HW 6

Math 4512
Differential Equations with Applications

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Contents

1 Section 3.8, problem 12

Solve

$$\dot{x} = \begin{pmatrix} 3 & 1 & -2 \\ -1 & 2 & 1 \\ 4 & 1 & -3 \end{pmatrix} x, \qquad x(0) = \begin{pmatrix} 1 \\ 4 \\ -7 \end{pmatrix}$$

Solution

The first step is to find the eigenvalues. For this we need to solve

$$\begin{aligned}
\det(A - \lambda I) &= 0 \\
\begin{vmatrix} 3-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 4 & 1 & -3-\lambda \end{vmatrix} &= 0 \\
(3-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} - \begin{vmatrix} -1 & 1 \\ 4 & -3-\lambda \end{vmatrix} - 2\begin{vmatrix} -1 & 2-\lambda \\ 4 & 1 \end{vmatrix} &= 0 \\
(3-\lambda)((2-\lambda)(-3-\lambda)-1) - ((3+\lambda)-4) - 2(-1-4(2-\lambda)) &= 0 \\
\lambda^3 - 2\lambda^2 - \lambda + 2 &= 0
\end{aligned}$$

Guessing a root at $\lambda = 1$ is verified to be correct since $1 - 2 - 1 + 2 = 0$. Now that we know one root, we can do long division $\frac{(\lambda^3 - 2\lambda^2 - \lambda + 2)}{(\lambda - 1)} = \lambda^2 - \lambda - 2$. Therefore the charaterstic polynominal factors to

$$\begin{aligned}
\lambda^3 - 2\lambda^2 - \lambda + 2 &= (\lambda - 1)\left(\lambda^2 - \lambda - 2\right) \\
&= (\lambda - 1)(\lambda - 2)(\lambda + 1)
\end{aligned}$$

Hence the eigenvalues are $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = -1$.

For $\lambda_1 = 1$

$$\begin{aligned}
(A - \lambda_1 I)\,v_1 &= 0 \\
\begin{pmatrix} 3-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 4 & 1 & -3-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 3-1 & 1 & -2 \\ -1 & 2-1 & 1 \\ 4 & 1 & -3-1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 2 & 1 & -2 \\ -1 & 1 & 1 \\ 4 & 1 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}$$

Let $v_1 = 1$. First equation gives $2 + v_2 - 2v_3 = 0$ and the second equation gives $-1 + v_2 + v_3 = 0$. Subtracting gives $3 - 3v_3 = 0$, giving $v_3 = 1$. Therefore $v_2 = 0$. Hence

$$v^1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 2$

$$(A - \lambda_2 I)\, \boldsymbol{v}^2 = 0$$

$$\begin{pmatrix} 3-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 4 & 1 & -3-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3-2 & 1 & -2 \\ -1 & 2-2 & 1 \\ 4 & 1 & -3-2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -2 \\ -1 & 0 & 1 \\ 4 & 1 & -5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Let $v_1 = 1$. Hence first equation gives $1 + v_2 - 2v_3 = 0$ and second equation gives $-1 + v_3 = 0$. Therefore $v_3 = 1$ and $v_2 = 2v_3 - 1 = 1$. Hence

$$\boldsymbol{v}^2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

For $\lambda_3 = -1$

$$(A - \lambda_3 I)\, \boldsymbol{v}^3 = 0$$

$$\begin{pmatrix} 3-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 4 & 1 & -3-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3+1 & 1 & -2 \\ -1 & 2+1 & 1 \\ 4 & 1 & -3+1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 1 & -2 \\ -1 & 3 & 1 \\ 4 & 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Let $v_1 = 1$. Hence first equation gives $4 + v_2 - 2v_3 = 0$ and second equation gives $-1 + 3v_2 + v_3 = 0$. Multiplying $4 + v_2 - 2v_3 = 0$ by -3 and adding it to $-1 + 3v_2 + v_3 = 0$ gives $(-12 - 3v_2 + 6v_3 + (-1 + 3v_2 + v_3)) = 0$ or $-13 + 7v_3 = 0$ Hence $v_3 = \frac{13}{7}$. Therefore $v_2 = 2v_3 - 4 = 2\left(\frac{13}{7}\right) - 4 = -\frac{2}{7}$. Hence

$$\boldsymbol{v}^3 = \begin{pmatrix} 1 \\ -\frac{2}{7} \\ \frac{13}{7} \end{pmatrix}$$

Therefore

$$\boldsymbol{x}^1(t) = e^{\lambda_1 t}\boldsymbol{v}^1 = e^{t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\boldsymbol{x}^2(t) = e^{\lambda_2 t}\boldsymbol{v}^2 = e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\boldsymbol{x}^3(t) = e^{\lambda_3 t}\boldsymbol{v}^3 = e^{-t} \begin{pmatrix} 1 \\ -\frac{2}{7} \\ \frac{13}{7} \end{pmatrix}$$

Hence the general solution is

$$\begin{aligned} \boldsymbol{x}(t) &= c_1\boldsymbol{x}^1(t) + c_2\boldsymbol{x}^2(t) + c_3\boldsymbol{x}^3(t) \\ &= e^t\begin{pmatrix}1\\0\\1\end{pmatrix} + c_2e^{2t}\begin{pmatrix}1\\1\\1\end{pmatrix} + c_3e^{-t}\begin{pmatrix}1\\-\frac{2}{7}\\\frac{13}{7}\end{pmatrix} \end{aligned}$$

Or

$$\boldsymbol{x}(t) = \begin{pmatrix} c_1e^t + c_2e^{2t} + c_3e^{-t} \\ c_2e^{2t} - \frac{2}{7}c_3e^{-t} \\ c_1e^t + c_2e^{2t} + \frac{13}{7}c_3e^{-t} \end{pmatrix} \tag{A}$$

Initial conditions are now used to find c_1, c_2, c_3. At $t = 0$ the above reduces to

$$\boldsymbol{x}(0) = \begin{pmatrix}1\\4\\-7\end{pmatrix}$$

$$\begin{pmatrix} c_1 + c_2 + c_3 \\ c_2 - \frac{2}{7}c_3 \\ c_1 + c_2 + \frac{13}{7}c_3 \end{pmatrix} = \begin{pmatrix}1\\4\\-7\end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -\frac{2}{7} \\ 1 & 1 & \frac{13}{7} \end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}1\\4\\-7\end{pmatrix} \tag{1}$$

Gaussian elimination on $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -\frac{2}{7} \\ 1 & 1 & \frac{13}{7} \end{pmatrix}\begin{pmatrix}1\\4\\-7\end{pmatrix}$. Replacing row 3 by row 3 - row 1 gives

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -\frac{2}{7} \\ 0 & 0 & \frac{13}{7}-1 \end{pmatrix}\begin{pmatrix}1\\4\\-7-1\end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -\frac{2}{7} \\ 0 & 0 & \frac{6}{7} \end{pmatrix}\begin{pmatrix}1\\4\\-8\end{pmatrix}$$

Hence (1) becomes

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -\frac{2}{7} \\ 0 & 0 & \frac{6}{7} \end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}1\\4\\-8\end{pmatrix}$$

Back substitution gives $\frac{6}{7}c_3 = -8$, or $c_3 = -\frac{28}{3}$. From second row

$$\begin{aligned} c_2 - \frac{2}{7}c_3 &= 4 \\ c_2 &= 4 + \frac{2}{7}c_3 \\ &= 4 + \frac{2}{7}\left(-\frac{28}{3}\right) \\ &= \frac{4}{3} \end{aligned}$$

From first row

$$\begin{aligned} c_1 + c_2 + c_3 &= 1 \\ c_1 &= 1 - c_2 - c_3 \\ &= 1 - \frac{4}{3} + \frac{28}{3} \\ &= 9 \end{aligned}$$

Using the above values of c_1, c_2, c_3, Eq (A) becomes

$$\begin{aligned}
\boldsymbol{x}(t) &= \begin{pmatrix} c_1 e^t + c_2 e^{2t} + c_3 e^{-t} \\ c_2 e^{2t} - \frac{2}{7} c_3 e^{-t} \\ c_1 e^t + c_2 e^{2t} + \frac{13}{7} c_3 e^{-t} \end{pmatrix} \\
&= \begin{pmatrix} 9e^t + \frac{4}{3} e^{2t} - \frac{28}{3} e^{-t} \\ \frac{4}{3} e^{2t} - \frac{2}{7}\left(-\frac{28}{3}\right) e^{-t} \\ 9e^t + \frac{4}{3} e^{2t} + \frac{13}{7}\left(-\frac{28}{3}\right) e^{-t} \end{pmatrix} \\
&= \begin{pmatrix} 9e^t + \frac{4}{3} e^{2t} - \frac{28}{3} e^{-t} \\ \frac{4}{3} e^{2t} + \frac{8}{3} e^{-t} \\ 9e^t + \frac{4}{3} e^{2t} - \frac{52}{3} e^{-t} \end{pmatrix} \qquad (2)
\end{aligned}$$

Therefore

$$\begin{aligned}
x_1(t) &= 9e^t + \frac{4}{3} e^{2t} - \frac{28}{3} e^{-t} \\
x_2(t) &= \frac{4}{3} e^{2t} + \frac{8}{3} e^{-t} \\
x_3(t) &= 9e^t + \frac{4}{3} e^{2t} - \frac{52}{3} e^{-t}
\end{aligned}$$

2 Section 3.9, problem 2 (complex roots)

Find general solution of

$$\dot{x} = \begin{pmatrix} 1 & -5 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix} x$$

Solution

The first step is to find the eigenvalues. For this we need to solve

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & -5 & 0 \\ 1 & -3-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\begin{vmatrix} -3-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} + 5\begin{vmatrix} 1 & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)((-3-\lambda)(1-\lambda)) + 5(1-\lambda) = 0$$

Factoring $(1-\lambda)$ gives

$$(1-\lambda)((-3-\lambda)(1-\lambda)+5) = 0$$

$$(1-\lambda)\left(\lambda^2 + 2\lambda - 3 + 5\right) = 0$$

$$(1-\lambda)\left(\lambda^2 + 2\lambda + 2\right) = 0$$

Hence one root is $\lambda_1 = 1$. Now we find roots of $\left(\lambda^2 + 2\lambda + 2\right)$. $\lambda = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac} = -1 \pm \frac{1}{2}\sqrt{4 - 4(2)} = -1 \pm \frac{1}{2}\sqrt{-4}$. Hence

$$\lambda = -1 \pm i$$

Therefore the roots are

$$\lambda_1 = 1$$
$$\lambda_2 = -1 + i$$
$$\lambda_3 = -1 - i$$

For $\lambda_1 = 1$

$$(A - \lambda_1 I)\, v^1 = 0$$

$$\begin{pmatrix} 1-\lambda_1 & -5 & 0 \\ 1 & -3-\lambda_1 & 0 \\ 0 & 0 & 1-\lambda_1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1-1 & -5 & 0 \\ 1 & -3-1 & 0 \\ 0 & 0 & 1-1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -5 & 0 \\ 1 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Hence v_3 is arbitrary, say $v_3 = 1$. And $v_2 = 0$ from first equation. And from second equation $v_1 = 0$. Therefore

$$v^1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Hence

$$\begin{aligned} \boldsymbol{x}^1(t) &= e^{\lambda_1 t}\boldsymbol{v}^1 \\ &= e^t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

<u>For $\lambda_2 = -1 + i$</u>

$$\begin{aligned} (A - \lambda_2 I)\boldsymbol{v}^2 &= 0 \\ \begin{pmatrix} 1-\lambda_2 & -5 & 0 \\ 1 & -3-\lambda_2 & 0 \\ 0 & 0 & 1-\lambda_2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1-(-1+i) & -5 & 0 \\ 1 & -3-(-1+i) & 0 \\ 0 & 0 & 1-(-1+i) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 2-i & -5 & 0 \\ 1 & -2-i & 0 \\ 0 & 0 & 2-i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

From last equation $v_3 = 0$. from second equation $v_1 = (2+i)v_2$. Hence

$$\boldsymbol{v}^2 = \begin{pmatrix} (2+i)v_2 \\ v_2 \\ 0 \end{pmatrix} = v_2 \begin{pmatrix} 2+i \\ 1 \\ 0 \end{pmatrix}$$

Choosing $v_2 = 1$ the above becomes

$$\boldsymbol{v}^2 = \begin{pmatrix} 2+i \\ 1 \\ 0 \end{pmatrix}$$

Hence

$$\boldsymbol{x}^2_{\lambda_2}(t) = e^{\lambda_2 t}\boldsymbol{v}^2 = e^{(-1+i)t}\begin{pmatrix} 2+i \\ 1 \\ 0 \end{pmatrix}$$

Since this is complex root, we will now find the real and imaginary parts of the above, and use these to generate $\boldsymbol{x}^2(t), \boldsymbol{x}^3(t)$ from the above.

$$
\begin{aligned}
e^{(-1+i)t}\begin{pmatrix}2+i\\1\\0\end{pmatrix} &= e^{-t}e^{it}\begin{pmatrix}2+i\\1\\0\end{pmatrix}\\
&= e^{-t}(\cos t + i\sin t)\begin{pmatrix}2+i\\1\\0\end{pmatrix}\\
&= \left(e^{-t}\cos t + ie^{-t}\sin t\right)\begin{pmatrix}2+i\\1\\0\end{pmatrix}\\
&= \begin{pmatrix}\left(e^{-t}\cos t + ie^{-t}\sin t\right)(2+i)\\ \left(e^{-t}\cos t + ie^{-t}\sin t\right)\\ 0\end{pmatrix}\\
&= \begin{pmatrix}2e^{-t}\cos t + ie^{-t}\cos t + 2ie^{-t}\sin t - e^{-t}\sin t\\ e^{-t}\cos t + ie^{-t}\sin t\\ 0\end{pmatrix}\\
&= \begin{pmatrix}\left(2e^{-t}\cos t - e^{-t}\sin t\right) + i\left(e^{-t}\cos t + 2e^{-t}\sin t\right)\\ e^{-t}\cos t + ie^{-t}\sin t\\ 0\end{pmatrix}
\end{aligned}
$$

The real of the above is

$$
\boldsymbol{x}^2(t) = \begin{pmatrix}2e^{-t}\cos t - e^{-t}\sin t\\ e^{-t}\cos t\\ 0\end{pmatrix}
$$

And imaginary part is

$$
\boldsymbol{x}^3(t) = \begin{pmatrix}e^{-t}\cos t + 2e^{-t}\sin t\\ e^{-t}\sin t\\ 0\end{pmatrix}
$$

We have now obtain the three eigenvectors we want. Hence the general solution is

$$
\begin{aligned}
\boldsymbol{x}(t) &= c_1\boldsymbol{x}^1(t) + c_2\boldsymbol{x}^2(t) + c_3\boldsymbol{x}^3(t)\\
&= c_1e^{t}\begin{pmatrix}0\\0\\1\end{pmatrix} + c_2e^{-t}\begin{pmatrix}2\cos t - \sin t\\ \cos t\\ 0\end{pmatrix} + c_3e^{-t}\begin{pmatrix}\cos t + 2\sin t\\ \sin t\\ 0\end{pmatrix}
\end{aligned}
$$

3 Section 3.10, problem 6 (Equal roots)

Solve

$$\dot{x} = \begin{pmatrix} -4 & -4 & 0 \\ 10 & 9 & 1 \\ -4 & -3 & 1 \end{pmatrix} x, \qquad x(0) = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

<u>Solution</u>

The first step is to find the eigenvalues. For this we need to solve

$$\begin{aligned}
\det(A - \lambda I) &= 0 \\
\begin{vmatrix} -4-\lambda & -4 & 0 \\ 10 & 9-\lambda & 1 \\ -4 & -3 & 1-\lambda \end{vmatrix} &= 0 \\
(-4-\lambda) \begin{vmatrix} 9-\lambda & 1 \\ -3 & 1-\lambda \end{vmatrix} + 4 \begin{vmatrix} 10 & 1 \\ -4 & 1-\lambda \end{vmatrix} &= 0 \\
(-4-\lambda)((9-\lambda)(1-\lambda)+3) + 4((10)(1-\lambda)+4) &= 0 \\
(\lambda-2)^3 &= 0
\end{aligned}$$

Hence root is $\lambda = 2$ of multiplicity 3.

To eigenvectors we start as before, using $\lambda = 2$.

$$\begin{aligned}
(A - \lambda I) v^1 &= 0 \\
\begin{pmatrix} -4-\lambda & -4 & 0 \\ 10 & 9-\lambda & 1 \\ -4 & -3 & 1-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} -4-2 & -4 & 0 \\ 10 & 9-2 & 1 \\ -4 & -3 & 1-2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} -6 & -4 & 0 \\ 10 & 7 & 1 \\ -4 & -3 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}$$

Now we check if the eigenvalue is complete or defective. Using the first 2 rows we obtain

$$\begin{aligned}
-6v_1 - 4v_2 &= 0 \\
10v_1 + 7v_2 + v_3 &= 0
\end{aligned}$$

Solving gives $v_1 = 2v_3, v_2 = -3v_3$. Hence

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 2v_3 \\ -3v_3 \\ v_3 \end{pmatrix} = v_3 \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

Choosing $v_3 = 1$ gives

$$v^1 = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

Lets see now if we can obtain another linearly independent eigenvector. Using the first row and the third row

$$\begin{aligned}
-6v_1 - 4v_2 &= 0 \\
-4v_1 - 3v_2 - v_3 &= 0
\end{aligned}$$

Solving gives $v_1 = 2v_3, v_2 = -3v_3$. Which is the same as the one found above. Finally using the second and third row

$$\begin{aligned} 10v_1 + 7v_2 + v_3 &= 0 \\ -4v_1 - 3v_2 - v_3 &= 0 \end{aligned}$$

Solving gives $v_1 = 2v_3, v_2 = -3v_3$ which is the same as above. So the eigenvalue 2 is defective.

$$\boldsymbol{x}^1(t) = e^{2t}\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

Since the eigenvalue is defective, to find the second and third eigenvectors we do the following. To find $\boldsymbol{v}^2$. We need to solve

$$(A - \lambda I)^2 \boldsymbol{v}^2 = 0 \tag{1}$$

But $A - \lambda I = \begin{pmatrix} -6 & -4 & 0 \\ 10 & 7 & 1 \\ -4 & -3 & -1 \end{pmatrix}$ from earlier. Hence

$$\begin{aligned} (A - \lambda I)^2 &= \begin{pmatrix} -6 & -4 & 0 \\ 10 & 7 & 1 \\ -4 & -3 & -1 \end{pmatrix}\begin{pmatrix} -6 & -4 & 0 \\ 10 & 7 & 1 \\ -4 & -3 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -4 & -4 & -4 \\ 6 & 6 & 6 \\ -2 & -2 & -2 \end{pmatrix} \end{aligned}$$

Therefore (1) becomes

$$\begin{pmatrix} -4 & -4 & -4 \\ 6 & 6 & 6 \\ -2 & -2 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Using the first equation $-4v_1 - 4v_2 - 4v_3 = 0$ or equivalently $v_1 + v_2 + v_3 = 0$. Therefore $v_1 = -v_2 - v_3$. Hence

$$\begin{aligned} \boldsymbol{v}^2 &= \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \\ &= \begin{pmatrix} -v_2 - v_3 \\ v_2 \\ v_3 \end{pmatrix} \\ &= v_2\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + v_3\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

Taking $v_2 = 1, v_3 = 0$ gives

$$\boldsymbol{v}^2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

Let us check the above choice is valid: $(A - \lambda I)\boldsymbol{v}^2 = \begin{pmatrix} -6 & -4 & 0 \\ 10 & 7 & 1 \\ -4 & -3 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ which is not zero. Good, so we can use it. Therefore

$$\begin{aligned}
x^2(t) &= e^{\lambda t}\left(v^2 + t(A-\lambda I)v^2\right)\\
&= e^{2t}\left(\begin{pmatrix}-1\\1\\0\end{pmatrix} + t\begin{pmatrix}-6 & -4 & 0\\10 & 7 & 1\\-4 & -3 & -1\end{pmatrix}\begin{pmatrix}-1\\1\\0\end{pmatrix}\right)\\
&= e^{2t}\left(\begin{pmatrix}-1\\1\\0\end{pmatrix} + t\begin{pmatrix}2\\-3\\1\end{pmatrix}\right)\\
&= e^{2t}\begin{pmatrix}-1+2t\\1-3t\\t\end{pmatrix}
\end{aligned}$$

Now we find the third eigenvector v^3. We need to solve

$$(A-\lambda I)^3 v^3 = 0 \tag{1}$$

But $(A-\lambda I)^2 = \begin{pmatrix}-4 & -4 & -4\\6 & 6 & 6\\-2 & -2 & -2\end{pmatrix}$ from earlier. Hence

$$\begin{aligned}
(A-\lambda I)^3 &= \begin{pmatrix}-4 & -4 & -4\\6 & 6 & 6\\-2 & -2 & -2\end{pmatrix}\begin{pmatrix}-6 & -4 & 0\\10 & 7 & 1\\-4 & -3 & -1\end{pmatrix}\\
&= \begin{pmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix}
\end{aligned}$$

Therefore v_3 is arbitrary as long as $(A-\lambda I)^2 v_3 \neq 0$. Let us pick $v_3 = \begin{pmatrix}1\\0\\0\end{pmatrix}$. Checking this choice is valid: $\begin{pmatrix}-4 & -4 & -4\\6 & 6 & 6\\-2 & -2 & -2\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}-4\\6\\-2\end{pmatrix}$. Not zero. Good, so we can use it. Therefore

$$\begin{aligned}
x^3(t) &= e^{\lambda t}\left(v^3 + t(A-\lambda I)v^3 + \frac{t^2}{2}(A-\lambda I)^2 v^3\right)\\
&= e^{2t}\left(\begin{pmatrix}1\\0\\0\end{pmatrix} + t\begin{pmatrix}-6 & -4 & 0\\10 & 7 & 1\\-4 & -3 & -1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} + \frac{t^2}{2}\begin{pmatrix}-4 & -4 & -4\\6 & 6 & 6\\-2 & -2 & -2\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}\right)\\
&= e^{2t}\left(\begin{pmatrix}1\\0\\0\end{pmatrix} + t\begin{pmatrix}-6\\10\\-4\end{pmatrix} + \frac{t^2}{2}\begin{pmatrix}-4\\6\\-2\end{pmatrix}\right)\\
&= e^{2t}\begin{pmatrix}1-6t-2t^2\\10t+3t^2\\-4t-t^2\end{pmatrix}
\end{aligned}$$

Therefore the general solution is

$$
\begin{aligned}
\boldsymbol{x}(t) &= c_1\boldsymbol{x}^1(t) + c_2\boldsymbol{x}^2(t) + c_3\boldsymbol{x}^3(t)\\
&= c_1e^{2t}\begin{pmatrix}2\\-3\\1\end{pmatrix} + c_2e^{2t}\begin{pmatrix}-1+2t\\1-3t\\t\end{pmatrix} + c_3e^{2t}\begin{pmatrix}1-6t-2t^2\\10t+3t^2\\-4t-t^2\end{pmatrix}\\
&= \begin{pmatrix}e^{2t}\left(2c_1 + c_2(-1+2t) + c_3\left(1-6t-2t^2\right)\right)\\ e^{2t}\left(-3c_1 + c_2(1-3t) + c_3\left(10t+3t^2\right)\right)\\ e^{2t}\left(c_1 + tc_2 + c_3\left(-4t-t^2\right)\right)\end{pmatrix}
\end{aligned}
$$

Now we find c_i from initial conditions. At $t=0$

$$
\begin{aligned}
\begin{pmatrix}2\\1\\-1\end{pmatrix} &= c_1\begin{pmatrix}2\\-3\\1\end{pmatrix} + c_2\begin{pmatrix}-1\\1\\1\end{pmatrix} + c_3\begin{pmatrix}1\\0\\0\end{pmatrix}\\
&= \begin{pmatrix}2c_1 - c_2 + c_3\\-3c_1 + c_2\\c_1\end{pmatrix}
\end{aligned}
$$

Or

$$
\begin{pmatrix}2 & -1 & 1\\-3 & 1 & 0\\1 & 0 & 0\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}2\\1\\-1\end{pmatrix} \tag{2}
$$

From last row, $c_1=-1$. From second row $-3c_1+c_2=1$, hence $c_2=1-3=-2$. From first row $2c_1-c_2+c_3=2$, hence $c_3=2-2+2=2$. Therefore the general solution becomes

$$
\begin{aligned}
\boldsymbol{x}(t) &= c_1\boldsymbol{x}^1(t) + c_2\boldsymbol{x}^2(t) + c_3\boldsymbol{x}^3(t)\\
&= -e^{2t}\begin{pmatrix}2\\-3\\1\end{pmatrix} - 2e^{2t}\begin{pmatrix}-1+2t\\1-3t\\t\end{pmatrix} + 2e^{2t}\begin{pmatrix}1-6t-2t^2\\10t+3t^2\\-4t-t^2\end{pmatrix}\\
&= e^{2t}\begin{pmatrix}-2-2(-1+2t)+2\left(1-6t-2t^2\right)\\3-2(1-3t)+2\left(10t+3t^2\right)\\-1-2t+2\left(-4t-t^2\right)\end{pmatrix}\\
&= e^{2t}\begin{pmatrix}-4t^2-16t+2\\6t^2+26t+1\\-2t^2-10t-1\end{pmatrix}
\end{aligned}
$$

Or

$$
\begin{aligned}
x_1(t) &= e^{2t}\left(-4t^2-16t+2\right)\\
x_2(t) &= e^{2t}\left(6t^2+26t+1\right)\\
x_3(t) &= e^{2t}\left(-2t^2-10t-1\right)
\end{aligned}
$$

This is a plot of the solutions. The solutions all blow up in time due to positive exponential terms.

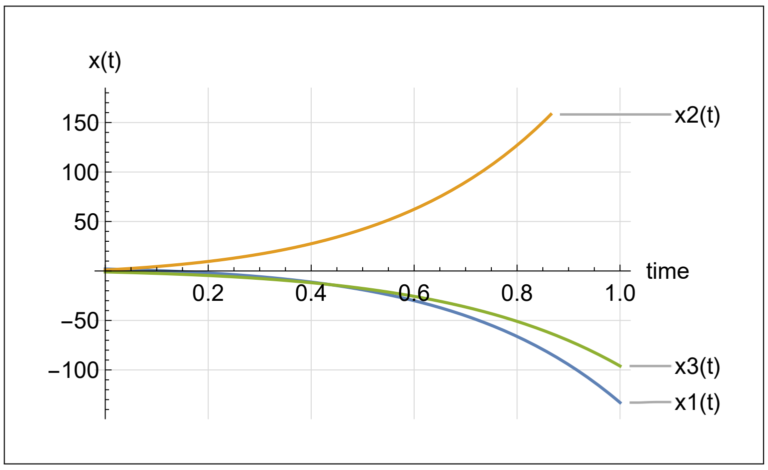

Figure 1: Plot of the solutions above