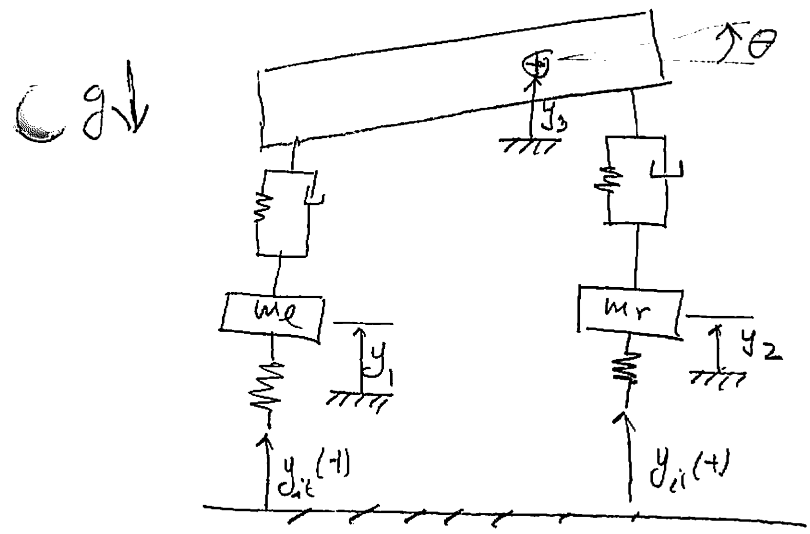
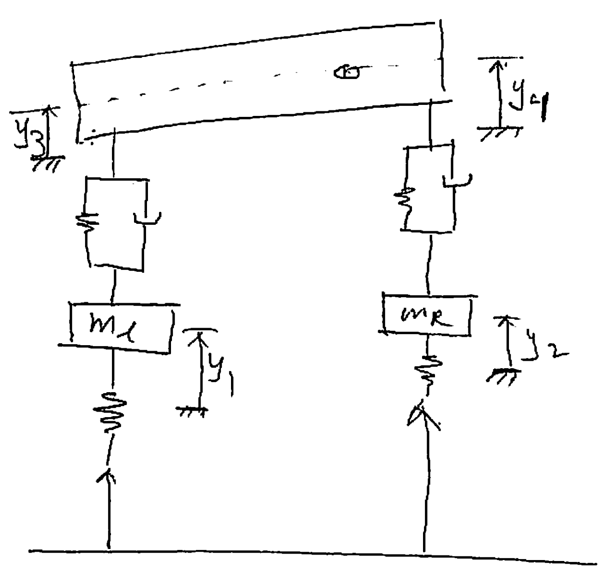
1.2

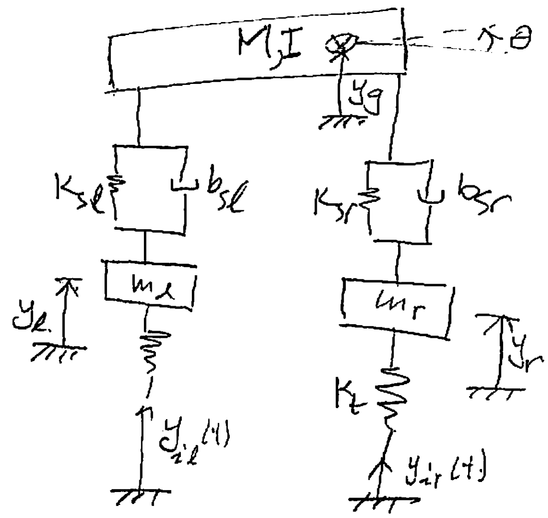


the above is 4 DOF system. 4 coordinates are shown. The coordinates are such that they are all zero when system is in static equilibrium under gravity only.

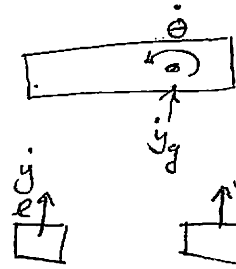
another possible 4 DOF coordinates is the following:



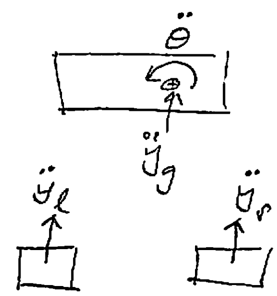
1.4



Velocity diagram



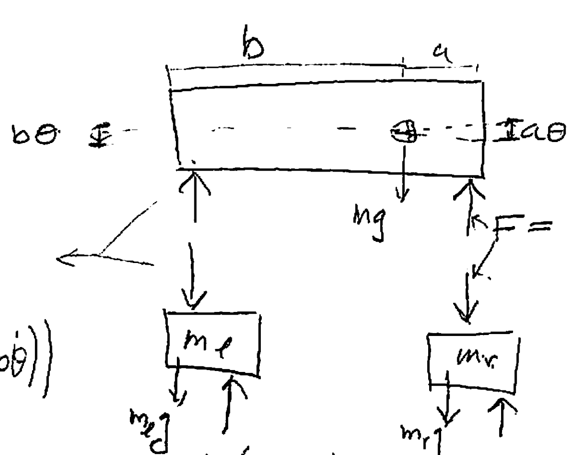
acceleration



Force diagram

assume m_r is above its equilibrium position.

assume m_ℓ is above its equilibrium position.



$$K_{s\ell}\left(y_\ell-(y_g-b\theta)\right)+b_{s\ell}\left(\dot{y}_\ell-(\dot{y}_g-b\dot{\theta})\right)$$

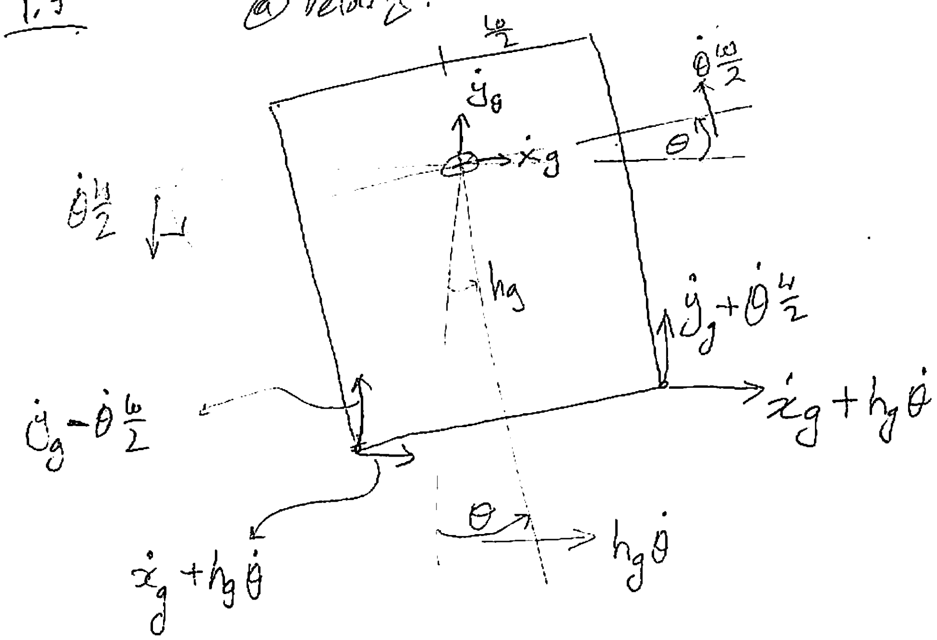
$$F=K_{sr}\left(y_r-(y_g+a\theta)\right)+b_{sr}\left(\dot{y}_r-(\dot{y}_g+a\dot{\theta})\right)$$

$$K_t\left(y_{i\ell}(t)-y_\ell\right)$$

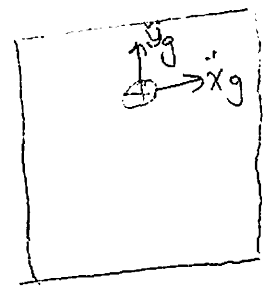
$$K_t\left(y_{ir}(t)-y_r\right)$$

1.5

@ velocity:

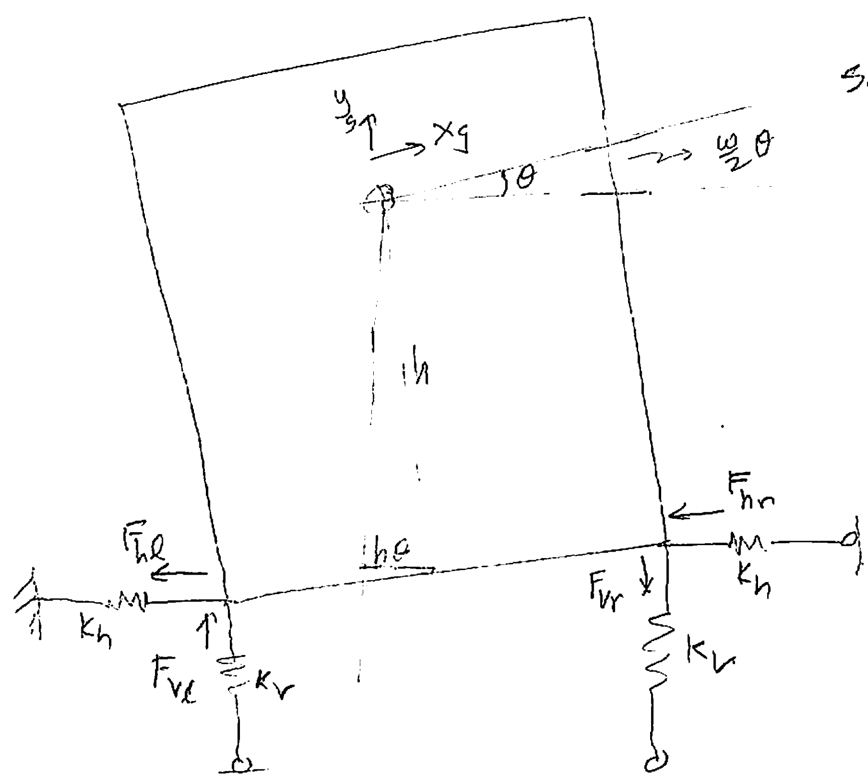


acc



Force diagram

small angle approximation



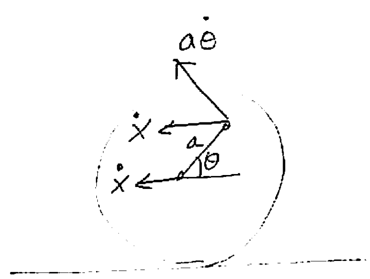
$$F_{hr} = K_h\left(x_g + h\theta\right)$$

$$F_{vr} = K_v\left(y_g + \frac{w}{2}\theta\right)$$

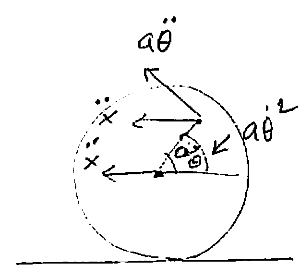
$$F_{hl} = K_h\left(x_g + h\theta\right)$$

$$F_{vl} = K_v\left(-y_g + \frac{w}{2}\theta\right)$$

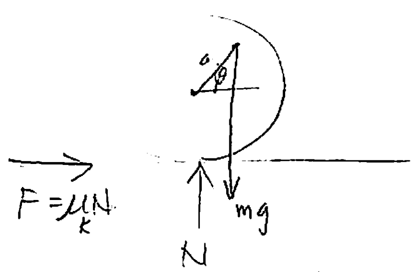
1.6



(a) velocity diagram



(b) acc diagram



(c) Force diagram

equation of motion

$$\overset{+}{\rightarrow}\sum F_x = 0 \quad \Rightarrow \quad F = m\left(-\ddot x - a\dot\theta^2\cos\theta\right)$$

or $$m\ddot x + ma\dot\theta^2\cos\theta + \underset{\to\, \mu_k mg}{F} = 0$$

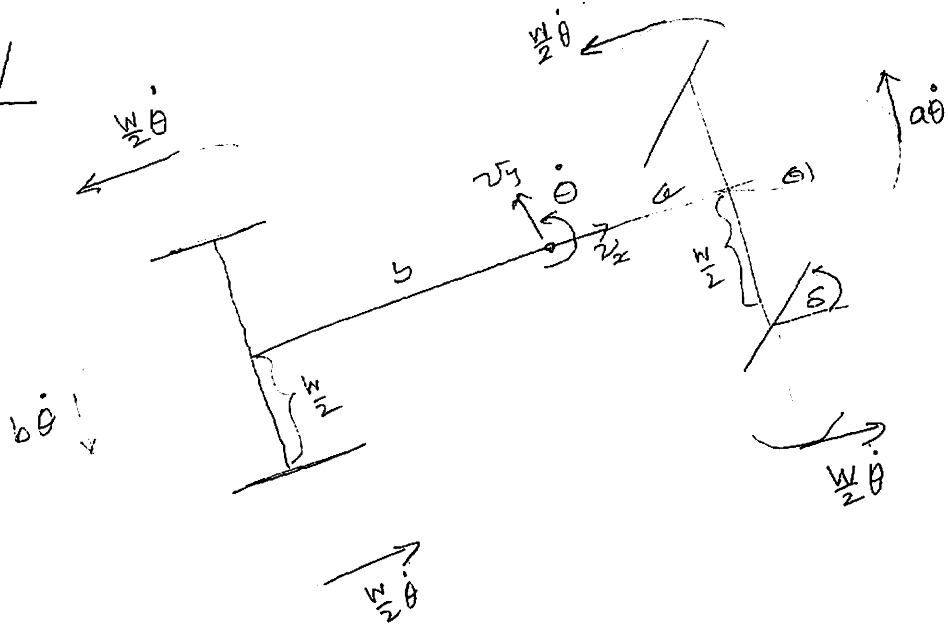
but $\theta = \dfrac{x}{R}$, hence

$$\ddot x + a\left(\frac{\dot x}{R}\right)^2\cos\left(\frac{x}{R}\right) + \mu_k g = 0$$

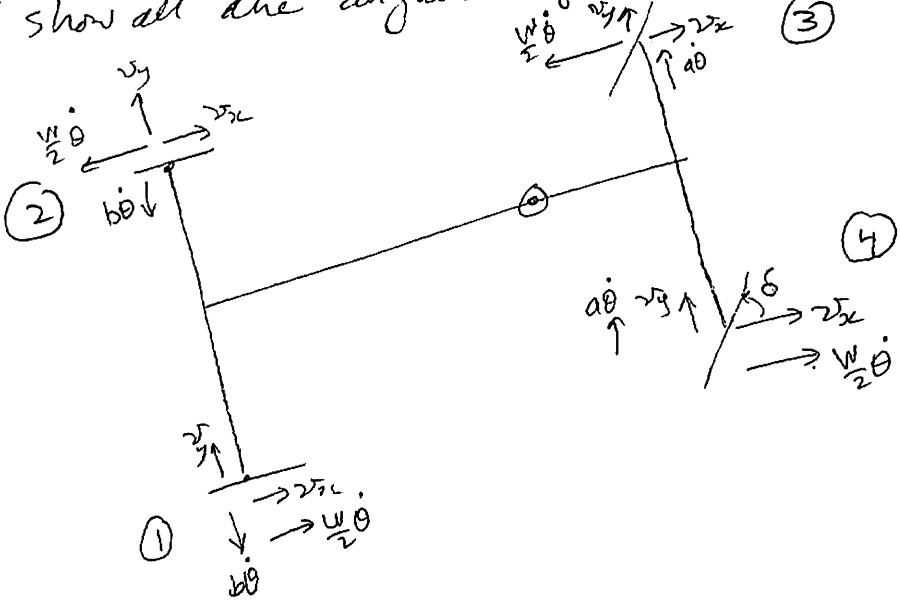
let
$$\boxed{\begin{array}{l} x_1 = x \\ x_2 = \dot x \end{array}\Bigg\} \quad \begin{array}{l} \dot x_1 = x_2 \\ \dot x_2 = -a\left(\dfrac{x_2}{R}\right)^2\cos\left(\dfrac{x_1}{R}\right) - \mu_k g \end{array}}$$
equation of motion

IC: $x_1 = 0$, x_2 = Initial velocity given to move it to left. say $v(0)$.

1.11



First, I show all the angular tangential velocities. next:

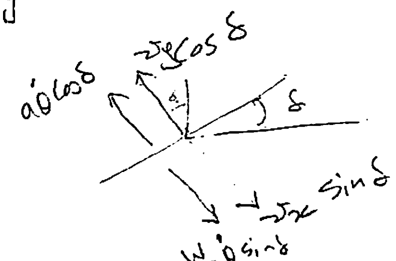


Constraints

Wheel ① $\quad v_y - b\dot\theta = 0$

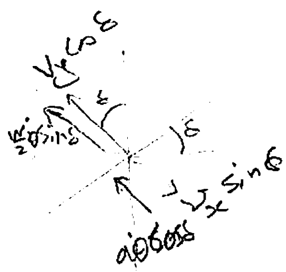
Wheel ② $\quad v_y - b\dot\theta = 0$

Wheel ④



$$\Rightarrow (v_y + a\dot\theta)\cos\delta - (v_x + \tfrac{W}{2}\dot\theta)\sin\delta = 0$$

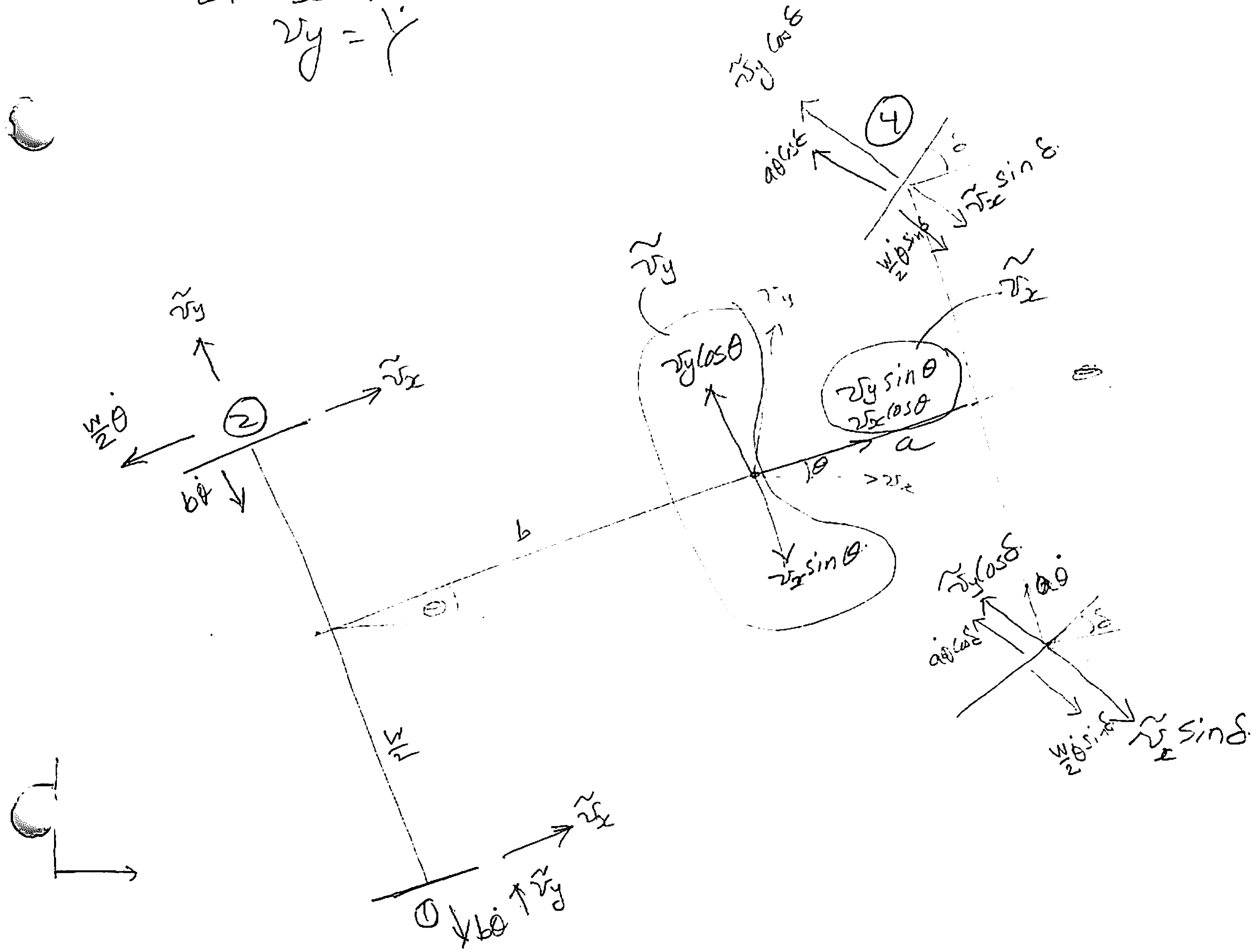
wheel ③



$$\Rightarrow (v_y + a\dot\theta)\cos\delta + (\tfrac{W}{2}\dot\theta - v_x)\sin\delta = 0$$

1.12

let $v_x = \dot{X}$
$v_y = \dot{Y}$



so this is the same as problem, but we replace v_x in the last problem with $(\underbrace{v_x\cos\theta + v_y\sin\theta}_{\tilde{v}_x})$, and replace v_y in the last problem with $(\underbrace{v_y\cos\theta - v_x\sin\theta}_{\tilde{v}_y})$.

Constraint on wheel ①

$b\dot{\theta} + v_y\cos\theta - v_x\sin\theta = 0$

note v_x, v_y here are $\dot{X}, \dot{Y}$

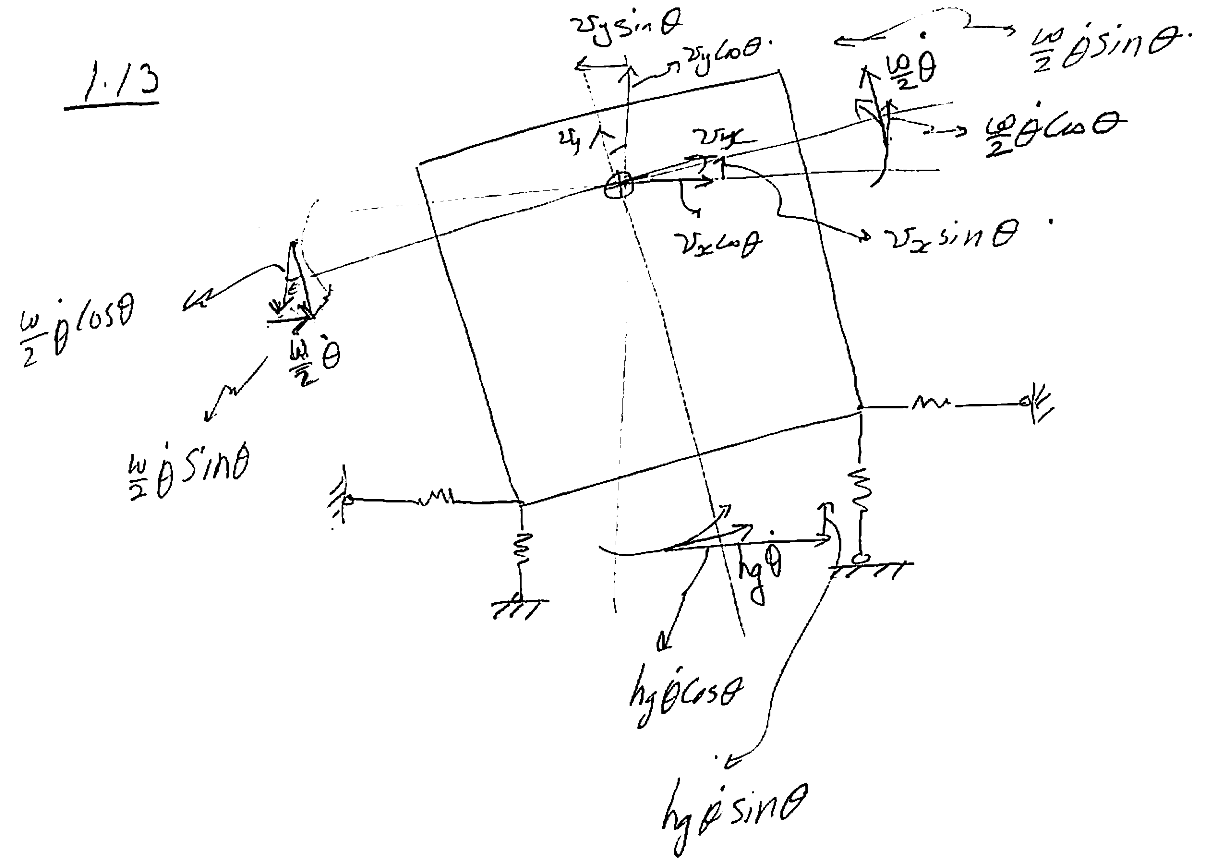
wheel ②

$\tilde{v}_y + b\dot{\theta} = 0 \Rightarrow v_y\cos\theta - v_x\sin\theta + b\dot{\theta} = 0$

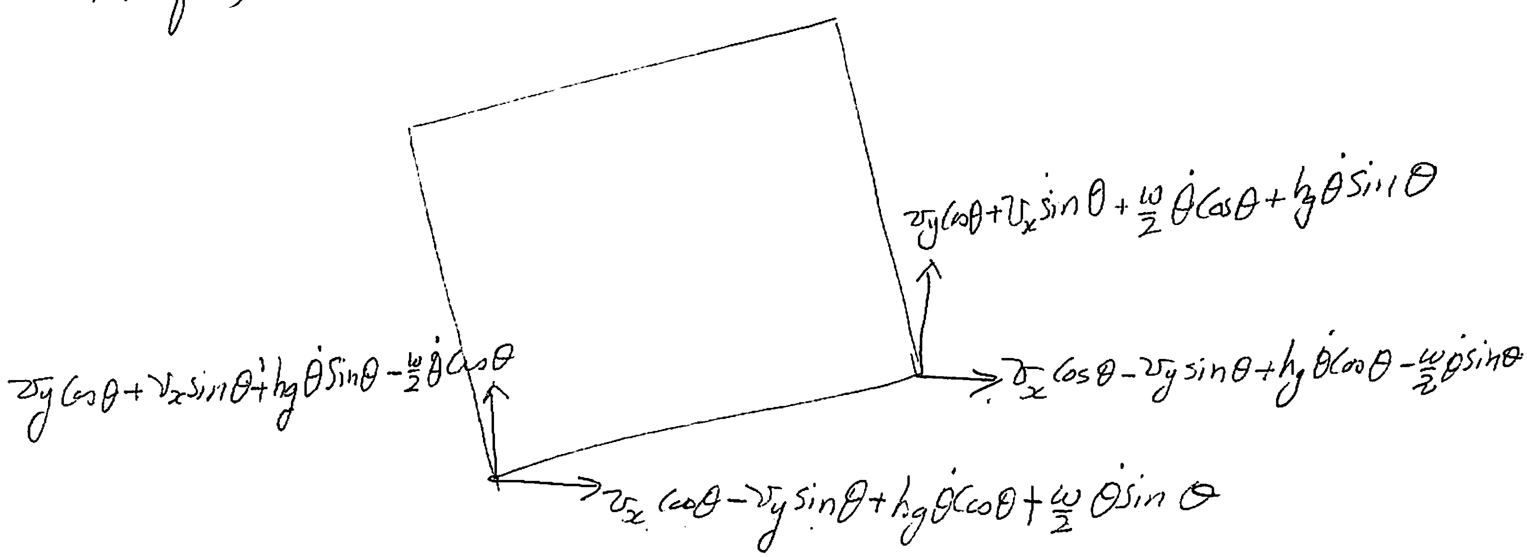
wheel ③ $-(\overbrace{v_y\sin\theta + v_x\cos\theta}^{\tilde{v}_x})\sin\delta + (\overbrace{v_y\cos\theta - v_x\sin\theta}^{\tilde{v}_y})\cos\delta + a\dot{\theta}\cos\delta - \frac{W}{2}\dot{\theta}\sin\delta = 0$

wheel ④ $-(\underbrace{v_y\sin\theta + v_x\cos\theta}_{\tilde{v}_x})\sin\delta + (\underbrace{v_y\cos\theta - v_x\sin\theta}_{\tilde{v}_y})\cos\delta + a\dot{\theta}\cos\delta - \frac{W}{2}\dot{\theta}\sin\delta = 0$

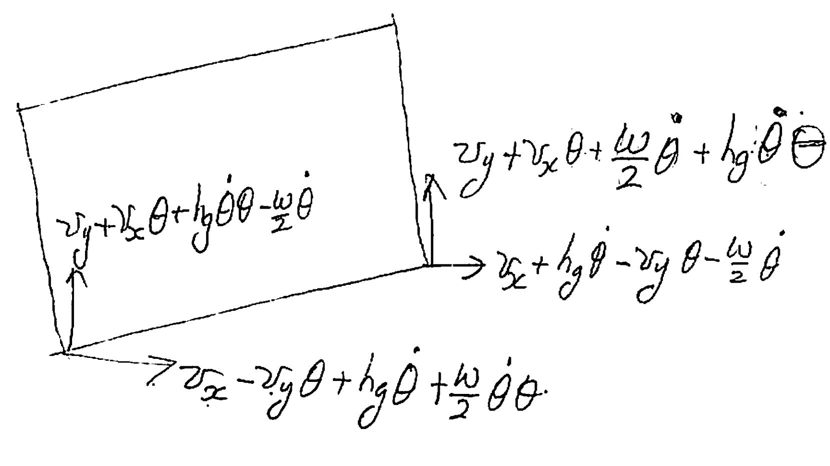
1.13



Therfor,

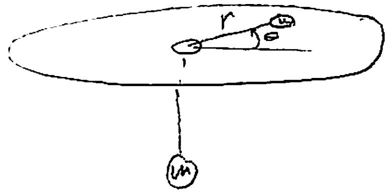


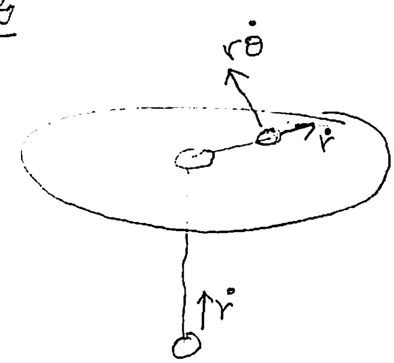
so using small angle approximation:



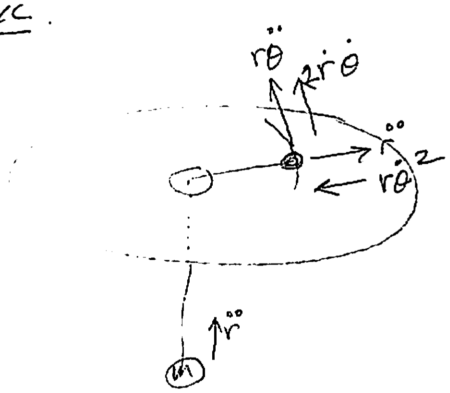
1.15

Velocity

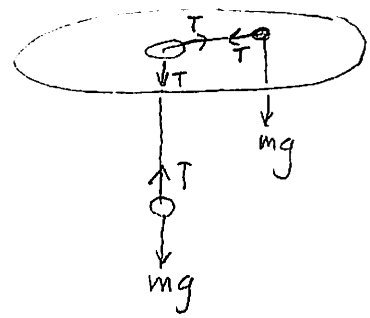




acc.



Force.



(b) since no force $\perp$ r, then acc $\perp$ to r is zero.

i.e.

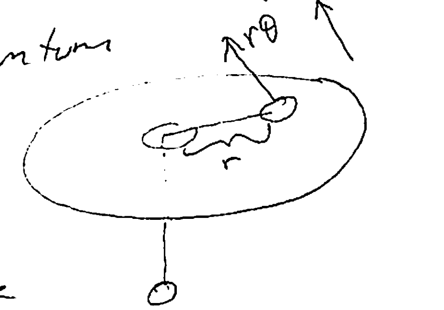
$$\boxed{r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0}$$

(c) let $\underline{P}$ be the linear momentum

let $\underline{L}$ be the angular momentum.

hence $\underline{L} = \underline{r} \times \underline{p}$. in this problem, $p = mr\dot{\theta}$, which is the tangential speed of the ball on the table

$p \equiv mv$, linear momentum



So, the moment of the linear momentum around the hole is

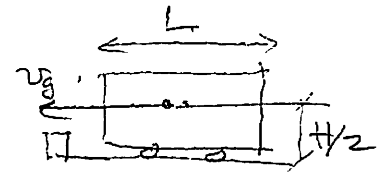
$$|\underline{r} \times \underline{p}| = r(mr\dot{\theta}) = \boxed{mr^2\dot{\theta}}$$

since this is constant, then product rule

let $mr^2\dot{\theta} = C$

so diff. $\Rightarrow m(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}) = 0$ or $\boxed{2\dot{r}\dot{\theta} + r\ddot{\theta} = 0}$ which is part (b)

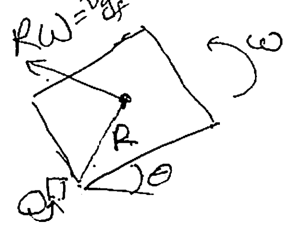
1.18



(a) before Impact, Linear Momentum $\underline{P} = m\bar{v}_g$

So angular momentum $= |\underline{r} \wedge \underline{P}| = \boxed{\frac{H}{2} m v_g}$

(b) after Impact



since body is rotating around cg, and also at same time, the whole body is rotating around O, which is taken as origin of Inertia coordinates, then

since we are finding angular momentum around point not the cg

angular Momentum $= \boxed{I_{cg}\omega + R^2 \omega M}$ or $I_{cg}\omega + R M v_{g_f}$.

where $R = \sqrt{\left(\frac{H}{2}\right)^2 + \left(\frac{L}{2}\right)^2}$ assuming cg is at Center of body.

(c) $v_{g_f} = R\omega$.

so from above equation

angular momentum $= I_{cg}\omega + R M v_{g_f}$

$= I_{cg}\omega + RM(R\omega) = I_{cg}\omega + R^2 M \omega$

$= \boxed{\omega\left(I_{cg} + R^2 M\right)}$

this is the same as using I_O since by parallel axis theorem

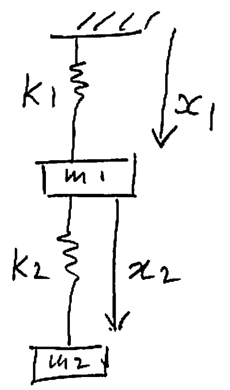
$\boxed{I_O = I_{cg} + R^2 M}$

(d) $\left.\begin{array}{l} \tilde{H}_O = v_g M \frac{H}{2} \\ \tilde{H}_{\text{after impact}} = \omega(I_{cg} + R^2 M) \end{array}\right\} \Rightarrow$ same.

Hence $\boxed{v_g M \frac{H}{2} = \omega I_O}$

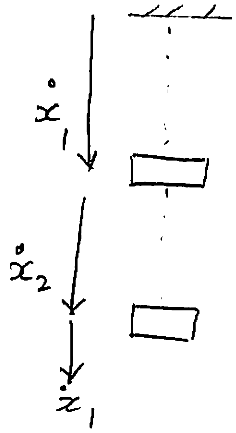
so $\omega_{\text{just after impact}} = \boxed{\dfrac{v_g M \frac{H}{2}}{I_O}}$

2.1

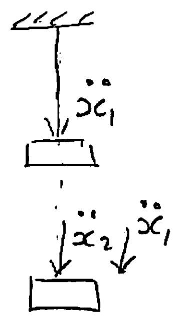


2 DOF.
Coordinate are x_1, x_2.

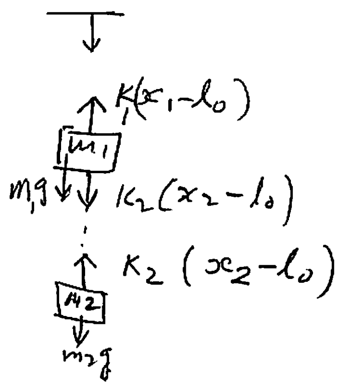
Velocity diagram:



acc.



Force diagram



assume l_0 is free length to both springs.

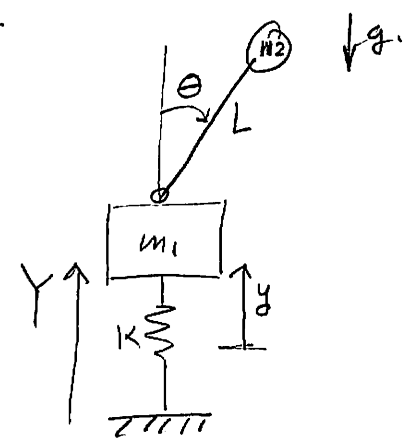
EQM for m_1:

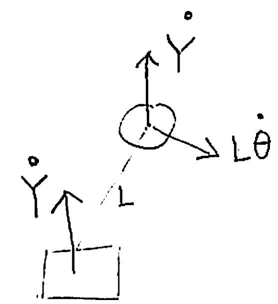
$$+\updownarrow \Sigma F = 0 \Rightarrow -m_1 g \cancel{+} k_2(x_2-l_0) + k_1(x_1-l_0) = m_1 \ddot{x}_1$$

EQM for m_2

$$+\uparrow \Sigma F = 0 \Rightarrow k_2(x_2-l_0) - m_2 g = m_2 \ddot{x}_2$$

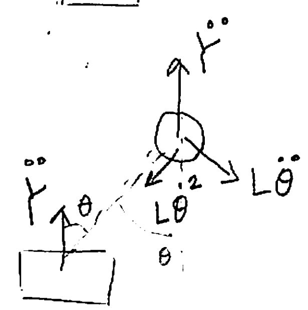
2.2



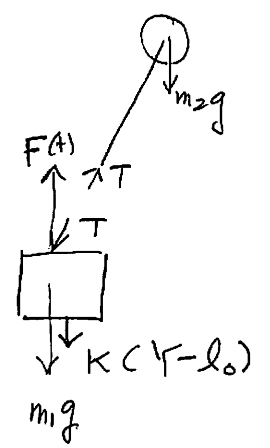


velocity diagram

acc. diagram



Force diagram.



if we measure from y, not Y, then use Ky instead of $K(Y-l_0)$. also $m_1 g$ is not included.

Equation of motion.

m_2. along the rod $\nearrow^{+}\Sigma = m_2(-L\dot{\theta}^2 + \ddot{Y}\cos\theta)$

so $T - m_2 g\cos\theta = m_2(-L\dot{\theta}^2 + \ddot{Y}\cos\theta)$

and $\perp$ to rod

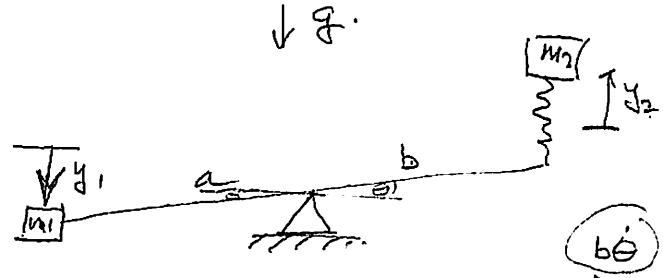
$\nwarrow^{+}\Sigma = m_2(-L\ddot{\theta} + \ddot{Y}\sin\theta)$

$-m_2 g\sin\theta = m_2(-L\ddot{\theta} + \ddot{Y}\sin\theta)$

so $m_2 g\sin\theta = m_2(-\ddot{Y}\sin\theta + L\ddot{\theta})$

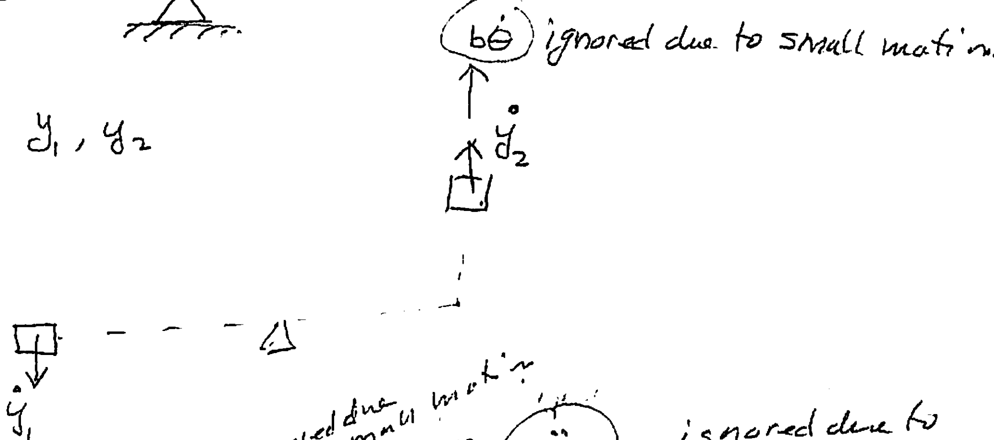
for m_1 $\uparrow\Sigma F = m_1(\ddot{Y}) \Rightarrow -m_1 g - K(Y-l_0) + F(t) - T\cos\theta = m_1\ddot{Y}$

2.57

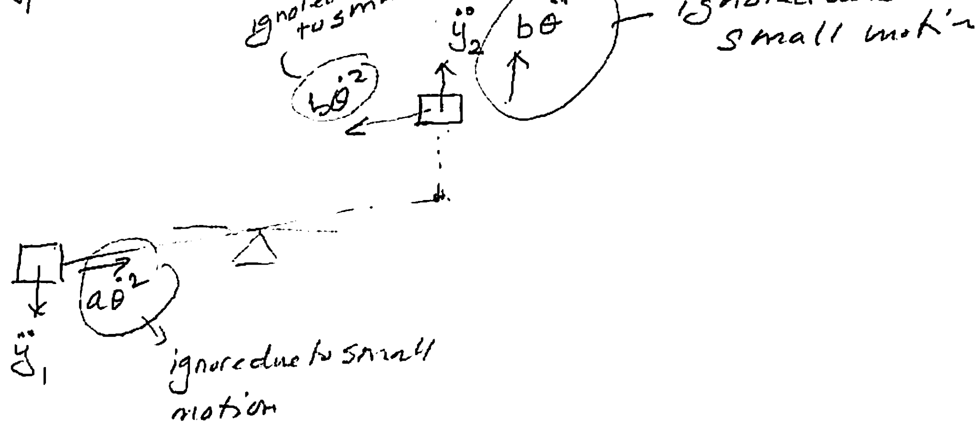


generalized coordinates: y_1, y_2

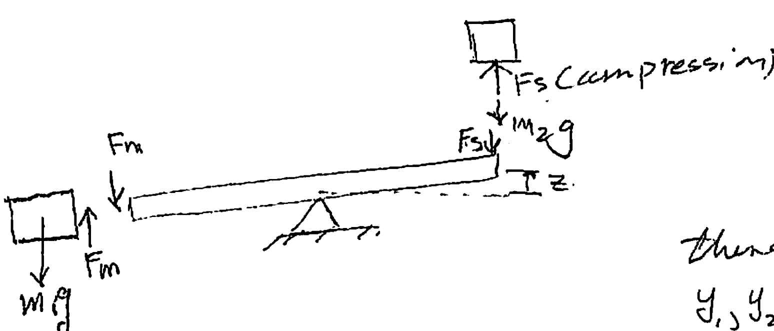
velocity diagram



acc. diagram



Force diagram   separate the masses from the system.



there are y_1, y_2, F_m i.e 3 unknowns.

m_1 $+\downarrow \Sigma = m_1 \ddot y_1 \Rightarrow m_1 g - F_m = m_1 \ddot y_1$ (1)

m_2 $+\downarrow \Sigma = m_2 \ddot y_2 \Rightarrow m_2 g - F_{spring} = m_2 \ddot y_2$ (2)

so need another equation to solve for Fm. so use moment equilibrium for lever. (no mass).

pivot ↺ $= 0 \Rightarrow F_m a = -(-F_s) b \Rightarrow F_m = F_s \frac{b}{a}$

where $F_s = m_2 g + k(z - y_2)$. but $z_2 = \frac{b}{a} y_1$ (small θ).

⟶

hence $F_s = m_2 g + k\left(\frac{b}{a}y_1 - y_2\right)$

so $F_m = F_s \frac{b}{a} = \boxed{m_2 g \frac{b}{a} + k\left(\frac{b}{a}y_1 - y_2\right)\frac{b}{a}}$

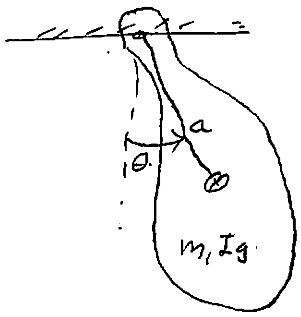
so (1) becomes

$$m_1 g - \left[m_2 g \frac{b}{a} + k\left(\frac{b}{a}y_1 - y_2\right)\frac{b}{a}\right] = m_1 \ddot{y}_1$$

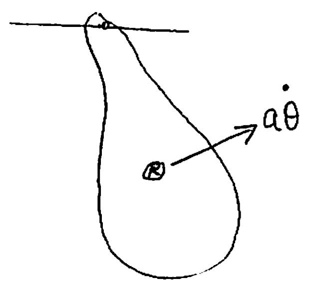
and (2) becomes

$$m_2 g - \left(m_2 g + k\left(\frac{b}{a}y_1 - y_2\right)\right) = m_2 \ddot{y}_2$$

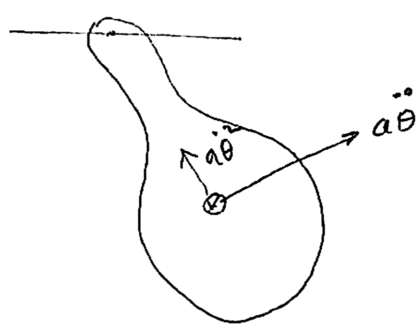
2.6



Velocity diagram



Acc diagram



Force diagram

(a) $\overset{+}{\rightarrow}\Sigma = m(a\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta)$.

so $F_x = m(a\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta)$. (1)

$+\uparrow\Sigma = m(a\ddot{\theta}\sin\theta + a\dot{\theta}^2\cos\theta)$

so $F_y - mg = m(a\ddot{\theta}\sin\theta + a\dot{\theta}^2\cos\theta)$ (2)

$\circlearrowleft_g \Sigma = I_g\ddot{\theta}$

so $\boxed{-F_x a\cos\theta - F_y a\sin\theta = I_g\ddot{\theta}}$ (3)

(b) so 3 equations, 3 unknowns $F_x, F_y, \ddot{\theta}$

Sub (1), (2), into (3) leads to

$$(I_g + ma^2)\ddot{\theta} = -mga\sin\theta$$

or $$\boxed{\ddot{\theta} = \frac{-mga\sin\theta}{I_g + ma^2}}$$

(c) $\circlearrowleft_o \Sigma = (I_g + ma^2)\ddot{\theta}$

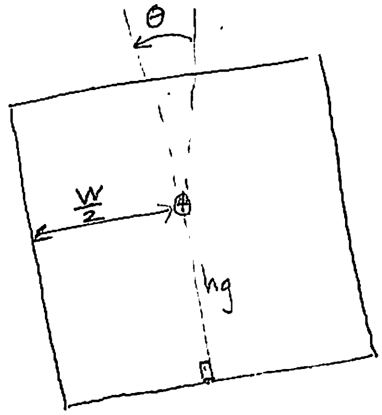
so $-mga\sin\theta = \underbrace{(I_g + ma^2)}_{I_o}\ddot{\theta}$

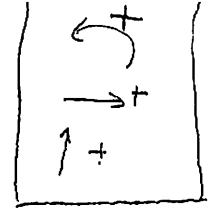
(D) yes. same results. this shows that taking the moment equation around any point will give same result.

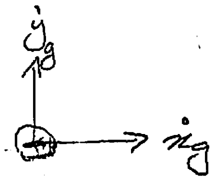
2.7.

Velocity diagram

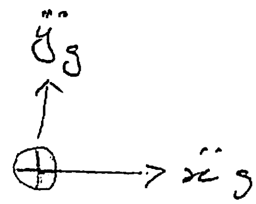
(a)



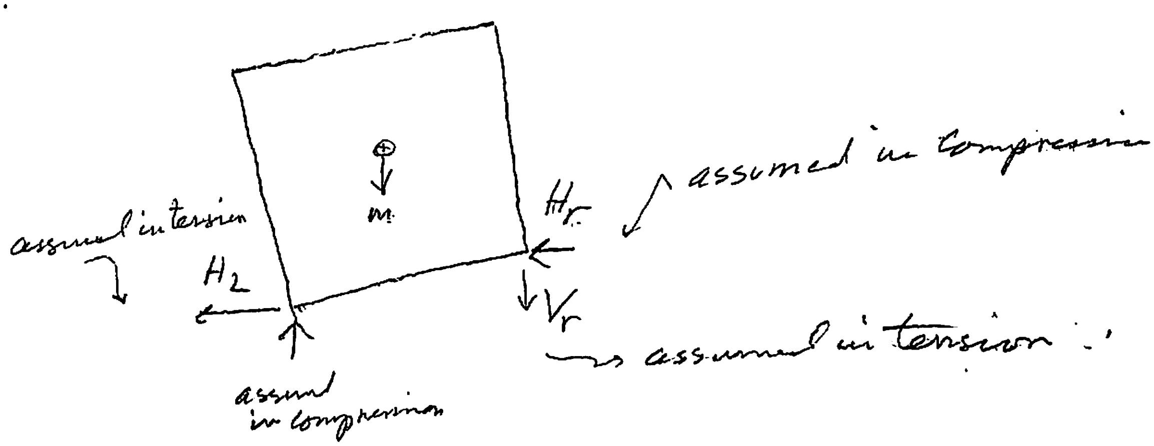




acc. diagram

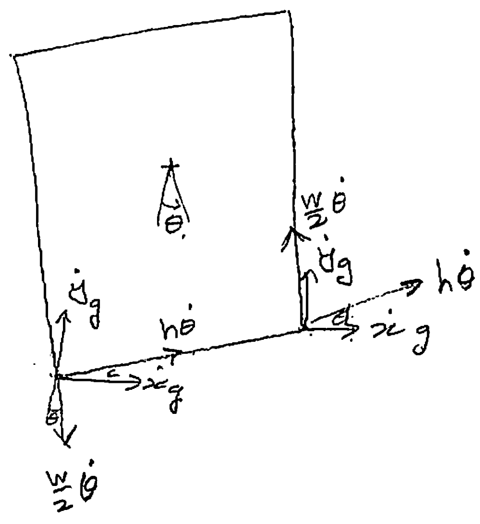


Force diagram.



(b)

$v_p = v_o + \omega \times r + v_{rel}$.



so $V_{hr} = \dot x_g + h\dot\theta\cos\theta - \frac{w}{2}\dot\theta\sin\theta$

$V_{vr} = \dot y_g + h\dot\theta\sin\theta + \frac{w}{2}\dot\theta\cos\theta$

$V_{hL} = \dot x_g + h\dot\theta\cos\theta - \frac{w}{2}\dot\theta\sin\theta$

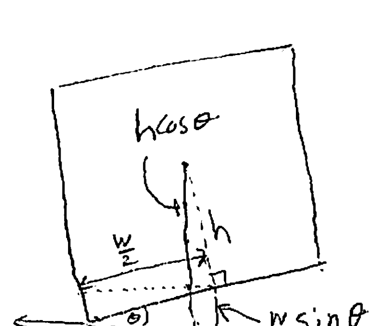
$V_{vL} = \dot y_g + h\dot\theta\sin\theta - \frac{w}{2}\dot\theta\cos\theta$

(c) $\overrightarrow{\Sigma} = m\ddot x_g$. we have 6 unknowns. $\ddot x, \ddot y, F_{hr}, F_{hL}, F_{vr}, F_{vL}$.

so $-F_{hr} - F_{hL} = m\ddot x_g$ (1)

$\uparrow\Sigma = m\ddot y_g$.

so $F_{vL} - F_{vr} = m\ddot y_g$ (2)



$\circlearrowleft_g \Sigma = I_g\ddot\theta$

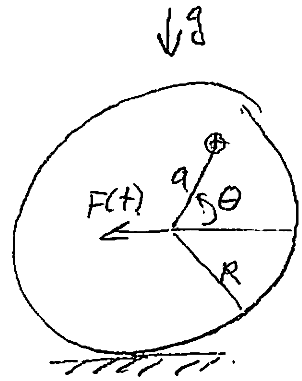
$-F_{hL}\left(\frac{w}{2}\sin\theta + h\cos\theta\right) - F_{hr}\left(h\cos\theta - \frac{w}{2}\sin\theta\right)$

$-F_{vr}\left(h\sin\theta + \frac{w}{2}\cos\theta\right) - F_{vR}\left(-h\sin\theta + \frac{w}{2}\cos\theta\right) = I_g\ddot\theta$ (3)

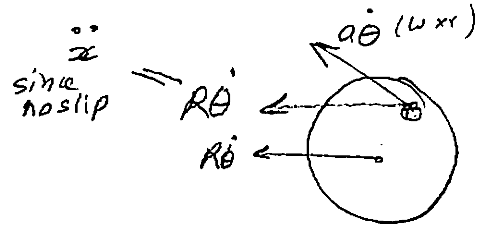
rest is simplification as shown in key.

2.8

(a)

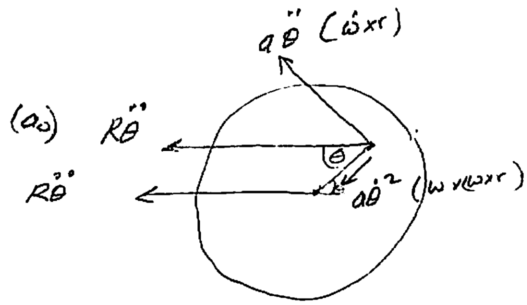


Velocity diagram

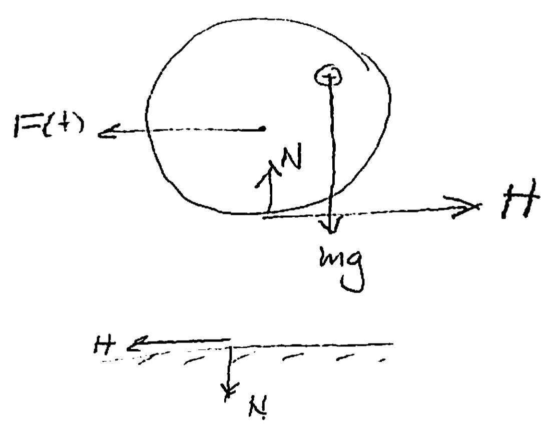


acc diagram

$a_p = a_0 + \dot{\omega} \times r + \omega(\omega \times r) + 2\omega \times V_{rel} + a_{rel}$



Force diagram



← called Rolling friction. force needed for disk to roll without slip.

(b) unknowns are $\ddot{\theta}$, H, N so 3 equations are needed.

$$\overset{+}{\rightarrow}\Sigma = m\left(-R\ddot{\theta} - a\ddot{\theta}\sin\theta - a\dot{\theta}^2\cos\theta\right)$$

$$\text{so } -F + H = m\left(-R\ddot{\theta} - a\ddot{\theta}\sin\theta - a\dot{\theta}^2\cos\theta\right) \quad (1)$$

$$+\uparrow\Sigma = m\left(a\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta\right)$$

$$\text{so } N - mg = m\left(a\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta\right) \quad (2)$$

→

take moments around c.g:

$$\circlearrowleft \Sigma = I\ddot{\theta}$$

so $H(R+a\sin\theta) - F\,a\sin\theta - N\,a\cos\theta = I\ddot{\theta}$ (3)

3 equation, 3 unknowns.

From (3) $H = \dfrac{I\ddot{\theta} + Na\cos\theta + Fa\sin\theta}{R+a\sin\theta}$

From (2), $N = m(a\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta) + mg$

so $H = \dfrac{I\ddot{\theta} + a\cos\theta(ma\ddot{\theta}\cos\theta - ma\dot{\theta}^2\sin\theta + mg) + Fa\sin\theta}{R+a\sin\theta}$

so (1) becomes

$-F + \quad = m(-R\ddot{\theta} - a\ddot{\theta}\sin\theta - a\dot{\theta}^2\cos\theta)$

simplifying gives

$$[I_g + m(R+a\sin\theta)^2 + ma^2\cos^2\theta]\ddot{\theta} = -mga\cos\theta - maR\dot{\theta}^2\cos\theta.$$

notice, using key solution but it is missing $F(t)$.

let $\left.\begin{array}{l} x_1 = \theta \\ x_2 = \dot{\theta} \end{array}\right\} \rightarrow \left.\begin{array}{l} \dot{x}_1 = \dot{\theta} \\ \dot{x}_2 = \ddot{\theta} \end{array}\right\} \rightarrow$

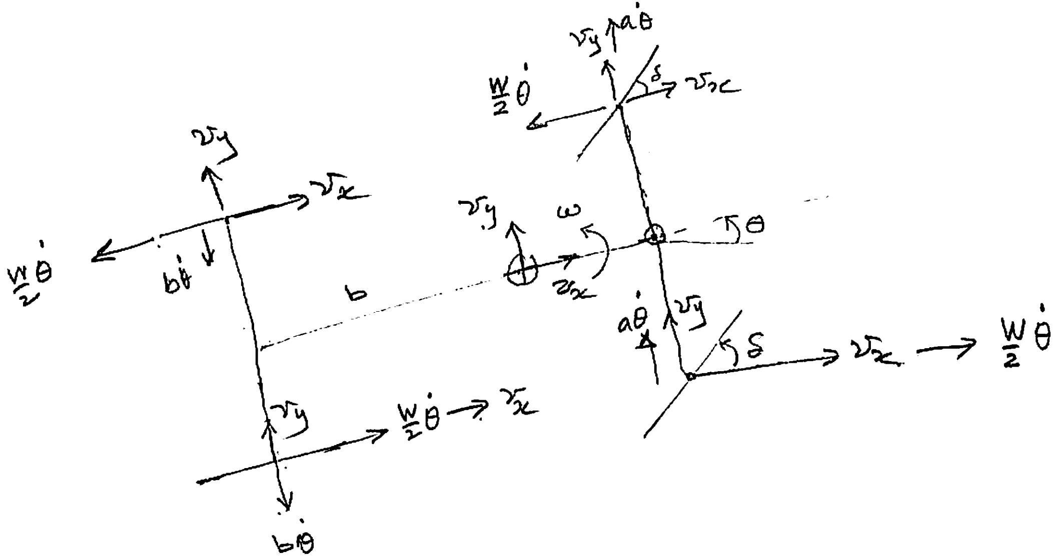
$$\boxed{\begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = \dfrac{-mga\cos x_1 - maRx_2^2\cos x_1}{I_g + m(R+a\sin x_1)^2 + ma^2\cos^2 x_1} \end{array}}$$

now ode45 can be used to solve it

2.13

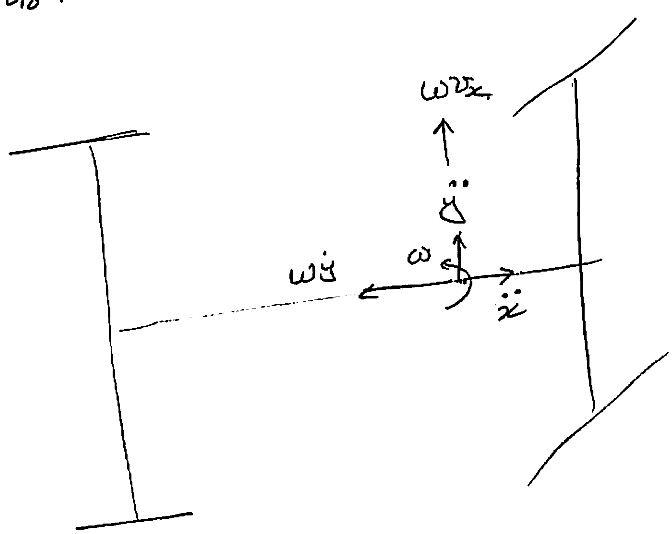
velocity diagram

(a)



$V_p = V_o + \omega \times r + V_{rel}$

acc. diagram $\quad a_p = a_o + \dot\omega \times r + \omega(\omega \times r) + 2\omega \times V_{rel} + a_{rel}$

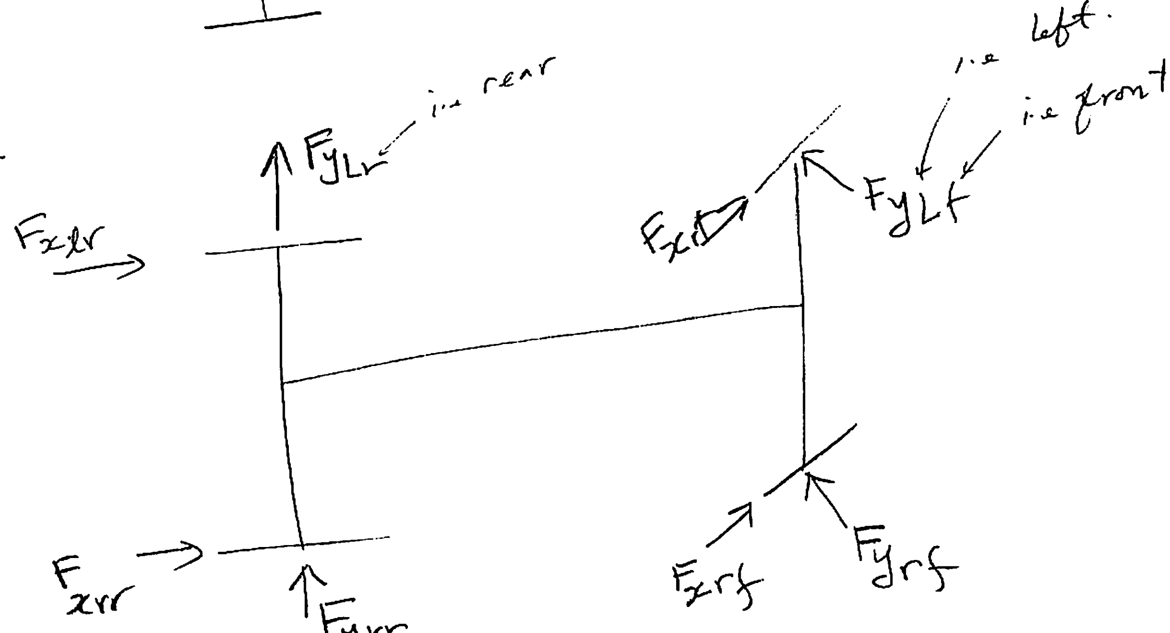


(b)

Force diagram

left

right



(c) $\overset{x}{\rightarrow}\sum = m(\ddot{x} - \omega\dot{y})$

so $(F_{x_{rf}} + F_{x_{Lf}})\cos\delta + (F_{x_{rr}}) + F_{x_{Lr}} - (F_{y_{rf}} + F_{y_{Lf}})\sin\delta = m(\ddot{x} - \omega\dot{y})$

$\uparrow\sum = m(\ddot{y} + \omega\dot{x})$

so $F_{y_{rr}} + F_{y_{Lr}} + (F_{y_{rF}} + y_{y_{LF}})\cos\delta + (F_{x_{rf}} + F_{x_{LF}})\sin\delta = m(\ddot{y} + \omega\dot{x})$

moment equation

$\overset{\curvearrowleft}{cs}\sum = I_g\ddot{\theta}$

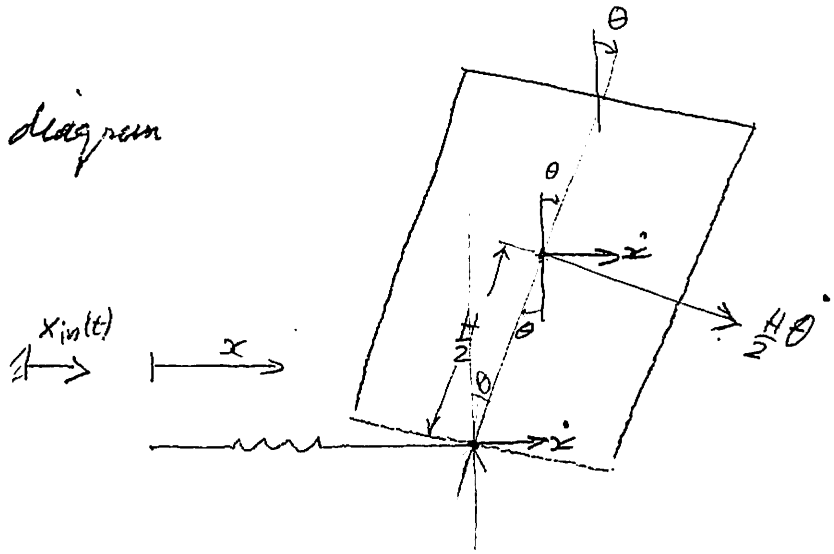
so $(F_{x_{rf}} - F_{x_{LF}})\frac{W}{2}\cos\delta + (F_{x_{rF}} + F_{x_{LF}})a\sin\delta + (F_{y_{rf}} + F_{y_{LF}})a\cos\delta +$

$(F_{y_{LF}} - F_{y_{RF}})\frac{W}{2}\sin\delta + (F_{x_{Lr}} - F_{x_{Lr}})\frac{W}{2} - (F_{y_{rr}} + F_{y_{Lr}})b = I_g\ddot{\theta}$

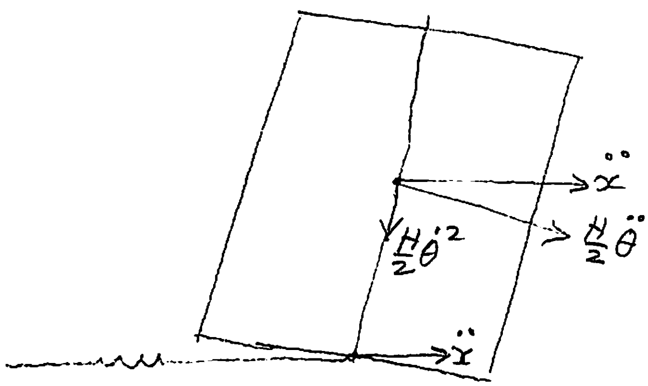
2.19

2 DOF. x, θ.

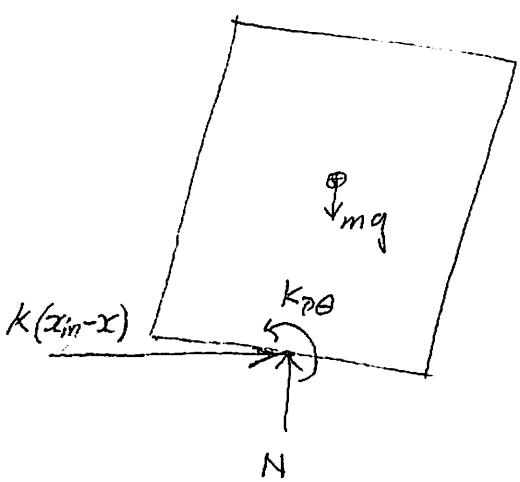
Velocity diagram



Acc diagram



Force diagram.



assume spring is in compression

Now that we have the force and accel. diagram we can generate the equation of motion →

$$\overset{+}{\rightarrow}\Sigma = m\left(\ddot{x} + \frac{H}{2}\ddot{\theta}\cos\theta - \frac{H}{2}\dot{\theta}^2\sin\theta\right) \quad \text{---} (1)$$

$$\Sigma = m\left(-\frac{H}{2}\ddot{\theta}\sin\theta - \frac{H}{2}\dot{\theta}^2\cos\theta\right) \quad \text{---} (2)$$

$$\circlearrowright + \Sigma = I_{cg}\ddot{\theta} \quad \text{---} (3)$$

so now add to forces to the above, we obtain

$$K(x_{in}-x) = m\left(\ddot{x} + \frac{H}{2}\ddot{\theta}\cos\theta - \frac{H}{2}\dot{\theta}^2\sin\theta\right) \quad \text{---} (1A)$$

$$N - mg = m\left(-\frac{H}{2}\ddot{\theta}\sin\theta - \frac{H}{2}\dot{\theta}^2\cos\theta\right) \quad \text{---} (2A)$$

$$-K_{\tau}\theta + N\frac{H}{2}\sin\theta - K(x_{in}-x)\frac{H}{2}\cos\theta = I_{cg}\ddot{\theta} \quad \text{---} (3A)$$

now apply small motion approximation.

i.e. $\cos\theta \rightarrow 1$

$\sin\theta \rightarrow \theta$ so the above becomes

$\dot{\theta}^2 \rightarrow 0$

$$K(x_{in}-x) = m\left(\ddot{x} + \frac{H}{2}\ddot{\theta}\right) \quad \text{---} (1C)$$

$$N - mg = 0 \quad \text{---} (2C)$$

replaced N by mg

$$-K_{\tau}\theta - K(x_{in}-x)\frac{H}{2} + mg\frac{H}{2}\theta = I_{cg}\ddot{\theta} \quad \text{---} (3C)$$

so only (1C) and (3C) are need to solve for $\ddot{x}, \ddot{\theta}$

put in matrix form.

$$\begin{pmatrix} m & \frac{H}{2} \\ & \end{pmatrix}\begin{pmatrix} \ddot{x} \\ \ddot{\theta} \end{pmatrix} + \begin{pmatrix} K & 0 \\ -K\frac{H}{2} & K_{\tau} - mg\frac{H}{2} \end{pmatrix}\begin{pmatrix} x \\ \theta \end{pmatrix} + \begin{pmatrix} K \\ -K\frac{H}{2} \end{pmatrix} x_{in}$$

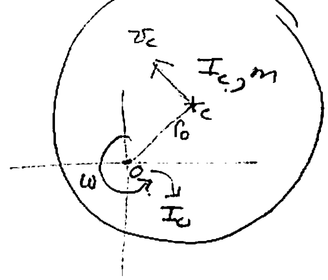
Convert to first order form:

$$\left.\begin{aligned} x_1 &= \theta \\ x_2 &= \dot{\theta} \\ x_3 &= x \\ x_4 &= \dot{x} \end{aligned}\right\} \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \ddot{\theta} = -\left(\frac{K\tau}{I_g} - \frac{mg}{I_g}\frac{H}{2}\right)x_1 + \frac{K}{I_g}\frac{H}{2}x_3 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \ddot{x} = \frac{H}{2}\left(\frac{K\tau}{I_g} - \frac{mg}{I_g}\frac{H}{2}\right)x_1 - \frac{K}{m}\left(1 + \frac{m\left(\frac{H}{2}\right)^2}{I_s}\right)x_3. \end{aligned}$$

now the above can be solved using ODE 45.

4.1

Velocity diagram



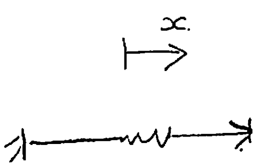
$T = \frac{1}{2} I_O \omega^2$ by definition.

$T = \frac{1}{2}(I_c + m r_0^2)\omega^2 = \frac{1}{2} I_c \omega^2 + \frac{1}{2} m (r_0\omega)^2 = \underbrace{\frac{1}{2} I_c \omega^2 + \frac{1}{2} m v_c^2}$

this last relation is the general equation, which says that

$T =$ sum of K.E. due to rotation + K.E. due to translation.

4.2



assume x is measured when spring is in relaxed position.

normally we write $V = \frac{1}{2} K x^2$.

but here $F = A x^3$.

so using the general formula where $W = \int_0^x F \, d\xi$

so $W = \int_0^x A x^3 \, d\xi = A \frac{x^4}{4}$

so $V = A \frac{x^4}{4}$

4.3

$$F = \frac{dV}{dx} = \begin{cases} K_1 x_0 + K_2(x - x_0) & x > 0 \\ K_1 x & -x_0 < x < x_0 \\ -K_1 x_0 + K_2(x + x_0) & x < -x_0 \end{cases}$$

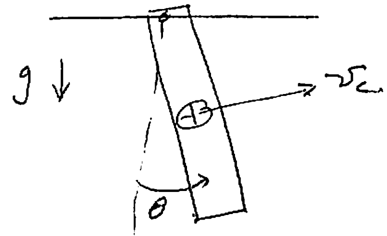
but equation of line is $F = \underbrace{m}_{\text{slope}} x + \underbrace{C}_{\text{intercept}}$

hence we see that

$$F = \begin{cases} \text{slope} = K_2, \quad \text{intercept} = x_0(K_1 - K_2) & x > 0 \\ \text{slope} = K_1, \quad \text{intercept} = 0 & -x_0 < x < x_0 \\ \text{slope} = K_2, \quad \text{intercept} = (K_2 - K_1)\, x_0 . & x < -x_0 \end{cases}$$

which matches what is shown in figure P4.3

4.4



ⓐ $KE = \frac{1}{2}mv_c^2 + \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}m\left(\frac{l}{2}\dot{\theta}\right)^2 + \frac{1}{2}\frac{1}{12}ml^2\dot{\theta}^2$

$= \frac{1}{2}m\frac{l^2}{4}\dot{\theta}^2 + \frac{1}{24}ml^2\dot{\theta}^2 = \left(ml^2\dot{\theta}^2\right)\left(\frac{1}{8}+\frac{1}{24}\right) = \left(\frac{4}{24}\right)\left(ml^2\dot{\theta}^2\right)$

$= \frac{1}{6}ml^2\dot{\theta}^2$

ⓑ assume $V=0$ at top. Then

$V = -mg\left(\frac{l}{2}\right)\cos\theta$

ⓒ $L = T - V = \frac{1}{6}ml^2\dot{\theta}^2 + mg\frac{l}{2}\cos\theta$.

$\frac{\partial L}{\partial\dot{\theta}} = \frac{1}{3}ml^2\dot{\theta}$

$\frac{\partial L}{\partial\theta} = -mg\frac{l}{2}\sin\theta$

$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = \frac{1}{3}ml^2\ddot{\theta}$

so $\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = Q_\theta$. $=0$ since no external forces

so $\frac{1}{3}ml^2\ddot{\theta} + mg\frac{l}{2}\sin\theta = 0$

or $l\ddot{\theta} + \frac{3}{2}g\sin\theta = 0$

or $\boxed{\ddot{\theta} + \frac{3}{2}\frac{g}{l}\sin\theta = 0}$

for small θ, the above becomes $\ddot{\theta} + \frac{3}{2}\frac{g}{l}\theta = 0$

Compare to $\ddot{\theta} + \omega_n^2\theta = 0 \Rightarrow \boxed{\omega_n = \sqrt{\frac{3}{2}\frac{g}{l}}}$

<u>4.5</u> we can add the torque by considering the work done by it, and use that as generalized force Q_θ. as we can write

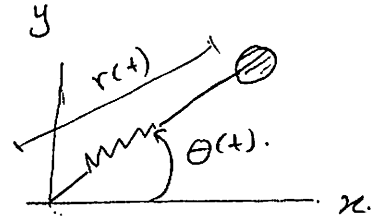
$$\frac{d}{dt}\underbrace{(T+V)}_{\text{stored energy}} = \underbrace{-\tau\dot{\theta}}_{\text{power dissipated}}$$

so from 4.4,

$$\frac{d}{dt}(T+V) = \frac{1}{3}ml^2\ddot{\theta} + mg\frac{l}{2}\sin\theta$$

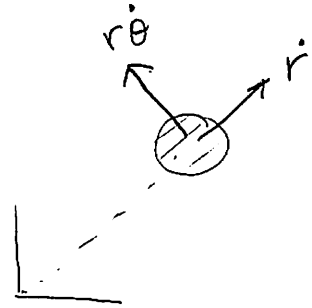
so $$\frac{1}{3}ml^2\ddot{\theta} + mg\frac{l}{2}\sin\theta + \tau_0 \,\mathrm{sgn}\,\dot{\theta} = 0$$

4.6



ⓐ generalized coordinates r, θ.

velocity diagram



$T = \frac{1}{2} m \left(\dot r^2 + (r\dot\theta)^2 \right)$

$V = \frac{1}{2} K (r - l_0)^2$

ⓑ $\underline{\theta}$:

$L = T - V = \frac{1}{2} m (\dot r^2 + r^2 \dot\theta^2) - \frac{1}{2} K (r - l_0)^2$

$\frac{\partial L}{\partial \dot\theta} = \frac{1}{2} m (2 r^2 \dot\theta) = m r^2 \dot\theta$

$\frac{\partial L}{\partial \theta} = 0$

$\frac{d}{dt} \frac{\partial L}{\partial \dot\theta} = m 2 r \dot r \dot\theta + m r^2 \ddot\theta$

$\Rightarrow$ EQM is $m 2 r \dot r \dot\theta + m r^2 \ddot\theta = Q_\theta \leftarrow = 0$

$$\boxed{\ddot\theta + \frac{2\dot r}{r} \dot\theta = 0}$$

$\underline{r}$:

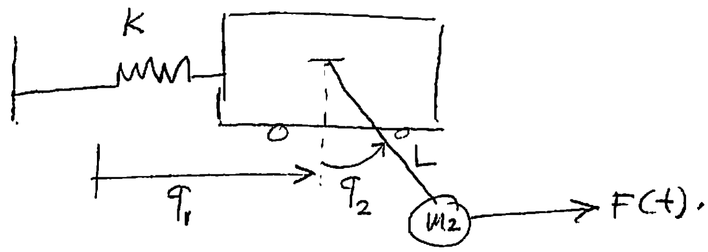
$\frac{\partial L}{\partial \dot r} = \frac{1}{2} m (2 \dot r) = m \dot r$

$\frac{\partial L}{\partial r} = \frac{1}{2} m (2 r \dot\theta^2) - K (r - l_0)$

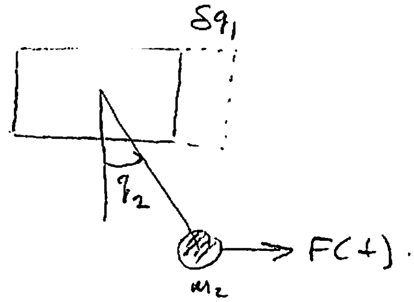
$\frac{d}{dt} \frac{\partial L}{\partial \dot r} = m \ddot r$

$\Rightarrow$ EQM $\Rightarrow$ $\boxed{m \ddot r - m r \dot\theta^2 + K (r - l_0) = 0}$ zero Q_r.

4.7

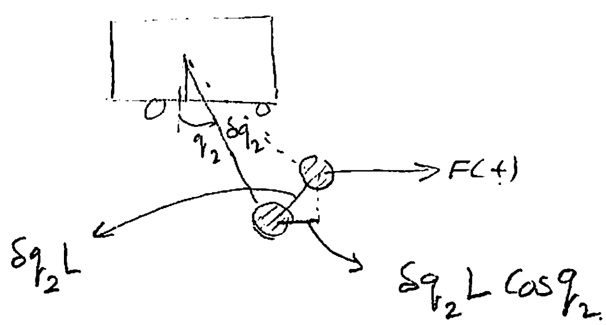


to find Q_{q_1}, we make small δq_1 while having q_2 fixed
then find the work done on the system by external forces.



$$\delta W = \frac{F\,\delta q_1}{\delta q_1} = F.$$

now, make δq_2 while keeping q_1 fixed



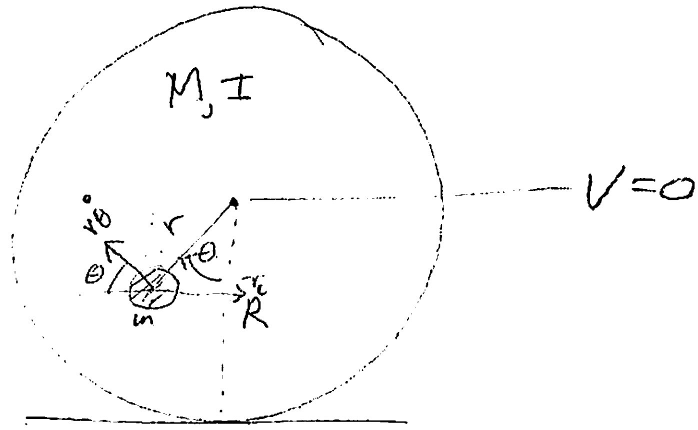
So, work done by F(t) is $\delta W = \dfrac{F(t) \cdot \delta q_2 L \cos q_2}{\delta q_2} = \boxed{FL\cos q_2}$

Since direction of force is same as increasing δq_2, then this is work done on system. so positive.

So $\boxed{\begin{aligned} Q_{q_1} &= F \\ Q_{q_2} &= FL\cos q_2 \end{aligned}}$

4.8

(a)



$$T = K.E(M) + K.E(m)$$

$$= \left(\frac{1}{2}Mv_c^2 + \frac{1}{2}I\dot\theta^2\right) + \frac{1}{2}m\left((v_c - r\dot\theta\cos\theta)^2 + (r\dot\theta\sin\theta)^2\right)$$

$$= \frac{1}{2}Mv_c^2 + \frac{1}{2}I\dot\theta^2 + \frac{1}{2}m\left(v_c^2 + r^2\dot\theta^2\cos^2\theta - 2v_c r\dot\theta\cos\theta + r^2\dot\theta^2\sin^2\theta\right)$$

$$= \frac{1}{2}Mv_c^2 + \frac{1}{2}I\dot\theta^2 + \frac{1}{2}m\left(v_c^2 - 2v_c r\dot\theta\cos\theta + r^2\dot\theta^2\right)$$

$$= \frac{1}{2}Mv_c^2 + \frac{1}{2}\frac{MR^2}{2}\dot\theta^2 + \frac{1}{2}mv_c^2 - mv_c r\dot\theta\cos\theta + \frac{1}{2}mr^2\dot\theta^2$$

but $v_c = R\dot\theta$ so

$$\boxed{T = \frac{1}{2}MR^2\dot\theta^2 + \frac{1}{2}\left(\frac{MR^2}{2}\right)\dot\theta^2 + \frac{1}{2}m(R\dot\theta)^2 - mR\dot\theta^2 r\cos\theta + \frac{1}{2}mr^2\dot\theta^2}$$

(b) $V = -mgr\cos\theta$

$\longrightarrow$

(c) Equation of motion is given by $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_{r_i}}\right) - \frac{\partial L}{\partial q_{r_i}} = Q_i$

so, For $q_{r_j} = q_{r_1} = \theta$, we obtain.

$$L = T - V$$

$$\text{so } \frac{\partial L}{\partial \dot\theta} = MR^2\dot\theta + \frac{MR^2}{2}\dot\theta + mR^2\dot\theta - 2m\dot\theta Rr\cos\theta + mr^2\dot\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = MR^2\ddot\theta + \frac{MR^2}{2}\ddot\theta + mR^2\ddot\theta - 2m\ddot\theta Rr\cos\theta + 2m\dot\theta Rr\dot\theta\sin\theta + mr^2\ddot\theta$$

$$\frac{\partial L}{\partial \theta} = mR\dot\theta^2 r\sin\theta - mgr\sin\theta$$

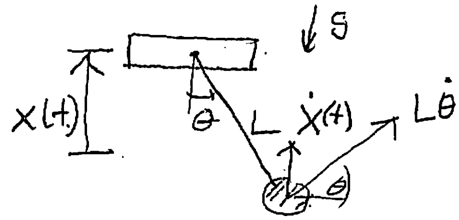
so EQM is

$$\ddot\theta\left(MR^2 + \frac{MR^2}{2} + mR^2 - 2mRr\cos\theta - mr^2\right) + \dot\theta^2(mRr\sin\theta) + mgr\sin\theta = 0$$

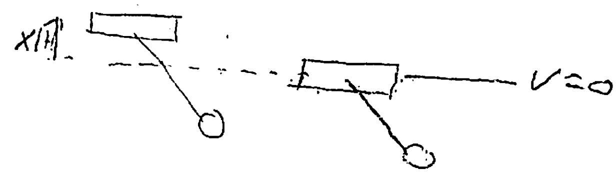
(c) yes, it is the Rolling friction force needed to prevent slip. but it does no work. hence no generalized force needed.

4.9

2DOF



assume V=0 when X(t)=0



(a) $T = \frac{1}{2}m\left((\dot{x} + L\dot{\theta}\sin\theta)^2 + (L\dot{\theta}\cos\theta)^2\right)$

$= \frac{1}{2}m\left(\dot{x}^2 + 2\dot{x}\dot{\theta}L\sin\theta + L^2\dot{\theta}^2\right)$

(b) $V = -mg\left(L\cos\theta - x(t)\right) = mg\left(X(t) - L\cos\theta\right)$

(c) $\mathcal{L} = T - V$

$\mathcal{L} = \frac{1}{2}m\left(\dot{x}^2 + 2\dot{x}\dot{\theta}L\sin\theta + L^2\dot{\theta}^2\right) - mg\left(X - L\cos\theta\right)$

$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}m\left(2\dot{x}L\sin\theta + 2L^2\dot{\theta}\right)$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}m\left(2\ddot{x}L\sin\theta + 2\dot{x}L\dot{\theta}\cos\theta + 2L^2\ddot{\theta}\right)$

$\frac{\partial L}{\partial \theta} = \frac{1}{2}m\left(2\dot{x}\dot{\theta}L\cos\theta\right) - mg\left(+L\sin\theta\right)$

so $m\ddot{x}L\sin\theta + m\dot{x}L\dot{\theta}\cos\theta + mL^2\ddot{\theta} - m\dot{x}\dot{\theta}L\cos\theta + mgL\sin\theta = Q_\theta$

generalized force $Q_\theta = 0$

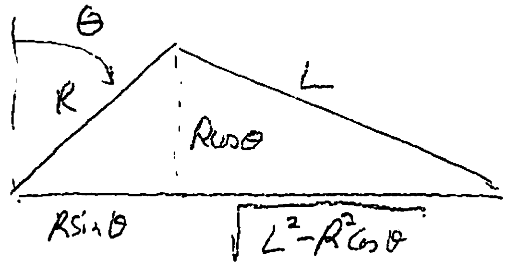
so $\boxed{\ddot{\theta} = \cancel{\frac{\dot{x}\dot{\theta}}{L}}\cos\theta - \frac{g}{L}\sin\theta - \frac{\ddot{x}}{L}\sin\theta - \cancel{\frac{\dot{x}\dot{\theta}\cos\theta}{L}}}$

$\ddot{\theta} = -\frac{g}{L}\sin\theta - \frac{\ddot{x}}{L}\sin\theta$

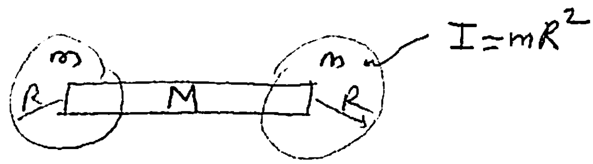
(d) when $\theta \approx 0$ $\sin\theta \approx \theta$

so $\boxed{\ddot{\theta} + \theta\left(\frac{g}{L} + \frac{\ddot{x}}{L}\right) = 0} \Rightarrow K = \frac{g}{L}$,

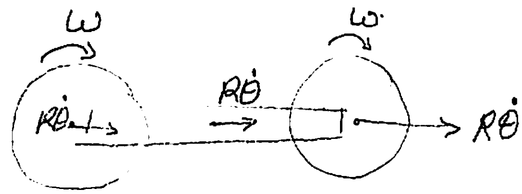
4.10



4.11



velocity diagram



$$T = \frac{1}{2}M(R\dot\theta)^2 + 2\left[\frac{1}{2}m(R\dot\theta)^2\right] + 2\left[\frac{1}{2}I\dot\theta^2\right]$$

$$= \frac{1}{2}MR^2\dot\theta^2 + mR^2\dot\theta^2 + mR^2\dot\theta^2$$

$$= \frac{1}{2}MR^2\dot\theta^2 + 2mR^2\dot\theta^2$$

$$= \left(\frac{M}{2} + 2m\right)V^2 = \frac{1}{2}\underbrace{(M+4m)}V^2 \approx \boxed{\frac{1}{2}\tilde{m}V^2}$$

so $F = \tilde{m}\dot{V}$ ← acceleration.

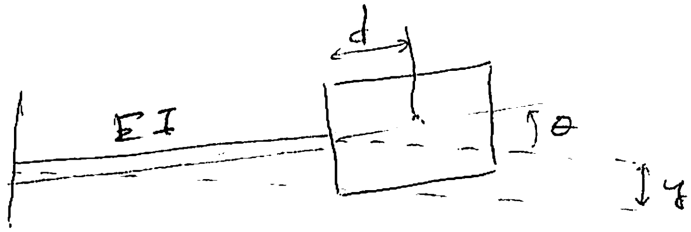
$$F = (M+4m)\dot{V}$$

so wheels appear 2 times as massive for acc. effect

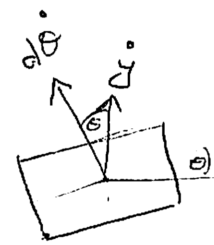
so he was right.

4.12

(a)



generalized Coordinate y, θ.



Velocity diagram

$$T = \frac{1}{2}M\left((\dot y \sin\theta)^2 + (\dot y\cos\theta + d\dot\theta)^2 \right) + \frac{1}{2} I_c \dot\theta^2$$

$$= \frac{1}{2}M\left(\dot y^2 \sin^2\theta + \dot y^2\cos^2\theta + d^2\dot\theta^2 + 2\dot y\cos\theta d\dot\theta \right) + \frac{1}{2} I_c \dot\theta^2$$

$$= \frac{1}{2}M\left(\dot y^2 + d^2\dot\theta^2 + 2\dot y\cos\theta d\dot\theta \right) + \frac{1}{2} I_c \dot\theta^2$$

for small θ, $\cos\theta \simeq 1$, $\dot\theta^2 \simeq 0$ so the above becomes

$$T = \frac{1}{2}M\left(\dot y^2 + 2\dot y\dot\theta d \right)$$

but for small angle approximation $\dot\theta^2 = 0$. why key solution shows it!!

(b)

$$V = \frac{1}{2} \underset{1\times 2}{\begin{bmatrix} y & \theta \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}} \underset{2\times 1}{\begin{pmatrix} y \\ \theta \end{pmatrix}}$$

$$= \frac{1}{2} \underset{1\times 2}{\begin{bmatrix} yK_{11}+\theta K_{21} & yK_{12}+\theta K_{22} \end{bmatrix}} \underset{2\times 1}{\begin{pmatrix} y \\ \theta \end{pmatrix}}$$

$$= \frac{1}{2}\left[y^2K_{11} + y\theta K_{21} + y\theta K_{12} + \theta^2 K_{22} \right]$$

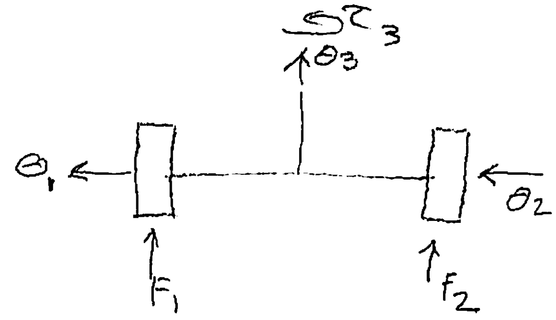
$$= \frac{1}{2}\left(K_{11}y^2 + \theta^2 K_{22} + 2y\theta K_{12} \right)$$

assuming $K_{12} = K_{21}$. which is true in this problem

(c) $\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} - \frac{\partial L}{\partial \theta} = 0$ and $\frac{d}{dt}\frac{\partial L}{\partial \dot y} - \frac{\partial L}{\partial y} = 0$ give

$$\begin{pmatrix} M & Md \\ Md & Md^2+I_c \end{pmatrix}\begin{pmatrix} \ddot y \\ \ddot\theta \end{pmatrix} + \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}\begin{pmatrix} y \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

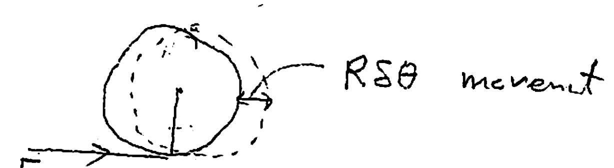
4.13



(a)

$$T = \tfrac{1}{2} I_1 \dot{\theta}_1^2 + \tfrac{1}{2} I_2 \dot{\theta}_2^2 + \tfrac{1}{2} I_3 \left(G \frac{(\dot{\theta}_1 + \dot{\theta}_2)}{2} \right)^2$$

(b) Hold θ_2 fixed.



$$\delta W = \frac{-F_1 (R \delta\theta_1) + \tau_3 \cdot \frac{G}{2} (\delta\theta_1)}{\delta\theta_1}$$

so $Q_1 = -F_1 R + \tau_3 \frac{G}{2}$

Hold θ_1 fixed, make $\delta\theta_2$, hence

$$Q_2 = -F_2 R + \tau_3 \frac{G}{2}$$

(c) $V = 0$ here, so

$\underline{\theta_1}$:
$$\frac{\partial L}{\partial \dot{\theta}} = I_1 \dot{\theta} + I_3 \frac{G}{4} \left((\dot{\theta}_1 + \dot{\theta}_2) \right)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = I_1 \ddot{\theta} + I_3 \frac{G}{4} (\ddot{\theta}_1 + \ddot{\theta}_2)$$

so $I_1 \ddot{\theta}_1 + I_3 \frac{G}{4} (\ddot{\theta}_1 + \ddot{\theta}_2) = -F_1 R + \tau_3 \frac{G}{2}$.

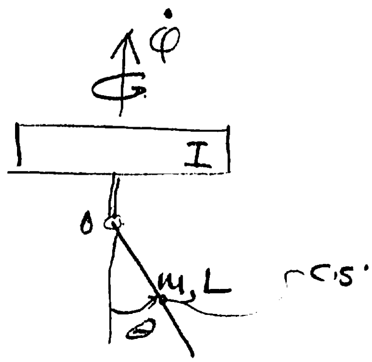
$$\ddot{\theta}_1 \left(I_3 \frac{G}{4} + I_1 \right) + I_3 \frac{G}{4} \ddot{\theta}_2 = -F_1 R + \tau_3 \frac{G}{2}$$

$\longrightarrow$

Similarly for θ_2. here

$$\begin{bmatrix} I_1 + \frac{G^2}{4} I_3 & \frac{G^2}{4} I_3 \\ \frac{G^2}{4} I_3 & I_2 + \frac{G^2}{4} I_3 \end{bmatrix} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = \begin{pmatrix} -R & 0 & \frac{G}{2} \\ 0 & -R & \frac{G}{2} \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ \tau_3 \end{pmatrix}$$

4.17

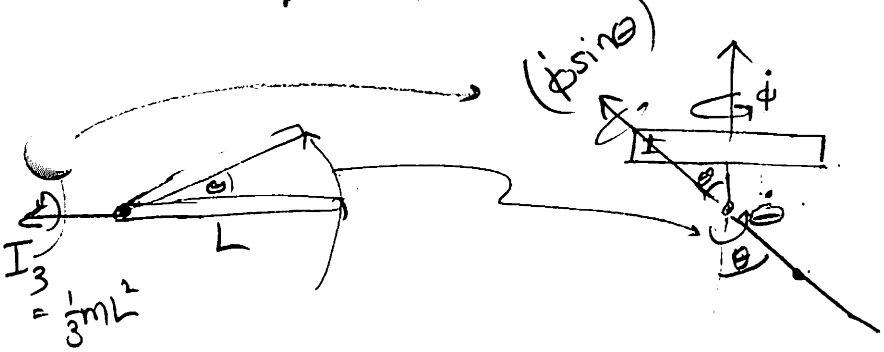


principle moment of inertia about 0 are $I_1 = 0, I_2 = I_3 = \frac{1}{3}mL^2$

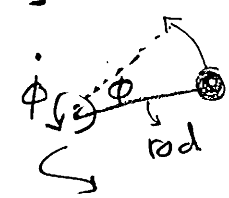
point 0 is fixed in space.

a) $T = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right)$

here we first need to resolve all angular velocity along the principle directions in order to find $\omega_1, \omega_2, \omega_3$.



top view



I of beam around this is $\frac{1}{3}mL^2$

For the flywheel, its $T = \frac{1}{2} I_{Flywheel}\dot\phi^2$

for the rod, it will have 2 components since it rotates in θ and ϕ. for θ, we obtain $\frac{1}{2} I_\theta \dot\theta^2$

$\frac{1}{3}mL^2$

and for ϕ for rod, we obtain $\frac{1}{2} I_3 (\dot\phi \sin\theta)^2$

a) hence total K.E. is $T = \frac{1}{2}I\dot\phi^2 + \frac{1}{2}(\frac{1}{3}mL^2)\dot\theta^2 + \frac{1}{2}(\frac{1}{3}mL^2)(\dot\phi\sin\theta)^2$

b) take $V=0$ to be level of flywheel. hence

$V = -mg\frac{L}{2}\cos\theta$

c). $L = T - V$

$$= \frac{1}{2} I \dot{\phi}^2 + \frac{1}{2}\left(\frac{1}{3} mL^2\right)\dot{\theta}^2 + \frac{1}{2}\left(\frac{1}{3} mL^2\right)(\dot{\phi}\sin\theta)^2 + mg\frac{L}{2}\cos\theta.$$

$\underline{\phi:}$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = Q_\phi \rightarrow = 0$$

$$\frac{\partial L}{\partial \dot{\phi}} = I\dot{\phi} + \left(\frac{1}{3} mL^2\right)\dot{\phi}\sin^2\theta$$

$$\frac{\partial L}{\partial \phi} = 0$$

and since $Q_\phi = 0$, hence we see that $\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0$

i.e $\frac{\partial L}{\partial \dot{\phi}} = \text{Constant}$.

i.e $I\dot{\phi} + \left(\frac{1}{3} mL^2\right)\dot{\phi}\sin^2\theta = \text{Constant}$.

$$\boxed{I\ddot{\phi} + \left(\frac{1}{3} mL^2\right)\left(\ddot{\phi}\sin^2\theta + 2\dot{\phi}\sin\theta\cos\theta\,\dot{\theta}\right) = 0}$$

$\underline{\theta}$ $\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{3} mL^2\dot{\theta}$

$$\frac{\partial L}{\partial \theta} = \left(\frac{1}{3} mL^2\right)\dot{\phi}^2\sin\theta\cos\theta - mg\frac{L}{2}\sin\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{3} mL^2\ddot{\theta}$$

so

$$\boxed{\frac{1}{3} mL^2\ddot{\theta} - \frac{1}{3} mL^2\dot{\phi}^2\sin\theta\cos\theta + mg\frac{L}{2}\sin\theta = 0}$$

d) if $\dot{\phi} = \omega_0 = \text{Constant}$ and $\theta = \theta_0 = \text{constant}$. then $\ddot{\phi} = 0$ and $\dot{\theta} = 0$ and $\ddot{\theta} = 0$

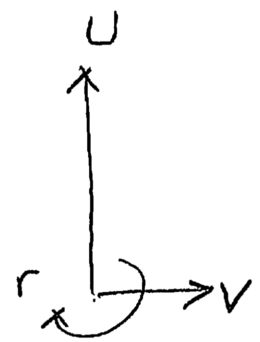
so eq. for θ becomes $\boxed{-\frac{1}{3} mL^2\omega_0^2\sin\theta_0\cos\theta_0 + mg\frac{L}{2}\sin\theta_0 = 0}$

hence $\left(-\frac{L}{3}\omega_0^2\cos\theta_0 + \frac{g}{2}\right) = 0$ or $(\sin\theta_0) = 0$ $\Rightarrow \theta_0 = 0$ stable or $\theta_0 = N\pi$ unstable $N=1$

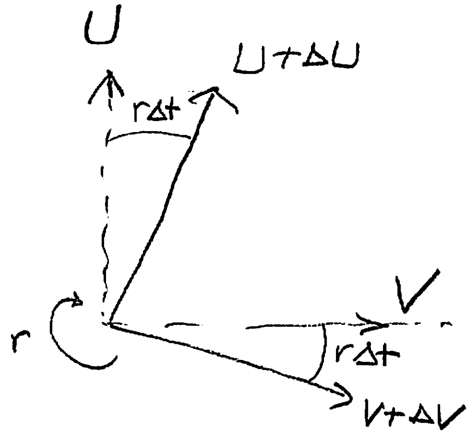
at $\theta_0 = 0$ becomes $-\frac{L}{3}\omega_{crit}^2 = -g/2$

or $\omega_{crit} = \sqrt{\frac{3g}{2L}}$.

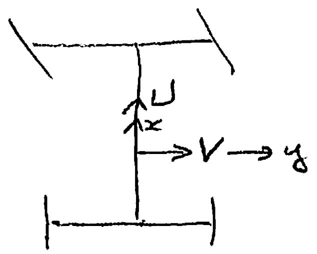
5.1



time t



time t+Δt



we need to Find/derive the following 2 equations

$$a_x = \dot{U} - rV$$

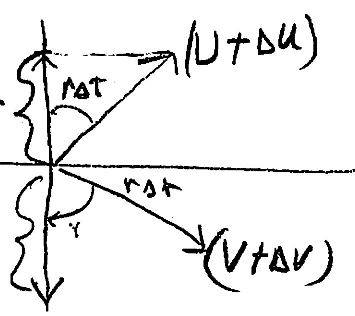
$$a_y = \dot{V} + rU$$

This is a rigid body problem, under rotation.

by definition, $a_x = \dfrac{U(t+\Delta t) - U(t)}{\Delta t}$ ①

but $U(t+\Delta t) = (U+\Delta U)\cos(r\Delta t) - (V+\Delta V)\sin(r\Delta t)$ ②

this is $U(t+\Delta t)$. $\left\{ \begin{array}{l} (U+\Delta U)\cos(r\Delta t) \\ (V+\Delta V)\sin(r\Delta t) \end{array} \right.$



So Plug ② into ① gives ⟶

$$a_x = \frac{(U+\Delta U)\cos(r\Delta t) - (V+\Delta V)\sin(r\Delta t) - U(t)}{\Delta t}.$$

for small Δt, $\cos(r\Delta t) \Rightarrow 1$, $\sin(r\Delta t) \to r\Delta t$

hence

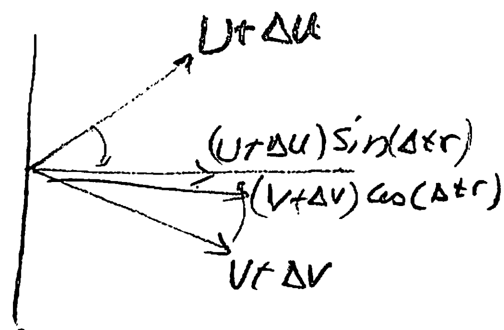
$$a_x = \frac{(U+\Delta U) - (V+\Delta V)r\Delta t - U}{\Delta t}$$

$$= \frac{\Delta U - Vr\Delta t - \Delta V r\Delta t}{\Delta t} =$$ but this $O(\Delta t)^2 \approx 0$.

$$= \frac{\Delta U - Vr\Delta t}{\Delta t} = \frac{\Delta U}{\Delta t} - Vr = \boxed{\dot{U} - Vr}$$

---

for a_y: resolve along y direction:



$$a_y = \frac{\left[(U+\Delta U)\sin(\Delta t r) + (V+\Delta V)\cos(\Delta t r)\right] - V}{\Delta t}.$$

so $\cos \Delta t r \to 1$, $\sin \Delta t r \to \Delta t r$.

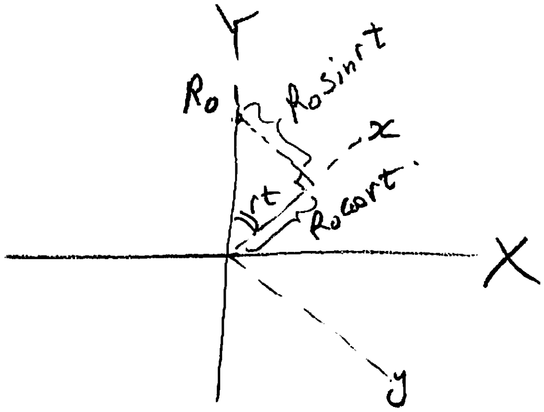
so

$$a_y = \frac{(U+\Delta U)\Delta t r + (V+\Delta V) - V}{\Delta t} = \frac{U\Delta t r + \Delta U\Delta t r + \Delta V}{\Delta t}.$$

but $\Delta U \Delta t r \to 0$ for small $\Delta U, \Delta t$.

$$\therefore a_y = \frac{U\Delta t r + \Delta V}{\Delta t} = \boxed{Ur + \dot{V}}$$

5.2



in the x-y frame, the point R_o has the coordinates

$$\{ R_o \cos rt \ , -R_o \sin rt \}$$

hence its speed is

$$\{ \underbrace{-rR_o \sin rt}_{U} \ , -\underbrace{rR_o \cos rt}_{V} \}$$

hence its acc. is

$$\{ \dot{U} - rV \ , \ \dot{V} + rU \}$$

or

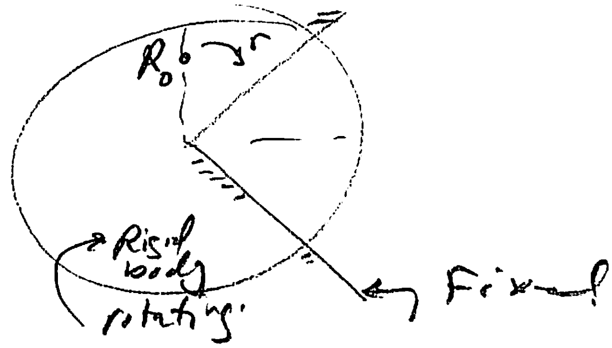
$$\{ -r^2 R_o \cos rt - r(-rR_o \cos rt) \ , \ +r^2 R_o \sin rt + r(-rR_o \sin rt) \}$$

=

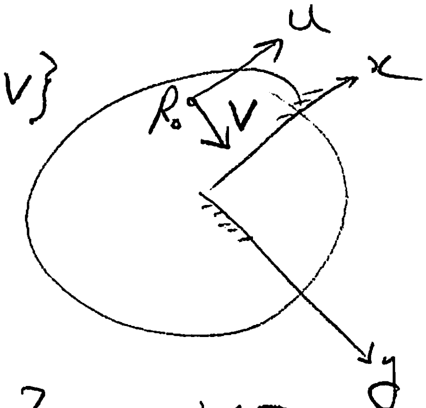
$$\{ 0 \ , \ 0 \}$$

hence point R_o also has zero acc. w.r.t to rotating frame. $\longrightarrow$

another way to look at this problem is to imagine that x,y are the fixed coordinates and the disk is moving. hence rigid body disk is rotating.



so R has speed $\{U,V\}$ relative to x,y.



here now the acc. of R_0 relative to xy is

$$\left\{\dot{U}-rV,\ \dot{V}+rU\right\}$$ as before

5.5

eq. 5.64 is $\delta = \frac{KU^2}{R} + \frac{a+b}{R}$

$r = \frac{U}{R}$.

starting from $\frac{r}{\delta} = \frac{U}{(a+b)+KU^2}$ eq. (5-56) in text.

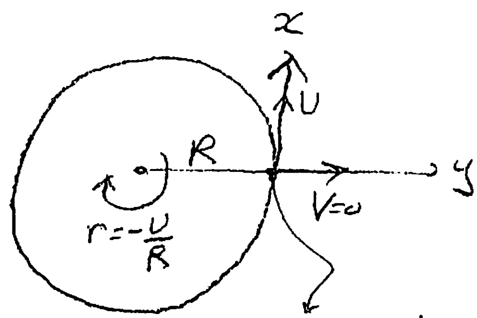
steering angle

hence $\frac{U/R}{\delta} = \frac{U}{(a+b)+KU^2} \Rightarrow \frac{1}{R\delta} = \frac{1}{(a+b)+KU^2}$

or $\delta = \frac{(a+b)+KU^2}{R}$

or $\boxed{\delta = \frac{(a+b)}{R} + \frac{KU^2}{R}}$

5.6



Velocity is given by $\{U, V\}$

acc is given by $\{\dot{U} - rV \; , \; \dot{V} + rU\}$

or $\{\dot{U}, rU\}$

since $V=0$, then $\dot{V}=0$

hence acc. is $\left\{\dot{U}, \left(-\frac{U}{R}\right)U\right\}$ or $\left\{\dot{U}, -\frac{U^2}{R}\right\}$

so the acc. in the y direction is $\frac{-U^2}{R}$.

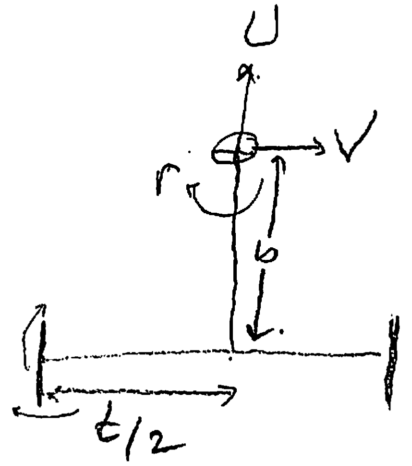
but $U = -rR$. so the above becomes

$$\frac{-(-rR)^2}{R} = \boxed{-r^2R}$$

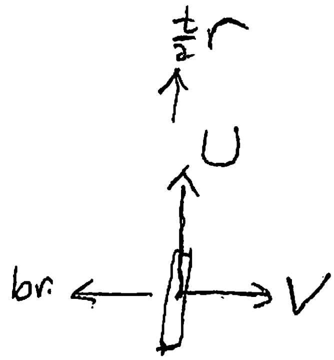
i.e. acc. is $-\dot{\theta}^2R$, where $r = \dot{\theta}$

this is the same as centripetal acc.

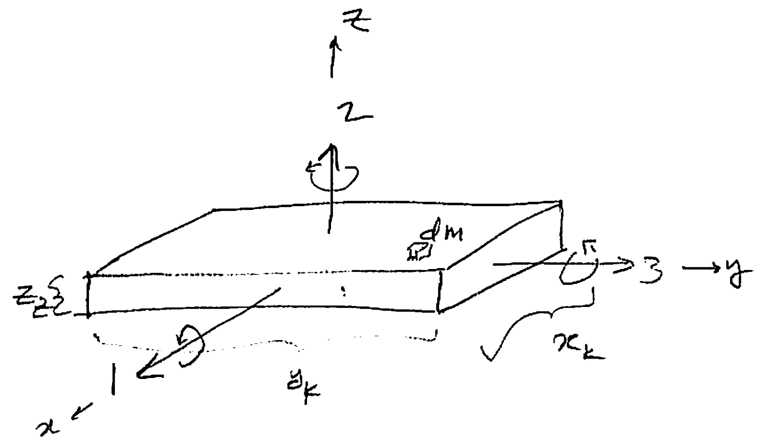
5.7

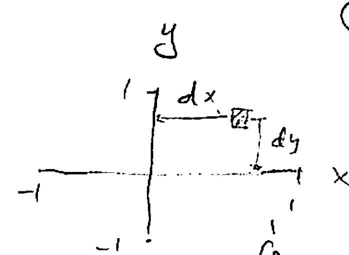


at left wheel



7.1 ①





$I_x = M\int_{-1}^{1} y^2 dy$

$I_y = M\int_{-1}^{1} dx\, x^2$

a) moment of inertia is defined as $I_1 = I_{xx} = \sum m_k (y_k^2 + z_k^2)$

$$I_2 = I_{yy} = \sum m_k (x_k^2 + z_k^2)$$

$$I_3 = I_{zz} = \sum m_k (x_k^2 + y_k^2)$$

so, we see that the dimensions, when sorted in terms of how large they are, become

y_k
x_k ↓ smaller
z_k

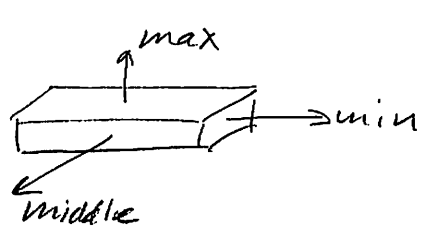
Therefore, the 3 sets of combinations are

y_k	y_k	y_k
smallest ← x_k	x_k	x_k
z_k	z_k	z_k
	second smallest	largest

now looking at the figure, we see that x_k, z_k are the dimensions around I_3. hence I_3 is the smallest

y_k, z_k are the dimension around I_1, hence I_1 is intermidiate

and y_k, x_k are dimensions around I_2, hence I_2 is largest

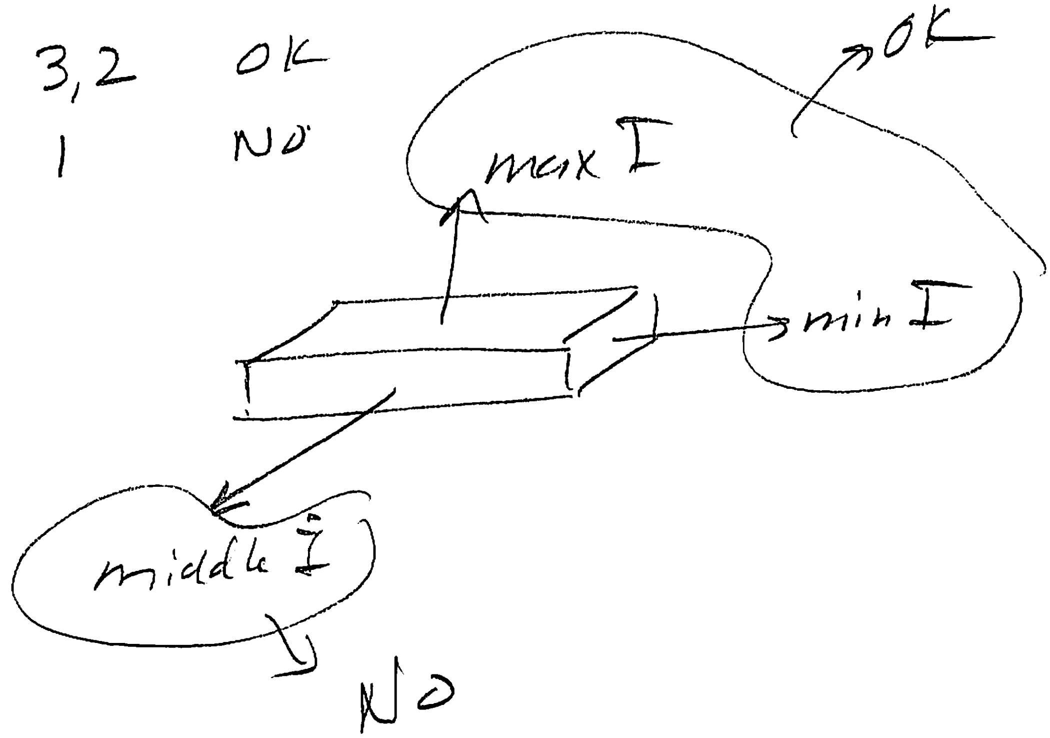


7.1

(b) as was done in class, lecture May 16, we found out the condition for it to successfully spin on which axes by finding K^2 and finding condition for it to be positive.

we found that either the axes with the Largest I or Smallest I will be successful, but the intermediate axes is not successful.

so 3,2 OK
1 NO



7.2 (a) $imp = \int_0^{\delta t} M\,dt$.

hence $imp = \int_0^{\delta t} \tau_y\,dt$.

but $\tau_y = I_{yy}\dot{\omega}_y$

so $\int \tau_y\,dt = I_{yy}\omega_y$.

or. $Imp = I_{yy}\omega_y$.

hence $\boxed{\omega_y = \frac{Imp}{I_{yy}}}$

called q

in these below, $\omega_x \equiv p$
$\omega_y \equiv q$
$\omega_z \equiv r$

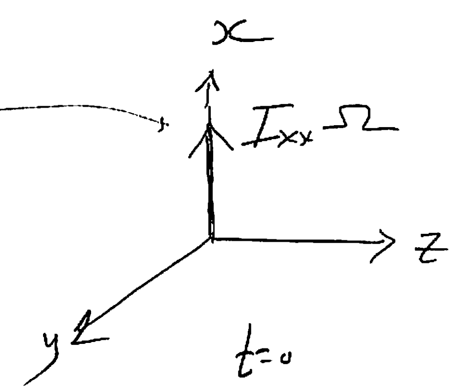
(b) assume $\boxed{\begin{matrix}\omega_x = \Omega \\ \omega_y = 0 \\ \omega_z = 0\end{matrix}}$ initial.

$\underline{H}_c$ is angular momentum of body about center of mass.

$$\underline{H}_c = \underline{I}_c\,\underline{\omega} = \underset{3\times 3}{\begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz}\end{pmatrix}} \underset{3\times 1}{\begin{pmatrix}\omega_x \\ \omega_y \\ \omega_z\end{pmatrix}}$$

so $\underline{H}_c = \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix} = \begin{pmatrix} I_{xx}\omega_x \\ I_{yy}\omega_y \\ I_{zz}\omega_z \end{pmatrix}$.

so at $t=0$, $\underline{H}_c = \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix} = \begin{pmatrix} I_{xx}\Omega \\ 0 \\ 0 \end{pmatrix}$



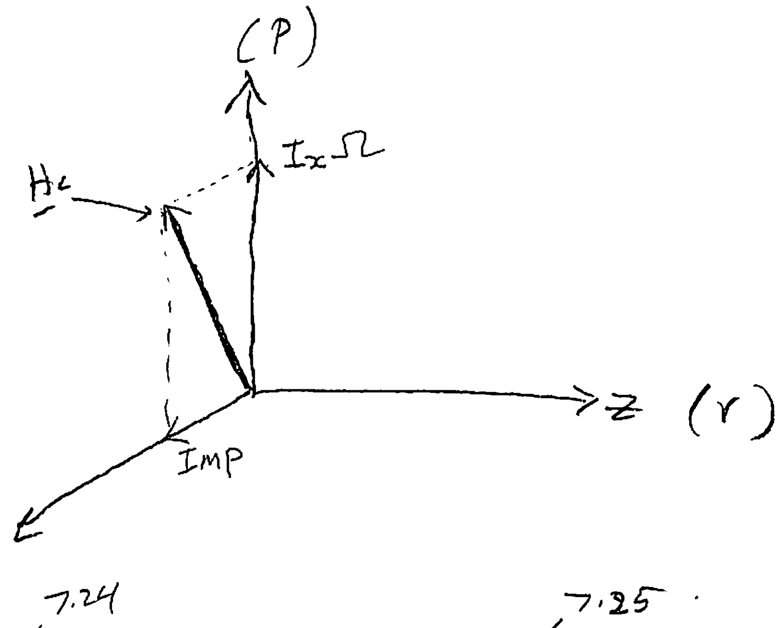
→

immediately after impulse, $\omega_x = I_{xx}\Omega$, $\omega_z = 0$, but

$\omega_y = \dfrac{Imp}{I_{yy}}$ as found in part (a). hence

$$\underline{H}_c = \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix} = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix} \begin{pmatrix} \Omega \\ \frac{Imp}{I_{yy}} \\ 0 \end{pmatrix} = \begin{pmatrix} I_{xx}\Omega \\ Imp \\ 0 \end{pmatrix}$$

hence



(c) $\boxed{\theta \approx \dfrac{Imp}{I_{xx}\Omega}}$

(d)

$$\underbrace{\Delta\dot{q} + \frac{(I_1 - I_3)\Omega}{I_2}\Delta r = 0}_{7.24} \; ; \quad \underbrace{\Delta\dot{r} + \frac{(I_2 - I_1)\Omega}{I_3}\Delta q = 0}_{7.25}$$

take derivative of 7.24, and sub (7.25) into result, this gives

$$\Delta\ddot{q} + \Omega^2\frac{(I_1 - I_3)}{I_2}\left[- \Delta q \frac{(I_2 - I_1)}{I_3} \right] = 0$$

or $\boxed{\Delta\ddot{q} + K\Delta q = 0}$ where $K = \dfrac{(I_1 - I_3)(I_1 - I_2)\Omega^2}{I_1 I_3}$

similarly

$$\boxed{\Delta\ddot{r} + K\Delta r = 0}$$

Problem 7.2 part (e)

Starting with

$$q'(t) + \frac{(I_1 - I_3)}{I_2}\Omega\, r(t) = 0 \tag{1}$$

$$r'(t) + \frac{(I_2 - I_1)}{I_3}\Omega\, q(t) = 0 \tag{2}$$

To decouple the ODE's, take derivatives and substituting back, we find

$$q''(t) + k\, q(t) = 0 \tag{3}$$

$$r''(t) + k\, r(t) = 0 \tag{4}$$

Where $k = \frac{(I_1-I_3)(I_2-I_1)}{I_2 I_3}\Omega^2$. The solution to the above is (for stability $k > 0$)

$$q(t) = A\cos\sqrt{k}t + B\sin\sqrt{k}t \tag{5}$$

$$r(t) = C\cos\sqrt{k}t + D\sin\sqrt{k}t \tag{6}$$

Now, $q(0) = \frac{\text{Imp}}{I_{yy}}$, hence $A = \frac{\text{Imp}}{I_{yy}}$. To find B take derivative

$$q'(t) = -\sqrt{k}\frac{\text{Imp}}{I_{yy}}\sin\sqrt{k}t + B\sqrt{k}\cos\sqrt{k}t \tag{7}$$

But at $t = 0$ then we go back and use Eq. (1) to find $q'(0)$, and equate the result to the above at $t = 0$. Eq (1), at $t = 0$, gives

$$q'(0) + \frac{(I_1 - I_3)}{I_2}\Omega\, r(0) = 0$$

But $r(0) = 0$, hence $q'(0) = 0$, and so this results in $B = 0$ in Eq.(7), hence the solution for $q(t)$ is

$$q(t) = \frac{\text{Imp}}{I_{yy}}\cos\sqrt{k}t$$

We do the same for $r(t)$. From Eq. (6) we find that $C = 0$ since $r(0) = 0$, so now $r(t)$ reduces to

$$r(t) = D\sin\sqrt{k}t$$

Hence

$$r'(t) = \sqrt{k}D\cos\sqrt{k}t \tag{8}$$

To find D, then we go back and use Eq. (2) to find $r'(0)$, and equate the result to the above at $t = 0$. Eq (2), at $t = 0$, gives

$$r'(0) + \frac{(I_2 - I_1)}{I_3}\Omega\, q(0) = 0$$

But $q(0) = \frac{\text{Imp}}{I_{yy}}$, hence

$$r'(0) = -\frac{(I_2 - I_1)}{I_3}\Omega\,\frac{\text{Imp}}{I_{yy}}$$

Therefore, equate the above to Eq. (8) evaluated at $t = 0$ gives

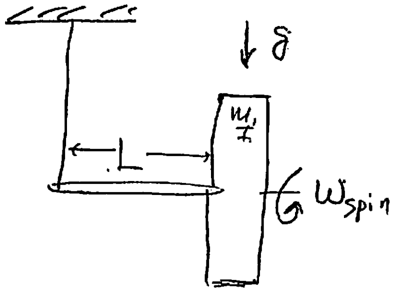
$$D = -\frac{(I_2 - I_1)}{I_3\sqrt{k}}\Omega\,\frac{\text{Imp}}{I_{yy}}$$

Hence Eq. (6) becomes

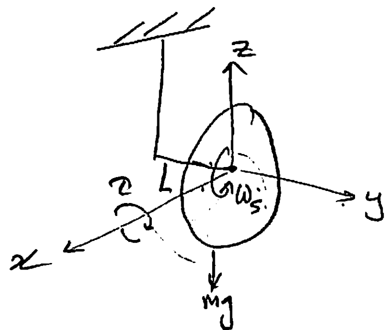
$$r(t) = \left(-\frac{(I_2 - I_1)}{I_3\sqrt{k}}\Omega\,\frac{\text{Imp}}{I_{yy}}\right)\sin\sqrt{k}t$$

so $r(t)$ is sinusoidal as well.

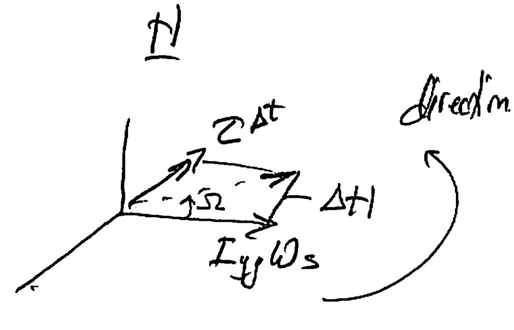
7.7



a) From Top/Front



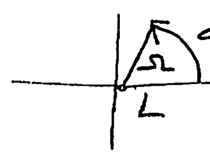
$\Rightarrow$



$\Delta H = \tau \Delta t$ i.e. $\boxed{\underline{H} \text{ after } \Delta t = I_{yy}\omega_s + \tau \Delta t}$

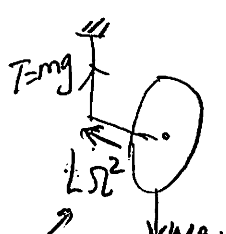
hence $\dfrac{\Delta H}{\Delta t} = \tau$ but $\dfrac{\Delta H}{\Delta t} = (I_{yy}\omega_s)\,\Omega$

hence $\boxed{\Omega = \dfrac{\tau}{I_{yy}\omega_s}}$ so looking from top

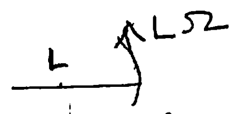


direction of precession anticlockwise.

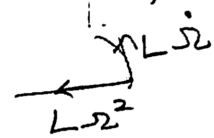
(b) looking from top



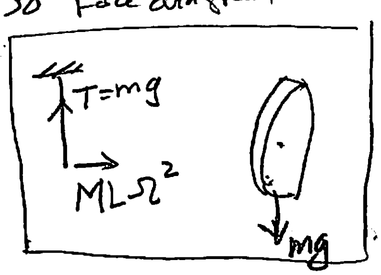
so velocity diagram is $\rightarrow$



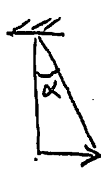
and acc. diagram is



so force diagram



$\Rightarrow$ angle



$$\Rightarrow \tan\alpha = \frac{ML\Omega^2}{Mg} = \frac{L\Omega^2}{g} = \frac{L}{g}\left(\frac{\tau}{I_{yy}\omega_s}\right)^2$$

$$= \frac{L}{g}\left(\frac{MgL}{I_{yy}\omega_s}\right)^2 = \left(\frac{M}{I_{yy}\omega_s}\right)^2 gL^3$$