = \begin{bmatrix} 2 & 1 \\ -5 & -2 \end{bmatrix}$ we find

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ -5 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) + 5 = -4 - 2\lambda + 2\lambda + \lambda^2 + 5$$

$$= \lambda^2 + 1 = 0 \quad , \qquad \lambda_1 = i \, , \quad \lambda_2 = \bar{\lambda}_1 = -i$$

Eigenvalues of $A$ are $\lambda_1 = i$, $\lambda_2 = -i$. They are complex, with zero real part, and consequently equilibrium point $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is center.

Let's just first pick few points and determine $\dot{x}, \dot{y}$.



| | $\dot{x}$ | $\dot{y}$ |
|---|---|---|
| $(1,1)$ | $3$ | $-7$ |
| $(-1,-1)$ | $-3$ | $7$ |
| $(1,0)$ | $2$ | $-5$ |
| $(-1,0)$ | $-2$ | $5$ |
| $(1,-1)$ | $1$ | $-3$ |
| $(0,2)$ | $2$ | $-4$ |

Direction of arrows points to clockwise orientation of orbits.