HW 8

Math 4512
Differential Equations with Applications

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Contents

1 Section 4.4, problem 1

In each of Problems 1-3, verify that $x(t), y(t)$ is a solution of the given system of equations, and find its orbit.

$$\begin{aligned}\dot{x} &= 1\\ \dot{y} &= 2(1-x)\sin\left((1-x)^2\right)\\ x(t) &= 1+t\\ y(t) &= \cos\left(t^2\right)\end{aligned}$$

solution

Since $x(t) = 1+t$ then $\dot{x} = 1$. Verified OK. And since $y(t) = \cos\left(t^2\right)$ then $\dot{y} = -2t\sin\left(t^2\right)$. But $t = x-1$, hence $\dot{y} = -2(x-1)\sin\left((1-x)^2\right)$ or

$$\dot{y} = 2(1-x)\sin\left((1-x)^2\right)$$

Verified OK. Both solutions verified. Now we need to find system orbit. The Orbit is given by the equation

$$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}$$

When we write the given system in the following form

$$\begin{aligned}\dot{x} &= f(x,y)\\ \dot{y} &= g(x,y)\end{aligned}$$

We see now that $f(x,y) = 1$ and $g(x,y) = 2(1-x)\sin\left((1-x)^2\right)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{2(1-x)\sin\left((1-x)^2\right)}{1}\\ &= 2(1-x)\sin\left((1-x)^2\right)\end{aligned}$$

This is first order ODE. Since separable, we can solve it by integration

$$y(x) = \int 2(1-x)\sin\left((1-x)^2\right)dx$$

Let $u = (1-x)^2$, then $\frac{du}{dx} = 2(1-x)(-1) = -2\sqrt{u}$. Substituting in the above gives

$$\begin{aligned}y(x) &= \int 2\sqrt{u}\sin(u)\frac{du}{-2\sqrt{u}}\\ &= -\int \sin(u)\,du\\ &= -(-\cos(u)) + C\\ &= \cos(u) + C\\ &= \cos\left((1-x)^2\right) + C\end{aligned}$$

Therefore the equation of the orbit is

$$y(x) = \cos\left((1-x)^2\right) + C$$

For different values of C, different orbit results.

2 Section 4.4, problem 2

In each of Problems l-3, verify that $x(t), y(t)$ is a solution of the given system of equations, and find its orbit.

$$\begin{aligned}\dot{x} &= e^{-x}\\ \dot{y} &= e^{e^{x}-1}\\ x(t) &= \ln(1+t)\\ y(t) &= e^{t}\end{aligned}$$

<u>Solution</u>

$$\begin{aligned}\frac{dx}{dt} &= \frac{d}{dt}\ln(1+t)\\ \dot{x} &= \frac{1}{1+t}\end{aligned}$$

But $e^{-x} = e^{-\ln(1+t)} = \frac{1}{1+t}$. Verified OK. And

$$\begin{aligned}\frac{dy}{dt} &= \frac{d}{dt}e^{t}\\ \dot{y} &= e^{t}\end{aligned}$$

But $x - 1 = \ln(1+t) - 1$. Hence $\ln(1+t) = x$. Therefore $1 + t = e^{x}$ or $t = e^{x} - 1$. Therefore $\dot{y} = e^{t} = e^{e^{x}-1}$. Verified OK.

Now we need to find system orbit. The Orbit is given by the equation

$$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}$$

When we write the given system in the following form

$$\begin{aligned}\dot{x} &= f(x,y)\\ \dot{y} &= g(x,y)\end{aligned}$$

We see now that $f(x,y) = e^{-x}$ and $g(x,y) = e^{e^{x}-1}$. Therefore

$$\frac{dy}{dx} = \frac{e^{e^{x}-1}}{e^{-x}}$$

Integrating

$$\int dy = \int \frac{e^{e^{x}-1}}{e^{-x}}dx$$

Let $e^{x} = u, du = e^{x}dx$. Hence the RHS $\int \frac{e^{e^{x}-1}}{e^{-x}}dx = \int \frac{e^{u-1}}{\frac{1}{u}}\frac{du}{u} = \int e^{u-1}du = e^{u-1} = e^{e^{x}-1}$. The above becomes

$$y = e^{e^{x}-1} + C$$

The orbits are given by the above equation for different C

3 Section 4.4, problem 3

In each of Problems l-3, verify that $x(t), y(t)$ is a solution of the given system of equations, and find its orbit.

$$\begin{aligned}\dot{x} &= 1 + x^2\\ \dot{y} &= \left(1 + x^2\right)\sec^2 x\\ x(t) &= \tan t\\ y(t) &= \tan(\tan t)\end{aligned}$$

solution

Orbits given by

$$\begin{aligned}\frac{dy}{dx} &= \frac{\left(1 + x^2\right)\sec^2 x}{1 + x^2}\\ &= \sec^2 x\end{aligned}$$

Hence

$$\int dy = \int \sec^2 x dx$$

But $\sec^2 x = \frac{1}{\cos^2 x}$. And $\frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$. Hence $\int \sec^2 x dx = \tan x$. Therefore the above gives

$$y = \tan x + C$$

The orbits are given by the above equation for different C. (do not know why book gives only $y = \tan x$)

4 Section 4.4, problem 8

Find the orbits of each of the following systems

$$\dot{x} = y + x^2 y$$
$$\dot{y} = 3x + xy^2$$

Solution

The Orbit is given by the equation

$$\frac{dy}{dx} = \frac{g\left(x,y\right)}{f\left(x,y\right)}$$

When we write the given system in the following form

$$\dot{x} = f\left(x,y\right)$$
$$\dot{y} = g\left(x,y\right)$$

We see now that $f\left(x,y\right) = y + x^2 y$ and $g\left(x,y\right) = 3x + xy^2$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{3x + xy^2}{y + x^2 y} \\ &= \frac{x\left(3 + y^2\right)}{y\left(1 + x^2\right)} \\ &= \frac{x}{\left(1 + x^2\right)}\frac{\left(3 + y^2\right)}{y}\end{aligned}$$

Hence it is separable.

$$\begin{aligned}\int \frac{y}{3 + y^2} dy &= \int \frac{x}{1 + x^2} dx \\ \frac{1}{2}\ln\left(3 + y^2\right) &= \frac{1}{2}\ln\left(1 + x^2\right) + C_2 \\ \ln\left(3 + y^2\right) &= \ln\left(1 + x^2\right) + C_1\end{aligned}$$

Therefore

$$\begin{aligned}3 + y^2 &= e^{\ln\left(1 + x^2\right) + C_1} \\ &= e^{C_1} e^{\ln\left(1 + x^2\right)} \\ &= C\left(1 + x^2\right)\end{aligned}$$

Hence

$$\begin{aligned}y^2 &= C\left(1 + x^2\right) - 3 \\ y(x) &= \pm\sqrt{C\left(1 + x^2\right) - 3}\end{aligned}$$

The above gives the equations for the orbit. For each C value, there is a different orbit curve.

Now we need to find equilibrium points, since these are orbits also. We need to solve

$$0 = y + x^2 y$$
$$0 = 3x + xy^2$$

Or

$$0 = y\left(1 + x^2\right)$$
$$0 = x\left(3 + y^2\right)$$

First equation gives $y = 0$ as only real solution. When $y = 0$ then second equation gives $x = 0$. Hence $(0, 0)$ is also an orbit. So the orbits are

$$y^2 = C\left(1 + x^2\right) - 3 \qquad C \neq 3$$
$$\left(x, y\right) = (0, 0)$$

And when $C = 3$ we obtain orbits $y^2 = 3\left(1 + x^2\right) - 3 = 3x^2$, with additional orbits (notice that we have to exclude $x = 0$ from each one below, since $x = 0$ is allready included in $\left(x, y\right) = (0, 0)$)

$$y = \sqrt{3}x \qquad x > 0$$
$$y = \sqrt{3}x \qquad x < 0$$
$$y = -\sqrt{3}x \qquad x > 0$$
$$y = -\sqrt{3}x \qquad x < 0$$

Hence there are 6 possible orbits in total.

5 Section 4.7, problem 3

Draw the phase portraits of each of the following systems of differential equations

$$\dot{x} = \begin{pmatrix} 4 & -1 \\ -2 & 5 \end{pmatrix} x$$

solution

$$\begin{aligned} \det\left(A - \lambda I\right) &= 0 \\ \begin{vmatrix} 4-\lambda & -1 \\ -2 & 5-\lambda \end{vmatrix} &= 0 \\ (4-\lambda)(5-\lambda) - 2 &= 0 \\ \lambda^2 - 9\lambda + 18 &= 0 \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= 6 \\ \lambda_2 &= 3 \end{aligned}$$

Case $\lambda_1 = 6$

$$\begin{aligned} \begin{pmatrix} 4-\lambda & -1 \\ -2 & 5-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 4-6 & -1 \\ -2 & 5-6 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -2 & -1 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

From first row $-2v_1 - v_2 = 0$. Hence $v_2 = -2v_1$. Therefore the first eigenvector is $\boldsymbol{v}^1 = \begin{pmatrix} v_1 \\ -2v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ by setting $v_1 = 1$

Case $\lambda_1 = 3$

$$\begin{aligned} \begin{pmatrix} 4-\lambda & -1 \\ -2 & 5-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 4-3 & -1 \\ -2 & 5-3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

From first row $v_1 - v_2 = 0$. Hence $v_2 = v_1$. Therefore the second eigenvector is $v^2 = \begin{pmatrix} v_1 \\ v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ by setting $v_1 = 1$

Since eigenvalues are both real and both are positive, then $(0,0)$ is <u>unstable node</u>. Here is a the Phase portrait. The lines marked red and blue are the two eigenvectors found above. The arrows are all leaving $(0,0)$ which means this is unstable equilibrium point.

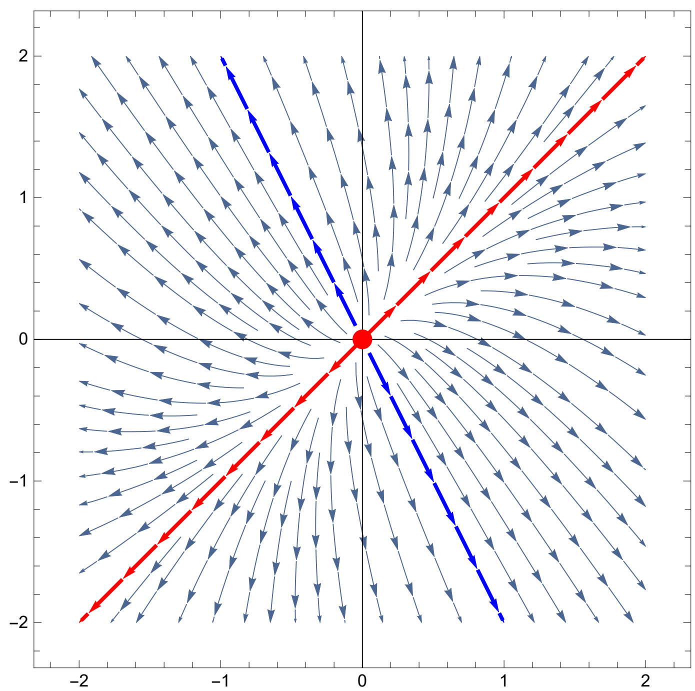

Figure 1: Phase portrait

```
p = StreamPlot[{4 x - y, -2 x + 5 y}, {x, -2, 2}, {y, -2, 2},
        StreamPoints → {
              {
                 {{1, 1}, {Thick, Red}},
                 {{1, -2}, {Thick, Blue}},
                 {{-1, -1}, {Thick, Red}},
                 {{-1, 2}, {Thick, Blue}},
                 Automatic}
           }, Epilog → {Red, PointSize[0.03], Point[{0, 0}]},
        Axes → True];
```

Figure 2: Code used

6 Section 4.7, problem 6

Draw the phase portraits of each of the following systems of differential equations

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 3 & -1 \\ 5 & -3 \end{pmatrix} \boldsymbol{x}$$

Solution

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \begin{vmatrix} 3-\lambda & -1 \\ 5 & -3-\lambda \end{vmatrix} &= 0 \\ (3-\lambda)(-3-\lambda) + 5 &= 0 \\ \lambda^2 - 4 &= 0 \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= 2 \\ \lambda_2 &= -2 \end{aligned}$$

We see that one eigenvalue is stable and one is not stable.

Case $\lambda_1 = 2$

$$\begin{aligned} \begin{pmatrix} 3-\lambda & -1 \\ 5 & -3-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 3-2 & -1 \\ 5 & -3-2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 & -1 \\ 5 & -5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

From first row $v_1 - v_2 = 0$. Hence $v_2 = v_1$. Therefore the first eigenvector is $\boldsymbol{v}^1 = \begin{pmatrix} v_1 \\ v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ by setting $v_1 = 1$

<u>Case $\lambda_1 = -2$</u>

$$\begin{pmatrix} 3-\lambda & -1 \\ 5 & -3-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 3+2 & -1 \\ 5 & -3+2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 5 & -1 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From first row $5v_1 - v_2 = 0$. Hence $v_2 = 5v_1$. Therefore the first eigenvector is $v^1 = \begin{pmatrix} v_1 \\ 5v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$ by setting $v_1 = 1$.

Since one eigenvalue is stable and one is not, then $(0,0)$ is unstable saddle point. Here is a the Phase portrait. The lines marked red and blue are the two eigenvectors found above.

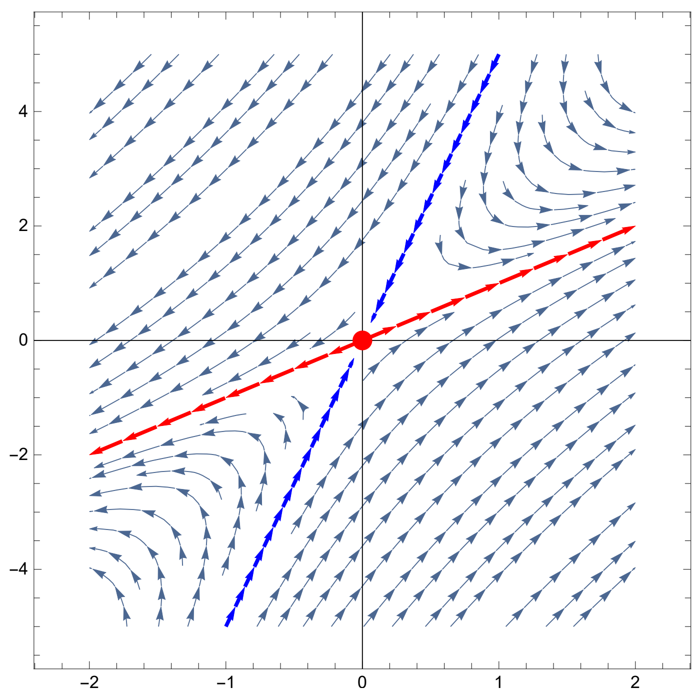

Figure 3: Phase portrait

```
p = StreamPlot[{3 x - y, 5 x - 3 y}, {x, -2, 2}, {y, -5, 5},
        StreamPoints → {
              {
                 {{1, 1}, {Thick, Red}},
                 {{1, 5}, {Thick, Blue}},
                 {{-1, -1}, {Thick, Red}},
                 {{-1, -5}, {Thick, Blue}},
                 Automatic}
           }, Epilog → {Red, PointSize[0.03], Point[{0, 0}]},
        Axes → True];
```

Figure 4: Code used

7 Section 4.7, problem 9

Draw the phase portraits of each of the following systems of differential equations

$$\dot{x} = \begin{pmatrix} 2 & 1 \\ -5 & -2 \end{pmatrix} x$$

solution

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \begin{vmatrix} 2-\lambda & -1 \\ -5 & -2-\lambda \end{vmatrix} &= 0 \\ (2-\lambda)(-2-\lambda) + 5 &= 0 \\ \lambda^2 + 1 &= 0 \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= i \\ \lambda_2 &= -i \end{aligned}$$

The real part is zero. Hence $(0,0)$ equilibrium point is called CENTER. it is stable, but not asymptotically stable.

Case $\lambda_1 = i$

$$\begin{aligned} \begin{pmatrix} 2-\lambda & 1 \\ -5 & -2-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 2-i & 1 \\ -5 & -2-i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

From second row $-5v_1 - (2+i)v_2 = 0$. Hence $v_2 = -\frac{5}{(2+i)}v_1$. Therefore the first eigenvector is $v^1 = \begin{pmatrix} v_1 \\ -\frac{5}{(2+i)}v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ -\frac{5}{(2+i)} \end{pmatrix} = \begin{pmatrix} -(2+i) \\ 5 \end{pmatrix}$ by setting $v_1 = 1$

Case $\lambda_1 = -i$

$$\begin{aligned} \begin{pmatrix} 2-\lambda & 1 \\ -5 & -2-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 2+i & 1 \\ -5 & -2+i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

From second row $-5v_1 + (-2+i)v_2 = 0$. Hence $v_2 = -\frac{5}{(-2+i)}v_1$. Therefore the first eigenvector is $v^1 = \begin{pmatrix} v_1 \\ -\frac{5}{(-2+i)}v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ -\frac{5}{(-2+i)} \end{pmatrix} = \begin{pmatrix} -2+i \\ 5 \end{pmatrix}$ by setting $v_1 = 1$

$(0,0)$ equilibrium point is called CENTER with curves making closed circles around $(0,0)$ as shown below

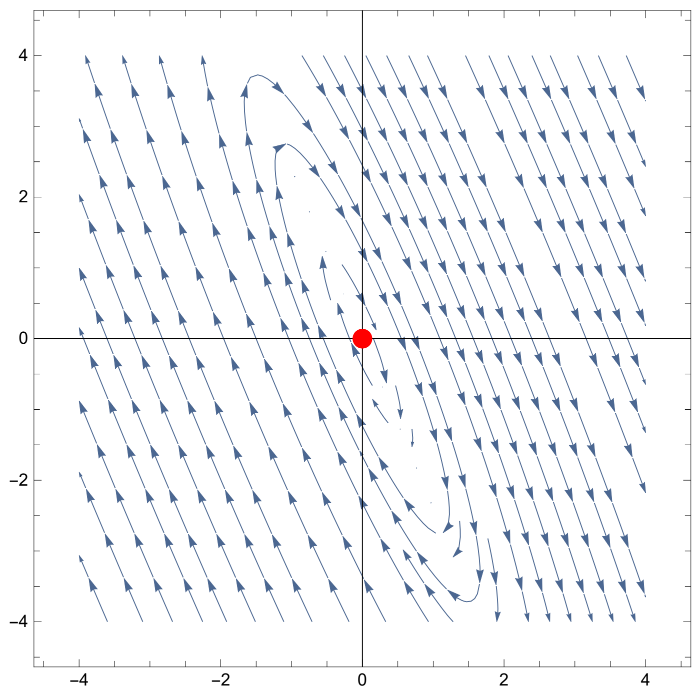

Figure 5: Phase portrait

```
p = StreamPlot[{2 x + y, -5 x - 2 y}, {x, -4, 4}, {y, -4, 4}
        , Epilog → {Red, PointSize[0.03], Point[{0, 0}]},
        Axes → True];
```

Figure 6: Code used