HW 8

MATH 121B

Spring 2004
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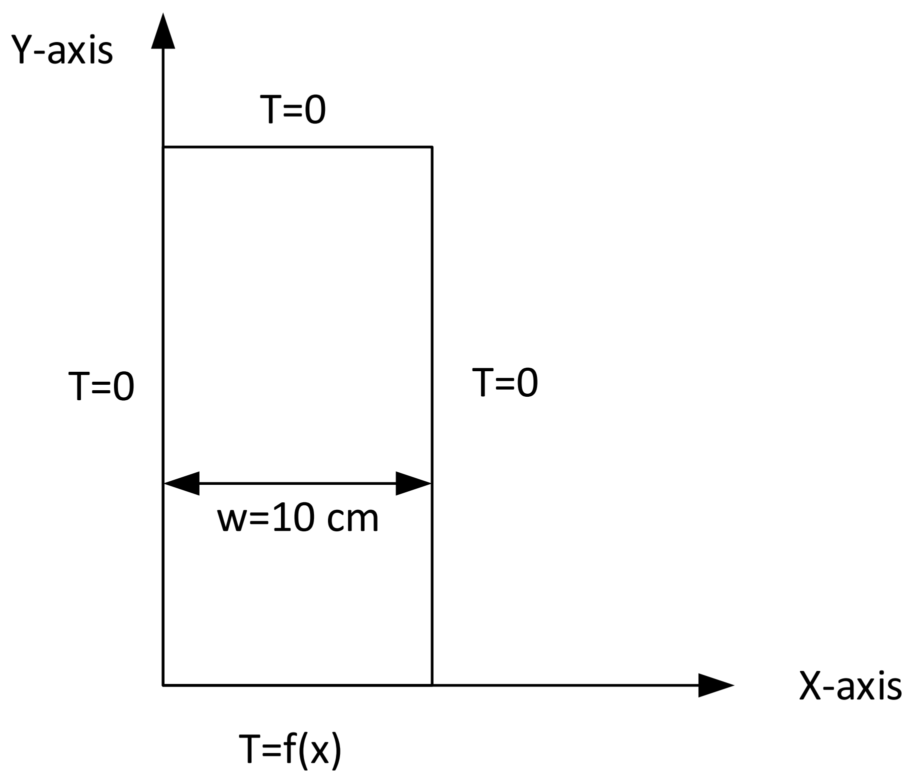
1 Chapter 13, problem 2.1 Mary Boas book. Second edition

Find the steady-state temperature distribution for the semi-infinite plate problem if the temp at the bottom edge is $T = f(x) = x$ (in degrees; that is the temp at x cm is x degrees). The temperature of the others sides is zero degrees and the width of the plate is 10 cm.



Semi-infinite plate

Solution

Since we are looking for a steady state heat distribution, which means there is no heat source, then we use Laplace PDE to represent the problem. We need to solve the following PDE

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

For a 2D problem as the above, we start by assuming that the solution $T(x, y)$ is of the form $T(x, y) = X(x)Y(y)$. We now substitute this assumed solution into the Laplace PDE and obtain

$$X''Y + Y''X = 0$$

dividing by XY gives

$$\frac{1}{X}X'' + Y''\frac{1}{Y} = 0$$
$$\frac{1}{X}X'' = -\frac{1}{Y}Y'' = -k^2$$

Since the left hand side in the above equation depends only on the independent variable x while the right hand side depends only on the independent variable y, and both sides are equal to each others, then each side must be equal to the same constant. This is called the separation of variables approach. Assuming this constant is $-k^2$ for $k \geq 0$ we obtain two ODE's to solve for X and Y

$$X'' + k^2 X = 0$$

and

$$Y'' - k^2 Y = 0$$

To solve the X ODE, we assume the solution is $X = Ae^{mx}$, for some constants A, m and substitute this in the ODE to obtain $m^2 Ae^{mx} + k^2 Ae^{mx} = 0$, or $m^2 + k^2 = 0$. This is the characteristic equation whose solution is $m = \pm ik$, hence $X = Ae^{\pm ikx}$.

A general solution is found by adding all the individual solutions, hence $X = Ae^{ikx} + Ae^{-ikx} = A\left(e^{ikx} + e^{-ikx}\right)$. But $\cos(kx) = \frac{e^{ikx}+e^{-ikx}}{2}$, hence $X = 2A\cos(kx) = \cos(kx)$ by taking constant $2A = 1$.

Another general solution can be obtained by taking the difference of the individual solutions, hence, $X = Ae^{ikx} - Ae^{-ikx} = A\left(e^{ikx} - e^{-ikx}\right)$. But $\sin kx = \frac{e^{ikx} - e^{-ikx}}{2i}$, hence $X = 2iA \sin kx = sin(kx)$ by taking $2iA = 1$. Therefore solutions to $\frac{1}{X}\frac{d^2X}{dx^2} = -k^2$ are

$$\begin{aligned} X_1(x) &= \cos(kx) \\ X_2(x) &= \sin(kx) \end{aligned}$$

Now we solve $Y'' - k^2 Y = 0$. Assuming solution is $Y = Ae^{my}$ hence the characteristic equation is $m^2 Ae^{my} - k^2 Ae^{my} = 0$, or $m^2 - k^2 = 0$, hence $m = \pm k$, then $Y = Ae^{\pm ky}$, and let $A = 1$, then $Y = e^{-ky}$ or $Y = e^{ky}$

Since $T\left(x, y\right) = X(x)Y(y)$, then the T solution is a combination of all the above solutions.

$$T\left(x, y\right) = \begin{cases} \sin kx \\ \cos kx \end{cases} \begin{cases} e^{ky} \\ e^{-ky} \end{cases}$$

Now we use the boundary conditions on the plate to find which of the above 4 solutions is the correct solution.

Since this is a semi-infinite plate, then as $y \to \infty$, $T\left(x, y\right) \to 0$, this means e^{ky} solution must be rejected since they have the positive power of y on the exponential function. (since $k > 0$). Therefore we now have

$$T\left(x, y\right) = \begin{cases} \sin kx\, e^{-ky} \\ \cos kx\, e^{-ky} \end{cases}$$

Looking now at the left boundary condition where we want $T = 0$ for $x = 0$, this means that solution $\cos kx\, e^{-ky}$ must be rejected since it is not zero at $x = 0$.

So, only solution left is

$$T\left(x, y\right) = \sin kx\, e^{-ky}$$

And we have two boundary conditions to satisfy yet, the right hand side, and the bottom side.

At the right side, where $x = w = 10\ cm$, we need $T = 0$, hence this can be achieved by having $\sin 10k = 0$ or $10k = n\pi$, or $k = \frac{n\pi}{10}$ for $n = 1, 2, 3, \cdots$ So the solution now looks like

$$T(x, y) = \sin\left(\frac{n\pi}{10}x\right)e^{-\frac{n\pi}{10}y} \quad n = 1, 2, 3, \cdots$$

We have the last boundary condition to satisfy, which is the bottom side. On that side we have $T = f(x) = x$ at $y = 0$ hence if we let $y = 0$ in the above the solution becomes

$$T(x, 0) = x = \sin\left(\frac{n\pi}{10}x\right)$$

This solution is not satisfied for any n. for example, for $x=5$, $n=1$, we have $\sin\left(\frac{\pi}{10}5\right)=\sin\frac{\pi}{2}=1\neq 5$

Hence we need to find another method to find this boundary condition. Since a sum of scaled solutions is also a solution (this is a linear system), then we write

$$T(x,y)=\sum_{n=1}^{\infty} b_n\, e^{-\frac{n\pi}{10}y}\sin\left(\frac{n\pi}{10}x\right)$$

Now we try to find b_n when $y=0$. This is the Fourier series expansion for $f(x)$.

Since sin functions are orthogonal to each others, i.e. $\int_0^w \sin ax\ \sin bx\ dx=0\ a\neq b$,the above can be written as

$$\begin{aligned}\int_0^w \sin\left(\frac{n\pi}{10}x\right) f(x)\,dx &= \int_0^{10}\sin\left(\frac{n\pi}{10}x\right)\left(\sum_{m=1}^{\infty} b_m\sin\left(\frac{m\pi}{10}x\right)\right)dx\\ &=\sum_{m=1}^{\infty} b_m\int_0^{10}\sin\left(\frac{n\pi}{10}x\right)\sin\left(\frac{m\pi}{10}x\right)dx\\ &= b_n\int_0^{10}\sin\left(\frac{n\pi}{10}x\right)\sin\left(\frac{n\pi}{10}x\right)dx\end{aligned}$$

Since all terms vanish expect when $m=n$ then

$$\begin{aligned}b_n &= \frac{\int_0^{10}\sin\left(\frac{n\pi}{10}x\right) f(x)\,dx}{\int_0^{10}\sin^2\left(\frac{n\pi}{10}x\right)\ dx}\\ &=\frac{\int_0^{10}\sin\left(\frac{n\pi}{10}x\right) x\,dx}{\int_0^{10}\sin^2\left(\frac{n\pi}{10}x\right)\ dx}\end{aligned}$$

But $\int_0^{10}\sin^2\left(\frac{n\pi}{10}x\right)dx=5$ for $n\neq 0$. Hence $b_n=\frac{2}{10}\int_0^{10}x\sin\left(\frac{n\pi}{10}x\right)dx$. integration by parts. $\int udvdx=uv-\int v\frac{du}{dx}dx$. Let $u=x,\ dv=\sin\left(\frac{n\pi}{10}x\right)$ then $\frac{du}{dx}=1,\ v=\frac{-w}{n\pi}\cos\left(\frac{n\pi}{10}x\right)$. Hence (using $w=10$)

$$\begin{aligned}\frac{b_n}{\frac{2}{w}} &= \left[-x\frac{w}{n\pi}\cos\left(\frac{n\pi}{w}x\right)\right]_0^w-\int_0^w\frac{-w}{n\pi}\cos\left(\frac{n\pi}{w}x\right)dx\\ &=\left[\frac{-w^2}{n\pi}\cos(n\pi)-0\right]+\frac{w}{n\pi}\left[\frac{1}{\frac{n\pi}{w}}\sin\frac{n\pi}{w}x\right]_0^w\\ &=\left[\frac{-w^2}{n\pi}\cos(n\pi)\right]+\frac{w}{n\pi}\left[\frac{w}{n\pi}\sin n\pi-0\right]\\ &=\left[\frac{-w^2}{n\pi}\cos(n\pi)\right]+\left[\frac{w^2}{n^2\pi^2}\sin n\pi\right]\\ &=\frac{w^2}{\pi}\left(\frac{-1}{n}\cos(n\pi)+\frac{1}{n^2\pi}\sin n\pi\right)\end{aligned}$$

Hence

$$b_n=2\frac{w}{\pi}\left(\frac{-1}{n}\cos(n\pi)+\frac{1}{n^2\pi}\sin n\pi\right)$$

Since n is an integer, all the $\sin n\pi$ terms vanish

$$b_n=2\frac{w}{\pi}\left(\frac{-1}{n}\cos(n\pi)\right)$$

Since $w=10$ then

$$b_n = \frac{20}{\pi}\left(\frac{-1}{n}\cos(n\pi)\right)$$

Then

$$\begin{aligned}
b_n &= \frac{20}{\pi}\left(\frac{-1}{1}(-1)\right), \frac{20}{\pi}\left(\frac{-1}{2}(1)\right), \frac{20}{\pi}\left(\frac{-1}{3}(-1)\right), \cdots \\
b_n &= \frac{20}{\pi}, \frac{20}{\pi}\left(\frac{-1}{2}\right), \frac{20}{\pi}\left(\frac{+1}{3}\right), \cdots \\
&= \frac{20}{\pi}\left(\frac{-1^{n+1}}{n}\right)
\end{aligned}$$

Hence

$$\begin{aligned}
T(x,y) &= \sum_{n=1}^{\infty} b_n \, e^{-\frac{n\pi}{10}y} \sin\left(\frac{n\pi}{10}x\right) \\
T(x,y) &= \frac{20}{\pi}\sum_{n=1}^{\infty}\left(\frac{-1^{n+1}}{n}\right) e^{-\frac{n\pi}{10}y} \sin\left(\frac{n\pi}{10}x\right)
\end{aligned}$$

Here is a plot of the solution for n up to 70.

```
T0[x_, y_, m_] :=20/Pi Sum[ (-1)^(n + 1)/n Exp[- (n Pi/10) y] Sin[ (n Pi
    /10) x], {n,1, m}%
]
p = Plot3D[T0[x, y, 100], {x, 0, 10}, {y, 0, 6}%
, PlotRange -> All,AxesLabel -> {x, y, z}, BaseStyle -> 20]
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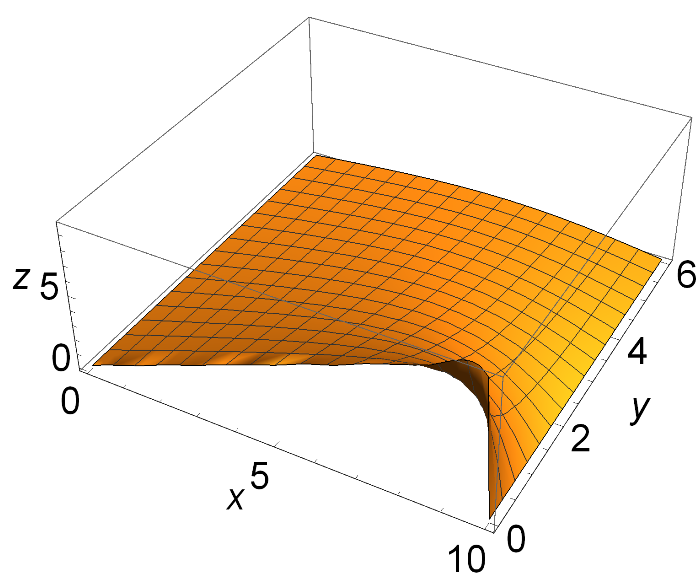
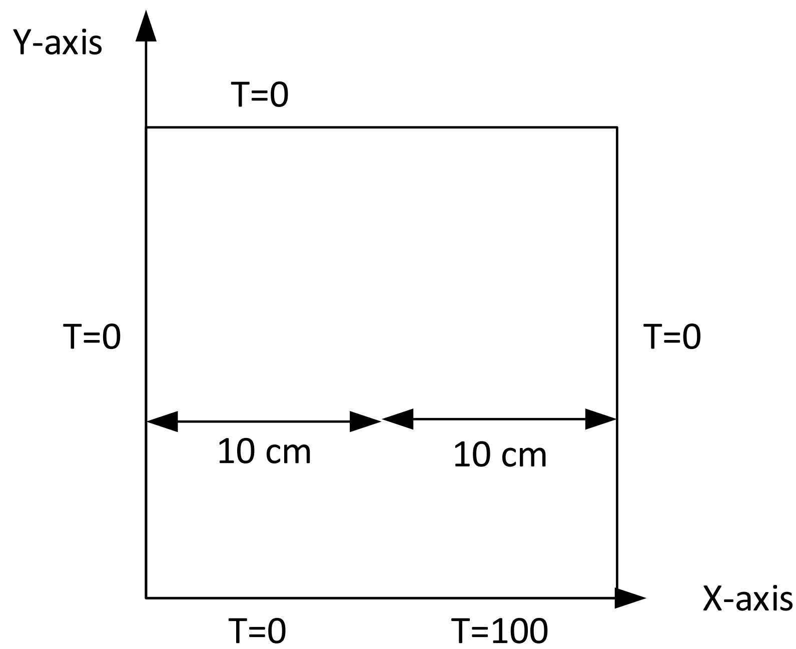


2 Chapter 13, problem 2.2 Mary Boas book. second edition

Find the steady-state temperature distribution for the semi-infinite plate with bottom edge of 20 cm if the temp at the bottom edge temp. is held at

$$T = \begin{cases} 0^0 & 0 < x < 10 \\ 100 & 10 < x < 20 \end{cases}$$

The others sides at zero degrees.



Semi-infinite plate

solution To solve this, I will follow the same steps as in 2.1, until I get to the step of trying to fit to the bottom edge conditions into the solution, and then I will use $f(x)$ as a step function:

$$f(x) = \begin{cases} 0 & 0 < x < 10 \\ 100 & 10 < x < 20 \end{cases}$$

Hence, as shown in problem 2.1, the candidate solutions for $T(x,y)$ are

$$T(x,y) = \begin{cases} \sin kx\, e^{ky} \\ \sin kx\, e^{-ky} \\ \cos kx\, e^{ky} \\ \cos kx\, e^{-ky} \end{cases}$$

Now we use the boundary conditions on the plate to find which of the above 4 solutions is the correct solution. We know by the uniqueness theorem of ODE solution that there will be one solution only out of the above 4, and by the existence theorem, that a solution will exist.

Since this is a semi-infinite plate, then as $y \to \infty$, $T(x,y) \to 0$, this means $\sin(kx)e^{ky}$ and $\cos(kx)e^{ky}$ solution must be rejected since they have the positive power of y on the exponential function. (since $k > 0$)

Looking now at the left boundary condition where we want $T = 0$ for $x = 0$, this means that solution $\cos(kx)e^{-ky}$ must be rejected since it is not zero at $x = 0$.

So, only solution left is $\sin(kx)e^{-ky}$ and we have 2 boundary conditions to satisfy yet, the right hand side, and the bottom side.

At the right side, where $x = w = 20\ cm$, we need $T = 0$, hence this can be achieved by having $kw = n\pi$, or $k = \frac{n\pi}{w}$ for $n = 1, 2, 3, \cdots$

so the solution now looks like

$$T(x, y) = \sin\left(\frac{n\pi}{w}x\right) e^{-\frac{n\pi}{w}y} \quad n = 1, 2, 3, \cdots$$

Now we have the last boundary condition to satisfy, Since a sum of scaled solutions is also a solution (this is a linear system), then we write

$$T(x, y) = \sum_{n=1}^{\infty} b_n\, e^{-\frac{n\pi}{w}y} \sin\left(\frac{n\pi}{w}x\right)$$

And now we try to find b_n when $y = 0$

This is the Fourier series expansion for $f(x)$.

Since sin functions are orthogonal to each others, i.e. $\int_0^w \sin ax\ \sin(bx)\, dx = 0$ for $a \neq b$ then the above can be written as

$$\begin{aligned}
\int_0^w \sin\left(\frac{n\pi}{w}x\right) f(x)\, dx &= \int_0^w \sin\left(\frac{n\pi}{w}x\right)\left(\sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi}{w}x\right)\right)dx \\
&= \sum_{m=1}^{\infty} b_m \int_0^w \sin\left(\frac{n\pi}{w}x\right)\sin\left(\frac{m\pi}{w}x\right)dx \\
&= b_n \int_0^w \sin\left(\frac{n\pi}{w}x\right)\sin\left(\frac{n\pi}{w}x\right)dx
\end{aligned}$$

Since all terms vanish expect when $m = n$, hence

$$\begin{aligned}
b_n &= \frac{\int_0^w \sin\left(\frac{n\pi}{w}x\right) f(x)\, dx}{\int_0^w \sin^2\left(\frac{n\pi}{w}x\right)\ dx} \\
&= \frac{\int_0^w \sin\left(\frac{n\pi}{w}x\right) x\, dx}{\int_0^w \sin^2\left(\frac{n\pi}{w}x\right)\ dx}
\end{aligned}$$

But $\int_0^w \sin^2\left(\frac{n\pi}{w}x\right)\ dx = \frac{w}{2}$ for $n \neq 0$ Hence

$$\begin{aligned}
b_n &= \frac{2}{w}\int_0^w f(x)\sin\left(\frac{n\pi}{w}x\right)\ dx \\
&= \frac{2}{20}\left\{\int_0^{10} f(x)\sin\left(\frac{n\pi}{w}x\right)\ dx + \int_{10}^{20} f(x)\sin\left(\frac{n\pi}{w}x\right)\ dx\right\}
\end{aligned}$$

But $f(x) = 0$ for $0 < x < 10$, and $f(x) = 100$ for $10 < x < 20$ therefore

$$
\begin{aligned}
b_n &= \frac{2}{20}\left\{\int_0^{10} 0\ \sin\left(\frac{n\pi}{20}x\right)\ dx + \int_{10}^{20} 100\sin\left(\frac{n\pi}{20}x\right)\ dx\right\}\\
&= \frac{200}{20}\int_{10}^{20} \sin\left(\frac{n\pi}{20}x\right)\ dx\\
&= 10\int_{10}^{20} \sin\left(\frac{n\pi}{20}x\right)\ dx\\
&= 10\,\frac{1}{\frac{n\pi}{20}}\left[-\cos\frac{n\pi}{20}x\right]_{10}^{20}\\
&= \frac{-200}{n\pi}\left[\cos\frac{n\pi}{20}x\right]_{10}^{20}\\
&= \frac{-200}{n\pi}\left[\cos\frac{n\pi}{20}20 - \cos\frac{n\pi}{20}10\right]\\
&= \frac{-200}{n\pi}\left[\cos n\pi - \cos\frac{n\pi}{2}\right]
\end{aligned}
$$

Looking at few n values starting from $n = 1$

$$
\begin{aligned}
b_n = {} & \frac{-200}{\pi}\left[\cos\pi - \cos\frac{\pi}{2}\right], \frac{-200}{2\pi}[\cos 2\pi - \cos\pi], \frac{-200}{3\pi}\left[\cos 3\pi - \cos\frac{3\pi}{2}\right], \frac{-200}{4\pi}[\cos 4\pi - \cos 2\pi],\\
& \frac{-200}{5\pi}\left[\cos 5\pi - \cos\frac{5\pi}{2}\right], \frac{-200}{6\pi}[\cos 6\pi - \cos 3\pi], \frac{-200}{7\pi}\left[\cos 7\pi - \cos\frac{7\pi}{2}\right], \frac{-200}{8\pi}[\cos 8\pi - \cos 4\pi]
\end{aligned}
$$

$$
\begin{aligned}
b_n = {} & \frac{-200}{\pi}[-1-0], \frac{-200}{2\pi}[1-(-1)], \frac{-200}{3\pi}[-1-0], \frac{-200}{4\pi}[1-1], \frac{-200}{5\pi}[-1-0], \frac{-200}{6\pi}[1-(-1)],\\
& \frac{-200}{7\pi}[-1-0], \frac{-200}{8\pi}[1-1]
\end{aligned}
$$

$$
b_n = \frac{-200}{\pi}[-1], \frac{-200}{2\pi}[2], \frac{-200}{3\pi}[-1], \frac{-200}{4\pi}[0], \frac{-200}{5\pi}[-1], \frac{-200}{6\pi}[2], \frac{-200}{7\pi}[-1], \frac{-200}{8\pi}[0], \cdots
$$

We see a term multiplier is $-1, 2, -1, 0, -1, 2, -1, 0, \ldots$

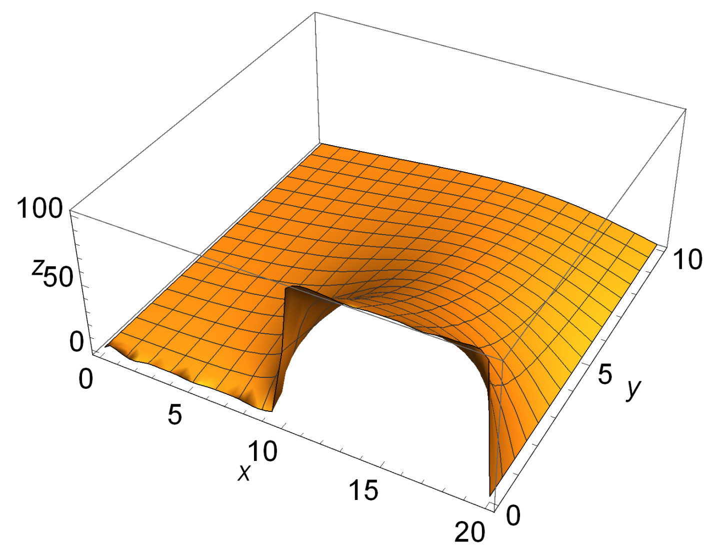
When n is multiple of 4, this multiplier is zero. when n is odd, the multiplier is -1, and when n is even (not multiple of 4), this multiplier is 2.

Solution is

$$
T(x,y) = \sum_{n=1}^{\infty} b_n\, e^{-\frac{n\pi}{20}y}\sin\left(\frac{n\pi}{20}x\right)
$$

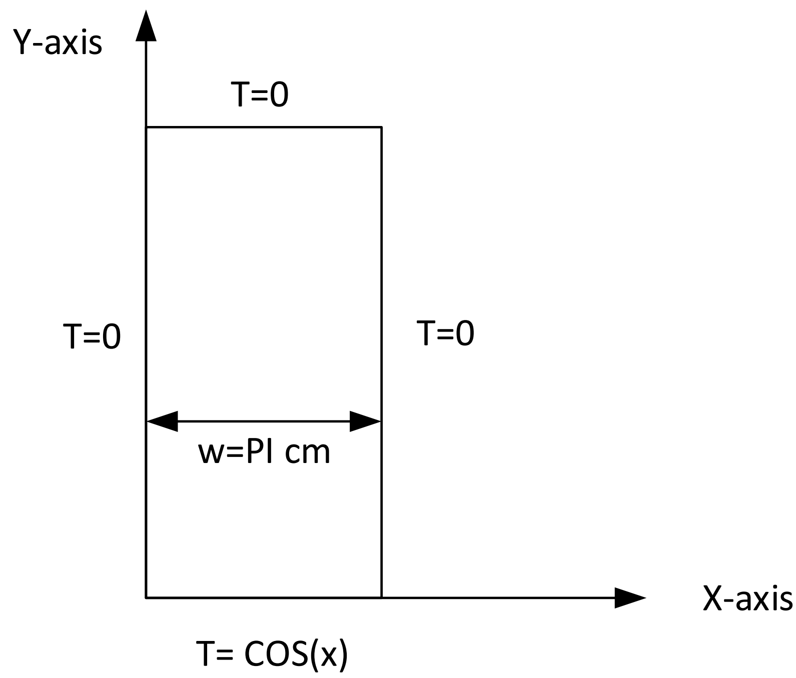
$$
T(x,y) = \begin{cases} \frac{200}{\pi}\sum\frac{1}{n}e^{-\frac{n\pi}{20}y}\sin\left(\frac{n\pi}{20}x\right) & \text{n odd}, 1,3,5,7,\ldots \\ -\frac{400}{\pi}\sum\frac{1}{n}e^{-\frac{n\pi}{20}y}\sin\left(\frac{n\pi}{20}x\right) & \text{n even } 2,6,10,14,18,\ldots \\ 0 & \text{Otherwise} \end{cases}
$$

```
T1[x_, y_, m_] :=200/Pi Sum[1/n Exp[- (n Pi/20) y] Sin[ (n Pi/20) x], {n,
    1, m, 2}];
T2[x_, y_, m_] := -400/Pi Sum[1/n Exp[- (n Pi/20) y] Sin[ (n Pi/20) x], {
   n, 2, m, 4}];
p = Plot3D[T1[x, y, 200] + T2[x, y, 100], {x, 0, 20}, {y, 0, 10},
PlotRange -> All, AxesLabel -> {x, y, z}, BaseStyle -> 15]
```


100
z 50
0
0
5
x 10
15
20
0
5 y
10

3 Chapter 13, problem 2.3 Mary Boas book. second edition

Find the steady-state temperature distribution for the semi-infinite plate problem if the temp at the bottom edge is $T = f(x) = \cos(x)$ (The temp. of the others sides is zero degrees, and the width of the plate is π cm.



Semi-infinite plate

Solution

This problem is similar to problem 2.1, but for a different boundary function at the bottom edge.

As shown in problem 2.1, $T(x,y)$ is given by one of these solutions:

$$T(x,y) = \begin{cases} \sin kx\, e^{ky} \\ \sin kx\, e^{-ky} \\ \cos kx\, e^{ky} \\ \cos kx\, e^{-ky} \end{cases}$$

Now we use the boundary conditions on the plate to find which of the above 4 solutions is the correct solution. We know by the uniqueness theorem of ODE solution that there will be one solution only out of the above 4, and by the existence theorem, that a solution will exist.

Since this is a semi-infinite plate, then as $y \to \infty$, $T(x,y) \to 0$, this means $\sin kx\, e^{ky}$ and $\cos kx\, e^{ky}$ solution must be rejected since they have the positive power of y on the exponential function. (since $k > 0$)

Looking now at the left boundary condition where we want $T = 0$ for $x = 0$, this means that solution $\cos kx\, e^{-ky}$ must be rejected since it is not zero at $x = 0$.

Only solution left is $\sin kx\, e^{-ky}$ and we have 2 boundary conditions to satisfy yet, the right hand side, and the bottom side.

At the right side, where $x = w = \pi\ cm$, we need $T = 0$, hence this can be achieved by having $k\pi = n\pi$, or $k = n$ for $n = 1, 2, 3, \cdots$

so the solution now looks like

$$T(x,y) = \sin(nx)\, e^{-ny} \quad n = 1, 2, 3, \cdots$$

Now we have the last boundary condition to satisfy, which is the bottom side. On that side we have $T = f(x) = \cos(x)$ at $y = 0$ hence if we let $y = 0$ in the above the solution becomes

$$T\left(x, y\right) = \cos(x) = \sin(nx)$$

This solution is not satisfied for any n.

Hence we need to find another method to find this boundary condition. Since a sum of scaled solutions is also a solution (this is a linear system), then we write

$$T\left(x, y\right) = \sum_{n=1}^{\infty} b_n \, e^{-ny} \sin(nx)$$

Now we try to find b_n when $y = 0$, i.e. at $y = 0$

$$T\left(x, y\right) = f(x) = \cos(x) = \sum_{n=1}^{\infty} b_n \; \sin(nx)$$

This is the Fourier series expansion for $f(x)$. Since sin functions are orthogonal to each others, i.e. $\int_0^{\pi} \sin ax \; \sin bx \quad dx = 0 \;\; a \neq b$,the above can be written as (taking inner product of RHS and LHS w.r.t. $\sin(nx)$) :

$$\begin{aligned}
\cos(x) &= \sum_{n=1}^{\infty} b_n \; \sin(nx) \\
\int_0^{\pi} \sin(nx)\, f(x)\, dx &= \int_0^{\pi} \sin(nx) \left(\sum_{m=1}^{\infty} b_m \sin(mx) \right) dx \\
\int_0^{\pi} \sin(nx) \; \cos(x)\, dx &= \sum_{m=1}^{\infty} b_m \int_0^{\pi} \sin(nx) \sin(mx) dx \\
\int_0^{\pi} \sin(nx) \; \cos(x)\, dx &= b_n \int_0^{\pi} \sin(nx) \sin(nx)\, dx
\end{aligned}$$

Where on the RHS we simplified it since all terms vanish expect when $m = n$. The above now becomes

$$\begin{aligned}
\int_0^{\pi} \sin(nx) \; \cos(x)\, dx &= b_n \int_0^{\pi} \sin^2(nx) \;\; dx \\
\int_0^{\pi} \sin(nx) \; \cos(x)\, dx &= b_n \frac{\pi}{2}
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{2}{\pi} \int_0^{\pi} \sin(nx) \; \cos(x)\, dx \\
&= \frac{2}{\pi} \frac{n(1 + \cos(n\pi))}{n^2 - 1}
\end{aligned}$$

Looking at few values of $n = 2, 3, 4, \ldots$ (not defined for n=1).

$$\begin{aligned}
b_n &= \frac{2}{\pi} \frac{2(1 + \cos(2\pi))}{3}, \frac{2}{\pi} \frac{3(1 + \cos(3\pi))}{8}, \frac{2}{\pi} \frac{4(1 + \cos(4\pi))}{15} \ldots \\
b_n &= \frac{2}{\pi} \frac{2(2)}{3}, 0, \frac{2}{\pi} \frac{4(2)}{8}, 0, \ldots \\
&= \frac{4}{\pi} \frac{n}{n^2 - 1} \quad \text{for even n}
\end{aligned}$$

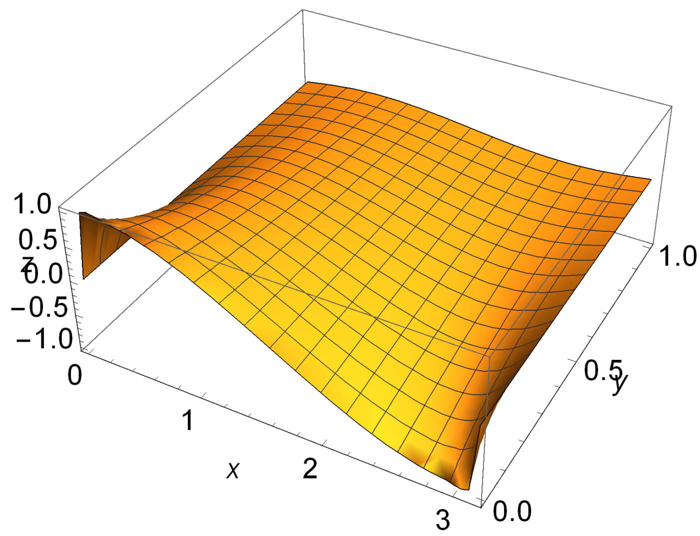
Since

$$T(x,y) = \sum_{n=1}^{\infty} b_n \, e^{-ny} \sin(nx)$$

Then the final solution is

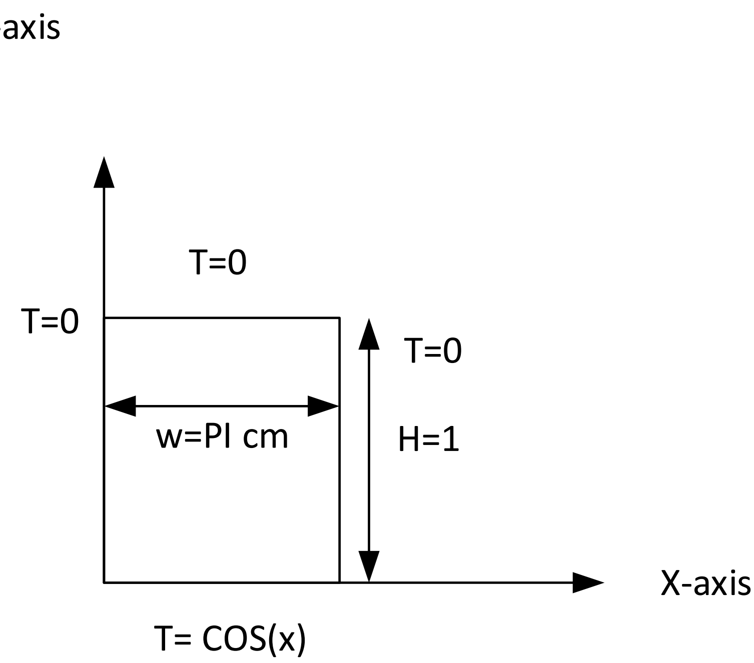
$$T(x,y) = \frac{4}{\pi} \sum_{n=even}^{\infty} \frac{n}{n^2-1} \, e^{-ny} \sin(nx)$$

```
T0[x_, y_, m_] :=4/Pi Sum[n/(n^2 - 1) Exp[-n  y] Sin[ n x], {n, 2, m,
   2}];
p = Plot3D[T0[x, y, 200], {x, 0, Pi}, {y, 0, 1}, PlotRange -> All,
AxesLabel -> {x, y, z}, BaseStyle -> 15]
```



4 Chapter 13, problem 2.7 Mary Boas book. second edition.

Find the steady-state temperature distribution of the following plate, height=1. Temp at the bottom edge is $T = \cos(x)$ (The temp. of the others sides is zero degree and width of the plate is π cm.



Solution As shown in problem 2.1, $T(x,y)$ is given by one of these solutions:

$$T(x,y) = \begin{cases} \sin kx\, e^{ky} \\ \sin kx\, e^{-ky} \\ \cos kx\, e^{ky} \\ \cos kx\, e^{-ky} \end{cases}$$

Now we use the boundary conditions on the plate to find which of the above 4 solutions is the correct solution. We know by the uniqueness theorem of ODE solution that there will be one solution only out of the above 4, and by the existence theorem, that a solution will exist.

Here we can not reject the 2 candidate solutions $\sin kx\, e^{ky}$ and $\cos kx\, e^{ky}$ as we did for the semi-infinite plate cases because as $y \to 1$, these solutions do not blow up.

But to use one of them , looking at $T(x,y) = \sin kx\, e^{ky}$, then at $y = 1$, where we require $T = 0$, we get $0 = \sin kx$, and this means we must have $k = n\pi$ for integer n. but this means that $T = 0$ everywhere in the plate and on the other boundaries, which is not correct.

Similarly if we try to fit $\cos(kx)e^{ky}$.

One way to avoid this problem is to use a linear combination of the exponential $ae^{-ky} + be^{ky}$ and now we try to find a, b. If we choose $a = \frac{1}{2}e^{hk}, b = -\frac{1}{2}e^{-hk}$, where h is the height of the plate, we get

$$\frac{1}{2}e^{hk}e^{-ky} - \frac{1}{2}e^{-hk}e^{ky} = \frac{1}{2}e^{k(h-y)} - \frac{1}{2}e^{-k(h-y)}$$

To verify, We want $\frac{1}{2}e^{k(h-y)} - \frac{1}{2}e^{-k(h-y)} = 0$ when $y = h$, Hence

$$\frac{1}{2}e^{k(h-y)} - \frac{1}{2}e^{-k(h-y)} = \frac{1}{2} - \frac{1}{2} = 0$$

The solutions to consider are now

$$T(x,y) = \begin{cases} \sin kx \left(\frac{1}{2}e^{k(h-y)} - \frac{1}{2}e^{-k(h-y)}\right) \\ \cos kx \left(\frac{1}{2}e^{k(h-y)} - \frac{1}{2}e^{-k(h-y)}\right) \end{cases}$$

The initial 4 candidate solutions are now 2 candidate solutions since we have combined a combination of two solutions together.

Looking now at the left boundary condition where we want $T = 0$ for $x = 0$, this means the second candidate solution above which is $\cos kx \left(\frac{1}{2}e^{k(h-y)} - \frac{1}{2}e^{-k(h-y)}\right)$ must be rejected since it is not zero at $x = 0$ for any y.

Only solution left is $\sin kx \left(\frac{1}{2}e^{k(h-y)} - \frac{1}{2}e^{-k(h-y)}\right)$. Write $\frac{1}{2}e^{k(h-y)} - \frac{1}{2}e^{-k(h-y)} = \sinh k(h-y)$ then the final candidate solution which we want to fit on the remaining boundary conditions can be written as

$$T(x,y) = \sinh k(h-y)\sin(kx)$$

We have 2 boundary conditions to satisfy yet, the right hand side, and the bottom side. At the right side, where $x = w = \pi cm$, we need

$$T = 0 = \sinh k(h-y)\ \sin k\pi$$

hence this can be achieved by having $k\pi = n\pi$, or $k = n$ for $n = 1,2,3,\cdots$ So the solution now looks like

$$T(x,y) = \sinh n(h-y)\sin(nx) \quad n = 1,2,3,\cdots$$

Now we have the last boundary condition to satisfy, which is the bottom side. On that side we have $T = f(x) = \cos(x)$ at $y = 0$ hence if we let $y = 0$ in the above the solution becomes

$$T(x,y) = \cos(x) = \sinh(n(h))\sin(nx)$$

This solution is not satisfied for any n. We need to find another method to find this boundary condition. Since a sum of scaled solutions is also a solution (this is a linear system), then we write

$$T(x,y) = \sum_{n=1}^{\infty} b_n \sinh(n(h-y))\sin(nx)$$

And now we try to find b_n when $y = 0$, i.e. at $y = 0$

$$T(x,y) = f(x) = \cos(x) = \sum_{n=1}^{\infty} b_n\ \sinh(nh)\ \sin(nx)$$

This is the Fourier series expansion for $f(x)$. Since sin functions are orthogonal to each others, i.e.

$$\int_0^{\pi} \sin(ax)\ \sin(bx)\,dx = 0 \quad a \neq b$$

The above can be written as (taking inner product of RHS and LHS w.r.t. $\sin nx$) :

$$\cos(x) = \sum_{m=1}^{\infty} b_m \ \sinh(mh) \ \sin(mx)$$
$$\int_0^{\pi} \sin(nx) \ f(x) \ dx = \int_0^{\pi} \sin(nx)\left(\sum_{m=1}^{\infty} b_m \ \sinh(mh) \ \sin(mx)\right)dx$$
$$\int_0^{\pi} \sin(nx) \ \cos(x) \ dx = \sum_{m=1}^{\infty} b_m \int_0^{\pi} \sin(nx) \sinh(mh) \ \sin(mx) dx$$
$$\int_0^{\pi} \sin(nx) \ \cos(x) \ dx = b_n \int_0^{\pi} \sin(nx) \sinh(nh) \sin(nx) \ dx$$

Where on the RHS we simplified it since all terms vanish expect when $m = n$. Above now becomes

$$\int_0^{\pi} \sin(nx) \ \cos(x) \ dx = b_n \int_0^{\pi} \sinh(nh) \ \sin^2(nx) \ dx$$
$$\int_0^{\pi} \sin(nx) \ \cos(x) \ dx = \sinh\left(ny\right) \ b_n \int_0^{\pi} \sin^2(nx) \ dx$$
$$= \frac{\pi}{2} b_n \ \sinh(nh)$$

$$b_n = \frac{2}{\pi} \frac{1}{\sinh(nh)} \int_0^{\pi} \sin(nx) \ \cos(x) \ dx$$
$$= \frac{2}{\pi} \frac{1}{\sinh(nh)} \frac{n(1 + \cos(n\pi))}{n^2 - 1}$$

Looking at few values of $n = 2, 3, 4, ...$ (not defined for n=1).

$$b_n = \frac{2}{\pi} \frac{1}{\sinh(h)} \frac{2(1 + \cos(2\pi))}{3}, \frac{2}{\pi} \frac{1}{\sinh(2h)} \frac{3(1 + \cos(3\pi))}{8}, \frac{1}{\sinh(2h)} \frac{2}{\pi} \frac{4(1 + \cos(4\pi))}{15} \cdots$$
$$b_n = \frac{2}{\pi} \frac{1}{\sinh(h)} \frac{2(2)}{3}, 0, \frac{1}{\sinh(3h)} \frac{2}{\pi} \frac{4(2)}{8}, 0, \ldots$$
$$= \frac{4}{\pi} \frac{1}{\sinh(nh)} \frac{n}{n^2 - 1} \quad \text{for even n}$$

Since

$$T\left(x, y\right) = \sum_{n=1}^{\infty} b_n \ \sinh\left(n\left(h - y\right)\right) \sin(nx)$$

The final solution becomes

$$T\left(x, y\right) = \frac{4}{\pi} \sum_{n=\text{even}}^{\infty} \frac{1}{\sinh(nh)} \frac{n}{n^2 - 1} \ \sinh\left(n\left(h - y\right)\right) \sin(nx)$$

```
h = 1;
T0[x_, y_, m_] := 4/Pi Sum[n/(n^2 - 1)  (1/Sinh[n h]) Sinh[n (h - y)] Sin
   [n x], {n, 2, m, 2}];
p = Plot3D[T0[x, y, 200], {x, 0, Pi}, {y, 0, 1}, PlotRange -> All,
   AxesLabel -> {x, y, z}, BaseStyle -> 15]aseStyle -> 15]
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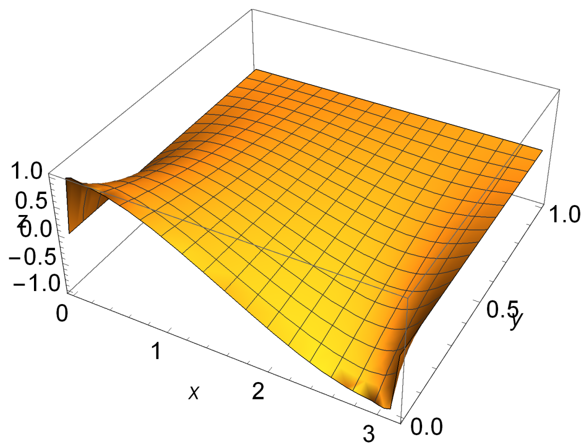
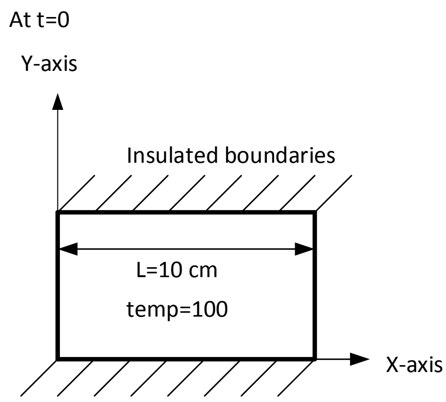
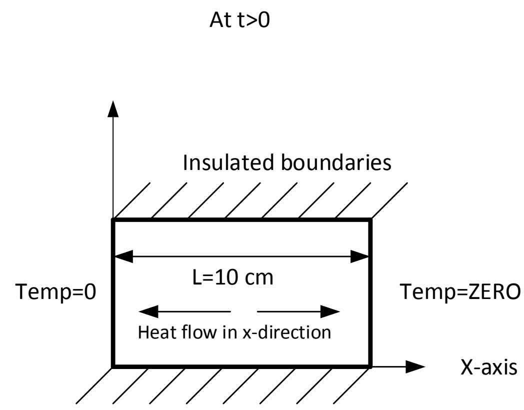

1.0
0.5
z
0.0
−0.5
−1.0
0
1
x
2
3
0.0
0.5
y
1.0

5 Chapter 13, problem 3.2 Mary Boas book. second edition

A bar length L=10 cm with insulated sides is initially at 100 degrees. starting at t=0, the ends are held at zero degree. Find the temperature distribution in the bar at time t.





Solution

This is a heat distribution problem governed by the diffusion or heat equation

$$\nabla^2 u(x,t) = \frac{1}{\alpha^2}\frac{\partial u(x,t)}{\partial t}$$

This is for a one spatial dimension.

To solve this PDE, assume the solution is

$$u(x,t) = F(x)T(t)$$

Where $F(x)$ is a function of the spatial x independent variable, and $T(t)$ is a function of the time t.

Solving using separation of variable as with the Laplace equation. By substituting in the original PDE, we get

$$\frac{1}{F}\frac{d^2F}{dx^2} = \frac{1}{\alpha^2}\frac{1}{T}\frac{dT}{dt}$$

Since RHS and LHS are both equal, and each is a function of a different independent variable, then both must be equal to a constant. Let this constant be $-k^2$. hence we get 2 ODE equations to solve

$$\begin{aligned}\frac{1}{F}\frac{d^2F}{dx^2} &= -k^2\\ \frac{1}{T}\frac{1}{\alpha^2}\frac{dT}{dt} &= -k^2\end{aligned}$$

To solve $\frac{1}{T}\frac{1}{\alpha^2}\frac{dT(t)}{dt} = -k^2$,

$$\begin{aligned}\frac{1}{T}\frac{dT}{dt} &= -\alpha^2 k^2\\ \frac{1}{T}dT &= -\alpha^2 k^2\, dt\\ \int \frac{1}{T}dT &= \int -\alpha^2 k^2\, dt\\ \ln T &= -\alpha^2 k^2 t\\ T &= e^{-\alpha^2 k^2 t}\end{aligned}$$

To solve $\frac{1}{F}\frac{d^2F}{dx^2} = -k^2$. Assume solution is $F(x) = e^{-mx}$ then$\frac{dF}{dx} = -me^{-mx}$, $\frac{d^2F}{dx^2} = m^2e^{-mx}$. Substituting in the ODE gives $m^2e^{-mx} = -e^{-mx}\,k^2$ or $m^2 = -k^2$, $m = \pm ik$, so $F(x) = e^{-ikx}$ or $F(x) = e^{ikx}$. By adding or subtracting these solutions we get a general solution that is either $\cos kx$ or $\sin kx$.

hence

$$F(x) = \begin{cases} \sin kx \\ \cos kx \end{cases}$$

So

$$u(x,t) = F(x)T(t)$$
$$u(x,t) = \begin{cases} e^{-\alpha^2 k^2 t}\sin kx \\ e^{-\alpha^2 k^2 t}\cos kx \end{cases}$$

Now we have 2 candidate solutions. Since these are solutions for $t > 0$, we need to find the conditions that $u = 0$ at $x = 0$ and $x = L$.

Since at $x = 0$, $u = 0$, then we can not use the $\cos kx$, solution because that will not go to zero at $x = 0$.

So we are left with the solution

$$u(x,t) = e^{-\alpha^2 k^2 t}\sin kx$$

Now apply the second boundary condition, which is $x = L, u = 0$.

This means $0 = e^{-\alpha^2 k^2 t}\sin kL$, then $kL = n\pi$ or $K = \frac{n\pi}{L}$ for $n = 1, 2, 3, ...$So our solution now looks like

$$u(x,t) = e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}\sin\frac{n\pi}{L}x \qquad n = 1, 2, 3, ...$$

Since a scaled sum of these solutions is a solution, then the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n\, e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}\sin\frac{n\pi}{L}x \qquad (1)$$

Now we need to find the b_n

For this we use the initial conditions, i.e. for $t = 0$. Then we had the sides at $u = 100$, and since no time was involved then (this is the initial steady state), the governing PDE is the Laplace equation with only the x spatial coordinate.

$\nabla^2 u_0(x) = 0$, a solution to this is $u_0(x) = ax + b$. when $x = 0$, $u_0 = 100$, hence $100 = b$.

When $x = L, u = 100$, hence $100 = La + b$, or $La = 100 - 100 = 0$. hence $a = 0$.

Hence at $t = 0$, $u_0 = 100$. So now from equation (1) above, we write

$$u(x,0) = u_0(x) = 100 = \sum_{n=1}^{\infty} b_n\ \sin\frac{n\pi}{L}x$$

Taking the inner product of the LHS and RHS w.r.t. $\sin\frac{n\pi x}{L}$ over $[0, L]$ gives

$$
\begin{aligned}
\int_0^L 100\ \sin\frac{n\pi}{L}x\,dx &= \int_0^L \left(\sum_{m=1}^{\infty} b_m\ \sin\frac{m\pi}{L}x\right)\sin\frac{n\pi}{L}x\ \ dx\\
100\int_0^L\ \sin\frac{n\pi}{L}x\,dx &= \sum_{m=1}^{\infty} b_m \int_0^L \sin\frac{m\pi}{L}x\sin\frac{n\pi}{L}xdx\\
100\int_0^L\ \sin\frac{n\pi}{L}x\,dx &= b_n\ \int_0^L \sin^2\frac{n\pi}{L}x\,dx\\
-100\left[\frac{L}{n\pi}\cos\frac{n\pi x}{L}\right]_0^L &= b_n\ \ \frac{L}{2}\\
-100\left[\frac{L}{n\pi}\cos n\pi-\frac{L}{n\pi}\right] &= b_n\ \ \frac{L}{2}\\
-\frac{100L}{n\pi}\left[\cos n\pi-1\right] &= b_n\ \ \frac{L}{2}\\
b_n &= -\frac{200}{n\pi}\left[\cos n\pi-1\right]
\end{aligned}
$$

Looking at few values of $n = 1, 2, 3, 4, \ldots$, $b_n = -\frac{200}{n\pi}[\cos n\pi - 1]$

$$
\begin{aligned}
b_n &= -\frac{200}{\pi}[\cos\pi-1], -\frac{200}{2\pi}[\cos 2\pi-1], -\frac{200}{3\pi}[\cos 3\pi-1], -\frac{200}{4\pi}[\cos 4\pi-1], \ldots\\
b_n &= -\frac{200}{\pi}[-1-1], -\frac{200}{2\pi}[1-1], -\frac{200}{3\pi}[-1-1], -\frac{200}{4\pi}[1-1], .\\
b_n &= \frac{400}{\pi}, 0, \frac{400}{3\pi}, 0,
\end{aligned}
$$

Hence

$$
b_n = \frac{1}{n}\frac{400}{\pi} \quad \text{for odd } n
$$

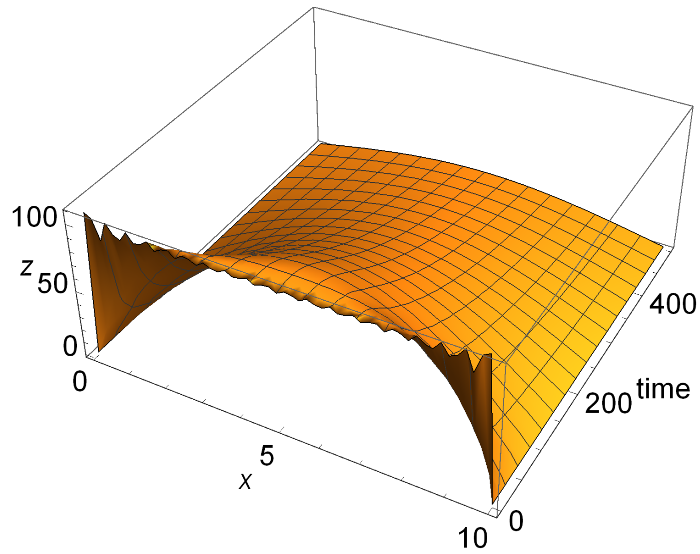
From equation (1) above we had

$$
u(x,t) = \sum_{n=1}^{\infty} b_n\, e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}\sin\frac{n\pi}{L}x
$$

Hence

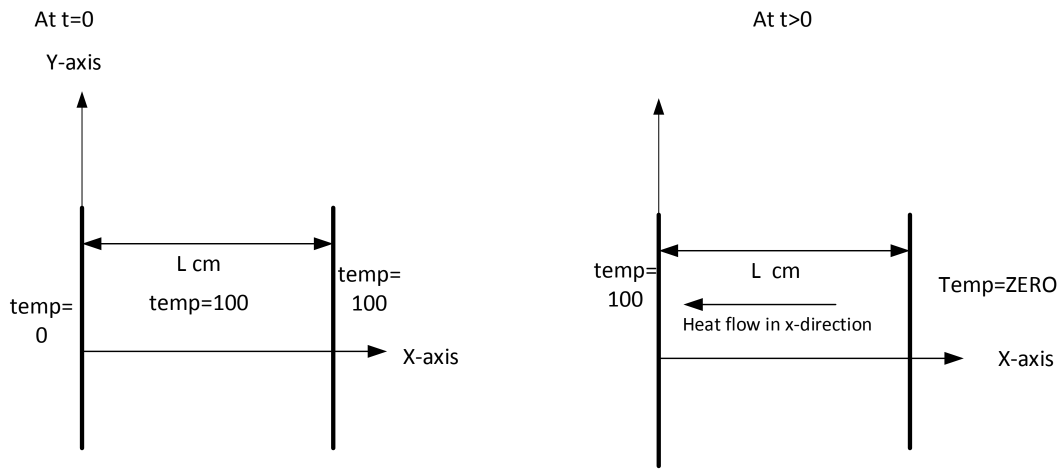
$$
u(x,t) = \frac{400}{\pi}\sum_{\text{odd } n}^{\infty}\frac{1}{n}\, e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}\sin\left(\frac{n\pi}{L}x\right)
$$

```
L0 = 10; alpha = 0.2;
T0[x_, t_, m_] := 400/Pi Sum[1/n Exp[-(n Pi/L0  alpha)^2 t] Sin[ n Pi/L0
   x], {n, 1, m, 2}];
p = Plot3D[T0[x, t, 40], {x, 0, 10}, {t, 0, 500}, PlotRange -> All,
AxesLabel -> {x, "time", z}, BaseStyle -> 15]
```


100
z
50
0
0
5
x
10
0
200
time
400

6 Chapter 13, problem 3.3 Mary Boas book. second edition

In the initial state of an infinite slab of thickness L, the face x=0 is at zero degrees, and the face at $x = l$ is at 100 degrees. from $t = 0$ on, the face at $x = 0$ is held at 100 degrees, and the face at $x = L$ at zero degrees. find the temp. distribution at time t.



Solution

This is a heat distribution problem governed by the diffusion or heat equation

$$\nabla^2 u(x,t) = \frac{1}{\alpha^2}\frac{\partial u(x,t)}{\partial t}$$

This problem is similar to problem 2.2, where an infinite slab is considered the same as a slab with 2 insulated sides. Similar to problem 3.2, we get the following 2 candidate solutions to the above PDE

$$u(x,t) = \begin{cases} e^{-\alpha^2 k^2 t}\sin kx \\ e^{-\alpha^2 k^2 t}\cos kx \end{cases}$$

Since these are solutions for $t > 0$, we need to find the conditions that $u = 100$ at $x = 0$ and $u = 0$ at $x = L$.

discard the $\cos kx$ solution because at $x = 0$ we want $u = 100$, which means $e^{-\alpha^2 k^2 t}\cos kx = e^{-\alpha^2 k^2 t} = 100$ which is not generally true for all t.

So the second solution is $e^{-\alpha^2 k^2 t}\sin kx$, which is 0 at $x = L$, hence $e^{-\alpha^2 k^2 t}\sin kL = 0$, i.e. $kL = n\pi$ or $k = \frac{n\pi}{L}$. So we start with the solution

$$u(x,t) = e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}\sin\frac{n\pi}{L}x$$

To make this solution fit at $x = 0$, we need to have $100 = e^{-\alpha^2 k^2 t}\sin\frac{n\pi}{L}x$. but $\sin(x)$ is zero at $x = 0$, hence to compensate, we start with the solution

$$u(x,t) = 100 - e^{-\left(\alpha\frac{n\pi}{L}\right)^2 t}\sin\frac{n\pi}{L}x$$

This solution gives 100 when $x = 0$.

But now we need to check it again for $x = L$, we see it gives $u = 100$ which is not correct. So need to subtract the term $\frac{100}{L}x$ (which is found below for the initial steady state). Now we have the candidate solution

$$u(x,t) = 100 - \frac{100}{L}x + e^{-\left(\alpha\frac{n\pi}{L}\right)^2 t}\sin\frac{n\pi}{L}x \tag{1}$$

To verify: at $x = 0$, this given $u = 100$, and at $x = L$, this gives $u(x,t) = 100 - \frac{100}{L}L = 0$, which is what we want.

Since a scaled sum of these solutions is a solution, then the general solution is

$$u(x,t) = 100 - \frac{100}{L}x + \sum_{n=1}^{\infty} b_n\, e^{-\left(\alpha\frac{n\pi}{L}\right)^2 t}\sin\frac{n\pi}{L}x \tag{2}$$

Now we need to find the b_n. For this we use the initial conditions, i.e. for $t = 0$.

The sides $x = 0$ is at $u = 0$, and since no time was involved then (this is the initial steady state), the governing PDE is the Laplace equation with only the x spatial coordinate. $\nabla^2 u_0(x) = 0$, a solution to this is $u_0(x) = ax + b$. when $x = 0$, $u_0 = 0$, hence $0 = b$.

When $x = L, u = 100$, hence $100 = La + 0$, or $a = \frac{100}{L}$.Hence at $t = 0$

$$u_0 = \frac{100}{L}x$$

So now from equation (2) above, we write

$$\begin{aligned}
u(x,0) = u_0(x) = \frac{100}{L}x &= 100 - \frac{100}{L}x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x \\
\frac{100}{L}x &= 100 - \frac{100}{L}x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x \\
\frac{200}{L}x - 100 &= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x
\end{aligned}$$

Take the inner product of the LHS and RHS w.r.t. $\sin\frac{n\pi}{L}x$ over $[0,L]$, we get

$$\begin{aligned}
\int_0^L \left(\frac{200}{L}x - 100\right)\sin\frac{n\pi}{L}x\,dx &= \int_0^L \left(\sum_{m=1}^{\infty} b_m \sin\frac{m\pi}{L}x\right)\sin\frac{n\pi}{L}x\;dx \\
100\int_0^L \left(\frac{2x}{L} - 1\right)\sin\frac{n\pi}{L}x\,dx &= \sum_{m=1}^{\infty} b_m \int_0^L \sin\frac{m\pi}{L}x\,\sin\frac{n\pi}{L}x\,dx \\
100\;\frac{-Ln\pi(1+\cos(n\pi)) + 2L\sin(n\pi)}{n^2\pi^2} &= b_n \int_0^L \sin^2\frac{m\pi}{L}x\,dx \\
100L\frac{-n\pi(1+\cos(n\pi)) + 2\sin(n\pi)}{n^2\pi^2} &= b_n\;\frac{L}{2} \\
b_n &= 200\left(\frac{-n\pi(1+\cos(n\pi)) + 2\sin(n\pi)}{n^2\pi^2}\right)
\end{aligned}$$

so looking at few values of $n = 1,2,3,4,\ldots$. Hence $b_n = 200\left(\frac{-n\pi(1+\cos(n\pi))+2\sin(n\pi)}{n^2\pi^2}\right)$

$$
\begin{aligned}
b_n &= 200\left(\frac{-\pi(1-1)+0}{\pi^2}\right),\ 200\left(\frac{-2\pi(1+1)+0}{2^2\pi^2}\right), 200\left(\frac{-3\pi(1-1)+0}{3^2\pi^2}\right),\\
&\quad 200\left(\frac{-4\pi(1+1)+0}{4^2\pi^2}\right), 200\left(\frac{-5\pi(1-1)+0}{5^2\pi^2}\right)\\
b_n &= 200\,(0),\ 200\left(\frac{-4\pi}{4\pi^2}\right), 200\,(0), 200\left(\frac{-8\pi}{16\pi^2}\right), 200\,(0)\ldots,\\
b_n &= 0, -400\,\frac{1}{2\pi}, 0, -400\frac{1}{4\pi}, 0,\\
b_n &= -400\,\frac{1}{n\pi}
\end{aligned}
$$

Hence

$$
b_n = -400\frac{1}{n\pi} \qquad n = 2, 4, 6, \ldots
$$

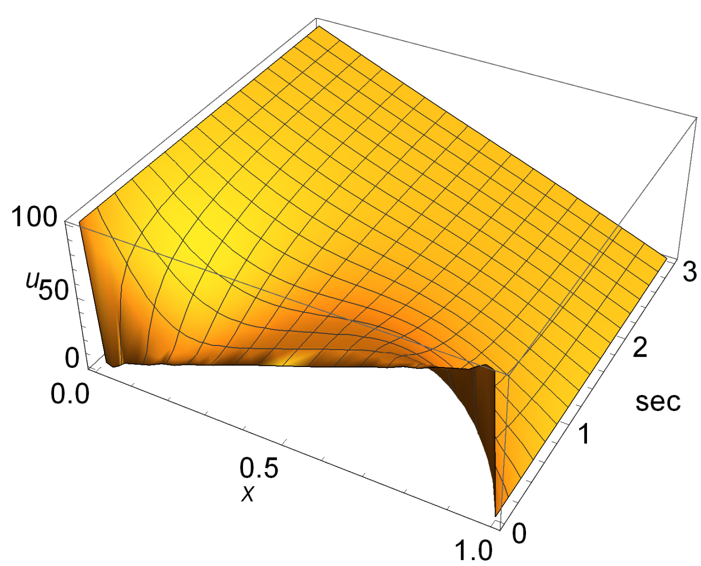
From equation (2) above we had

$$
u(x,t) = 100 - \frac{100}{L}x + \sum_{n=1}^{\infty} b_n\, e^{-\left(\alpha\frac{n\pi}{L}\right)^2 t} \sin\frac{n\pi}{L}x
$$

Hence

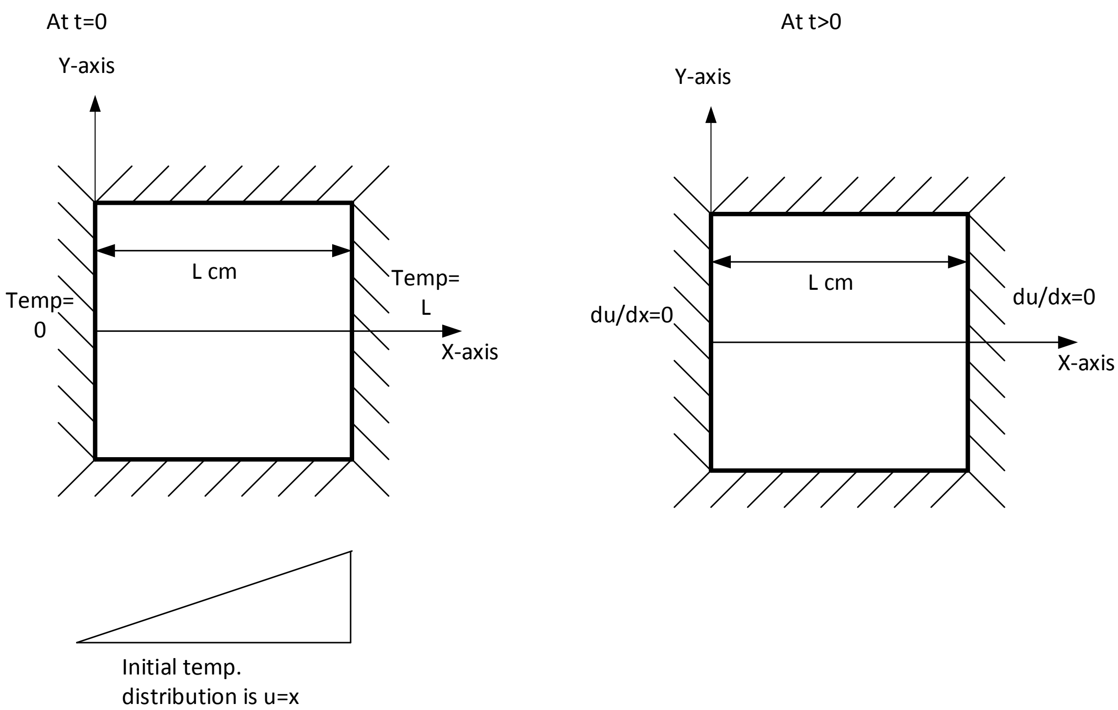
$$
\begin{aligned}
u(x,t) &= 100 - \frac{100}{L}x + \sum_{n \text{ even}}^{\infty} -400\,\frac{1}{n\pi}\ e^{-\left(\alpha\frac{n\pi}{L}\right)^2 t} \sin\frac{n\pi}{L}x\\
u(x,t) &= 100 - \frac{100}{L}x - \frac{400}{\pi}\sum_{n \text{ even}}^{\infty} \frac{1}{n}\, e^{-\left(\alpha\frac{n\pi}{L}\right)^2 t} \sin\frac{n\pi}{L}x
\end{aligned}
$$

```
L0 = 1; a = .2;
T0[x_, t_, m_] := 100 - 100/L0 x - 400/Pi  Sum[1/n Exp[-(a n Pi /L0)^2 t]
     Sin[n Pi/L0 x], {n, 2, m, 2}%
];
p = Plot3D[T0[x, t, 200], {x, 0, L0}, {t, 0, 3},
PlotRange -> All, AxesLabel -> {x, "sec", u}, BaseStyle -> 15]
```



7 Chapter 13, problem 3.7. Mary Boas book. second edition

A bar of length L with insulated sides has its ends also insulated from time $t = 0$. Initially the temp. is $u = x$, where is x is the distance from one end. Determine the temp. distribution inside the bar at time t.



<u>Solution</u>

In this problem, since all 4 sides are insulated, there will be no heat loss. Hence given the initial amount of heat inside the bar, we should obtain a solution that keeps this amount of heat the same. The solution should give a heat distribution at t large, such that it will be equally distributed over the length of the bar.

Since the two end sides are insulated, this is a Neumann type problem, so at $t > 0$ we will use $\frac{\partial u}{\partial x} = 0$ at both ends $(x = 0, x = L)$

This is a heat distribution problem governed by the diffusion or heat equation

$$\nabla^2 u(x,t) = \frac{1}{\alpha^2}\frac{\partial u(x,t)}{\partial t}$$

Similar to problem 3.2, we get the following two candidate solutions to the above PDE

$$u(x,t) = \begin{cases} e^{-\alpha^2 k^2 t}\sin kx \\ e^{-\alpha^2 k^2 t}\cos kx \end{cases}$$

Since these are solutions for $t > 0$ we need to find the conditions that $\frac{\partial u}{\partial x} = 0$ at $x = 0$ and $\frac{\partial u}{\partial x} = 0$ at $x = L$. The above conditions tells us to discard the $\sin kx$ solution because at $x = 0$ $\frac{\partial u}{\partial x} = e^{-\alpha^2 k^2 t}\cos kx = e^{-\alpha^2 k^2 t} \neq 0$. So the second solution we are left with is

$$e^{-\alpha^2 k^2 t}\cos(kx)$$

Which satisfies $\frac{\partial u}{\partial x} = 0$ at $x = 0$. Now at $x = L$, we also want $\frac{\partial u}{\partial x} = 0$, hence $\frac{\partial}{\partial x} e^{-\alpha^2 k^2 t}\cos kx = -k\, e^{-\alpha^2 k^2 t}\sin kL = 0$. For this to be true we need $k = 0$ or $\sin kL = 0$, i.e. $kL = n\pi$ or $k = \frac{n\pi}{L}$

So there are two solutions to look at, one for $k = 0$ and one for $k = \frac{n\pi}{L}$. Looking at the $k = \frac{n\pi}{L}$ solution first, we start with the solution

$$u(x,t) = e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t} \cos\frac{n\pi}{L}x$$

Now consider the initial conditions at $t = 0$. At $t = 0$, when $x = L/2$, $u = L/2$. Since we are told that $u_0 = x$. So from the above, we get

$$u\left(\frac{L}{2}, 0\right) = \cos\frac{n\pi}{L}\frac{L}{2} = \cos\frac{n\pi}{2}$$

Which is zero for integer n. Hence to force the outcome to be $\frac{L}{2}$, we need to add this term to the solution above. The solution now looks like

$$u(x,t) = \frac{L}{2} + e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t} \cos\frac{n\pi}{L}x \tag{1}$$

Now Since a scaled sum of these solutions is a solution, then the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \left(e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t} \cos\frac{n\pi}{L}x\right) \tag{2}$$

Now we need to find the b_n. For this we use the initial conditions, i.e. for $t = 0$. We are told that at $t = 0$, $u_0 = x$. From equation (1) above, we write, at time $t = 0$

$$\begin{aligned} u(x,0) &= x \\ &= \sum_{n=1}^{\infty} b_n \cos\frac{n\pi}{L}x \end{aligned}$$

Taking the inner product of the LHS and RHS w.r.t. $\cos\frac{n\pi}{L}x$ over $[0, L]$ gives

$$\begin{aligned} \int_0^L x \cos\frac{n\pi}{L}x &= \int_0^L \left(\sum_{m=1}^{\infty} b_m \cos\frac{m\pi}{L}x\right) \cos\frac{n\pi}{L}x\, dx \\ \int_0^L x \cos\frac{n\pi}{L}x &= \sum_{m=1}^{\infty} b_m \int_0^L \cos\frac{m\pi}{L}x \cos\frac{n\pi}{L}x\, dx \\ \frac{L^2(-1+\cos(n\pi)+n\pi\sin(n\pi))}{n^2\pi^2} &= b_n \int_0^L \cos^2\frac{m\pi}{L}x\, dx \\ \frac{L^2(-1+\cos(n\pi)+n\pi\sin(n\pi))}{n^2\pi^2} &= b_n \frac{L}{2} \\ b_n &= \frac{2L(-1+\cos(n\pi)+n\pi\sin(n\pi))}{n^2\pi^2} \end{aligned}$$

Looking at few values of $n = 1, 2, 3, 4 \ldots$

$$b_n = \frac{2L(-1+\cos(n\pi)+n\pi\sin(n\pi))}{n^2\pi^2}$$

Hence

$$
\begin{aligned}
b_n &= \frac{2L(-1+\cos(\pi)+\pi\sin(\pi))}{\pi^2}, \frac{2L(-1+\cos(2\pi)+2\pi\sin(2\pi))}{2^2\pi^2},\\
&\quad \frac{2L(-1+\cos(3\pi)+3\pi\sin(3\pi))}{3^2\pi^2}, \frac{2L(-1+\cos(4\pi)+4\pi\sin(4\pi))}{4^2\pi^2},\dots\\
b_n &= \frac{2L(-1-1)}{\pi^2}, \frac{2L(-1+1)}{2^2\pi^2}, \frac{2L(-1-1)}{3^2\pi^2}, \frac{2L(-1+1)}{4^2\pi^2},\dots\\
b_n &= \frac{2L(-2)}{\pi^2}, \frac{2L(0)}{2^2\pi^2}, \frac{2L(-2)}{3^2\pi^2}, \frac{2L(0)}{4^2\pi^2},\dots\\
b_n &= \frac{-4L}{\pi^2}, 0, \frac{-4L}{3^2\pi^2}, 0,\dots\\
b_n &= \frac{-4L}{n^2\pi^2}
\end{aligned}
$$

Hence

$$
b_n = \frac{-4L}{n^2\pi^2} \qquad n = 1,3,5,\dots
$$

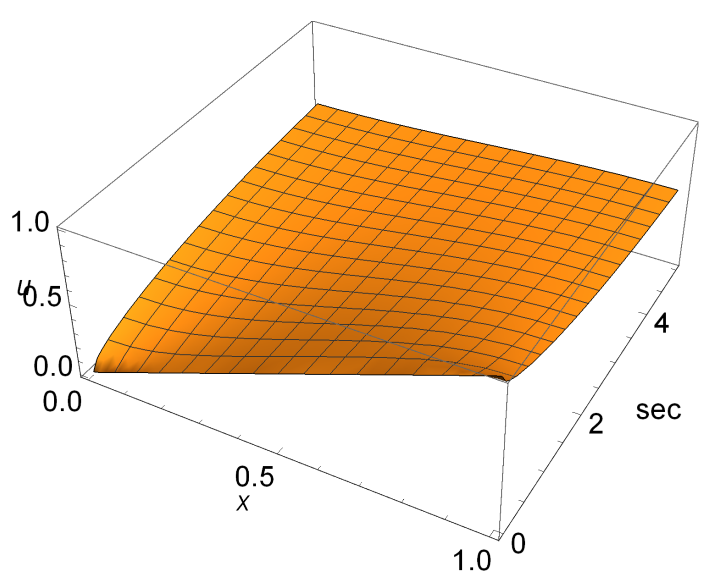
From equation (1) above we had

$$
\begin{aligned}
u(x,t) &= \frac{L}{2} + e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t}\cos\frac{n\pi}{L}x\\
&= \frac{L}{2} + \sum_{\text{odd}}^{\infty} b_n\left(e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t}\cos\frac{n\pi}{L}x\right)\\
u(x,t) &= \frac{L}{2} + \sum_{\text{odd}}^{\infty}\frac{-4L}{n^2\pi^2}\, e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t}\cos\frac{n\pi}{L}x
\end{aligned}
$$

Hence

$$
u(x,t) = \frac{L}{2} - \frac{4L}{\pi^2}\sum_{\text{odd}}^{\infty}\frac{1}{n^2}\, e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t}\cos\frac{n\pi}{L}x
$$

```
L0 = 1; a = .2;
T0[x_, t_, m_] := L0/2 - 4 L0/Pi^2  Sum[1/n^2 Exp[-(a n Pi /L0)^2 t] Cos[
    n Pi/L0 x], {n, 1, m, 2}%
];
p = Plot3D[T0[x, t, 50], {x, 0, L0}, {t, 0, 5}%
, PlotRange -> All,AxesLabel -> {x, "sec", u}, BaseStyle -> 15]
```



Now we need to consider the $k = 0$ solution we had at the beginning. Starting with $u(x,t) = e^{-\alpha^2 k^2 t} \cos kx$, for $k = 0$, we have $u(x,t) = 1$, and as before we want to look for a general solution as $u_g(x,t) = \sum_{n=1}^{\infty} b_n\, u(x,t)$, which is now will be

$$u_g(x,t) = \sum_{n=1}^{\infty} b_n$$

To find b_n, as before we use the conditions at $t = 0$, which is $u(x,0) = x$. Therefore

$$u(x,0) = x = \sum_{n=1}^{\infty} b_n$$

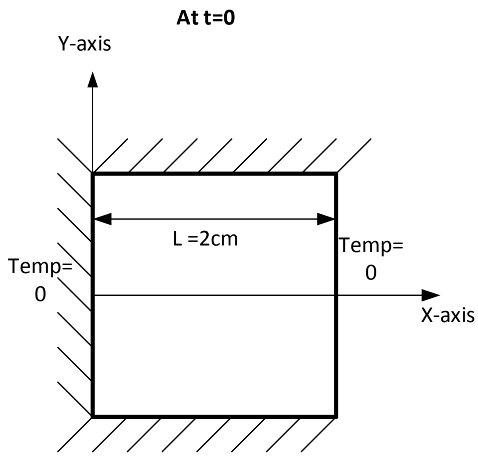
Therefore $\sum_{n=1}^{\infty} b_n = x$. The general solution in this case is given by

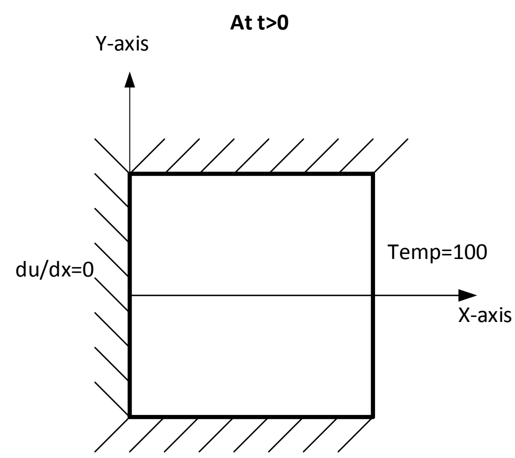
$$\begin{aligned} u_g(x,t) &= \sum_{n=1}^{\infty} b_n \\ u_g(x,t) &= x \end{aligned}$$

which is what we are required to show. What this means is that for $k = 0$, the heat distribution does not change. So this is the same as the time-independent initial conditions.

8 Chapter 13, problem 3.9. Mary Boas book. second edition

A bar of length $L = 2$ with insulated side at $x = 0$ only, at $t = 0$ held at zero temperature. at $t > 0$ the right side is held at $T = 100$ degrees. Determine the time dependent temperature distribution inside the bar





Solution

In this problem, since left side is insulated, this is a Neumann condition at the $x = 0$ side. So at $t > 0$ we will use $\frac{\partial u}{\partial x} = 0$ at $x = 0$. This is a heat distribution problem governed by the diffusion or heat equation

$$\nabla^2 u(x,t) = \frac{1}{\alpha^2}\frac{\partial u(x,t)}{\partial t}$$

Similar to problem 3.2, we get the following 2 candidate solutions to the above PDE

$$u(x,t) = \begin{cases} e^{-\alpha^2 k^2 t}\sin kx \\ e^{-\alpha^2 k^2 t}\cos kx \end{cases}$$

Since these are solutions for $t > 0$. We need to find the conditions that $\frac{\partial u}{\partial x} = 0$ at $x = 0$ and $u = 100$ at $x = L$. Since we want $\frac{\partial u}{\partial x} = 0$ at $x = 0$, then we can not use the $\sin kx$ solution. We are left with the solution $e^{-\alpha^2 k^2 t}\cos(kx)$ which satisfies $\frac{\partial u}{\partial x} = 0$ at $x = 0$. Now at $x = L$, we want $u = 100$, hence $e^{-\alpha^2 k^2 t}\cos(kL) = 100$. The way this is presented will not allow exact expression for k so we have to write $u(L,t) = 100 + e^{-\alpha^2 k^2 t}\cos(kL)$, and now we are able to set only the $e^{-\alpha^2 k^2 t}\cos(kL)$ term to zero, which means we need to have $\cos(kL) = 0$ or $kL = \frac{2n-1}{2}\pi$ or $k = \frac{2n-1}{2}\frac{\pi}{L}$. Hence we start with the solution

$$\begin{aligned} u(x,t) &= 100 + e^{-\alpha^2\left(\frac{2n-1}{2}\frac{\pi}{L}\right)^2 t}\cos\left(\frac{2n-1}{2}\frac{\pi}{L}x\right) \\ &= 100 + e^{-\left(\alpha\frac{2n-1}{2}\frac{\pi}{L}\right)^2 t}\cos\left(\frac{2n-1}{2}\frac{\pi x}{L}\right) \end{aligned} \tag{1}$$

Since a scaled sum of these solutions is a solution, then the general solution is

$$u(x,t) = 100 + \sum_{n=1}^{\infty} b_n\, e^{-\left(\alpha\frac{2n-1}{2}\frac{\pi}{L}\right)^2 t}\cos\left(\frac{2n-1}{2}\frac{\pi x}{L}\right) \tag{2}$$

Now we need to find the b_n. For this we use the initial conditions. We are told that at $t = 0$, $u_0 = 0$. So now from equation (1) above

$$\begin{aligned} u(x,0) &= 0 \\ &= 100 + \sum_{n=1}^{\infty} b_n \cos\left(\frac{2n-1}{2}\frac{\pi x}{L}\right) \\ -100 &= \sum_{n=1}^{\infty} b_n \cos\left(\frac{2n-1}{2}\frac{\pi x}{L}\right) \end{aligned}$$

Taking the inner product of the LHS and RHS w.r.t. $\cos\left(\frac{2n-1}{2}\frac{\pi x}{L}\right)$ over $[0, L]$ gives

$$\begin{aligned} -100 \int_0^L \cos\left(\frac{2n-1}{2}\frac{\pi x}{L}\right) dx &= \int_0^L \left(\sum_{m=1}^{\infty} b_m \cos\left(\frac{2m-1}{2}\frac{\pi x}{L}\right)\right) \cos\left(\frac{2n-1}{2}\frac{\pi x}{L}\right) \\ -100 \left[\frac{1}{\frac{2n-1}{2}\frac{\pi}{L}} \sin\left(\frac{2n-1}{2}\frac{\pi x}{L}\right)\right]_0^L &= \sum_{m=1}^{\infty} b_m \int_0^L \cos\left(\frac{2m-1}{2}\frac{\pi x}{L}\right) \cos\left(\frac{2n-1}{2}\frac{\pi x}{L}\right) dx \\ -100 \frac{2L}{(2n-1)\pi}\left[\sin\left(\frac{2n-1}{2}\frac{\pi x}{L}\right)\right]_0^L &= b_n \int_0^L \cos^2\left(\frac{2n-1}{2}\frac{\pi x}{L}\right) dx \\ \frac{-200L}{(2n-1)\pi}\left[\sin\left(\frac{2n-1}{2}\frac{\pi L}{L}\right) - \sin(0)\right] &= b_n \frac{L}{2} \\ \frac{-200L}{(2n-1)\pi}\left[\sin\left(\frac{2n-1}{2}\pi\right)\right] &= b_n \frac{L}{2} \\ b_n &= \frac{-400}{(2n-1)\pi}\left[\sin\left(\frac{2n-1}{2}\pi\right)\right] \end{aligned}$$

Looking at few values of n, $b_n = \frac{-400}{(2n-1)\pi}\left[\sin\left(\frac{2n-1}{2}\pi\right)\right]$, hence

$$\begin{aligned} b_n &= \frac{-400}{(2-1)\pi}\left[\sin\left(\frac{2-1}{2}\pi\right)\right], \frac{-400}{(4-1)\pi}\left[\sin\left(\frac{4-1}{2}\pi\right)\right], \\ &\quad \frac{-400}{(6-1)\pi}\left[\sin\left(\frac{6-1}{2}\pi\right)\right], \frac{-400}{(8-1)\pi}\left[\sin\left(\frac{8-1}{2}\pi\right)\right] \ldots \\ b_n &= \frac{-400}{\pi}\left[\sin\left(\frac{1}{2}\pi\right)\right], \frac{-400}{3\pi}\left[\sin\left(\frac{3}{2}\pi\right)\right], \frac{-400}{5\pi}\left[\sin\left(\frac{5}{2}\pi\right)\right], \ldots \\ b_n &= \frac{-400}{\pi}, \frac{+400}{3\pi}, \frac{-400}{5\pi}, \ldots \\ b_n &= -\frac{(-1)^n}{2n+1}\frac{400}{\pi} \qquad n = 0, 1, 2, 3, \ldots \end{aligned}$$

Therefore

$$b_n = -\frac{(-1)^n}{2n+1}\frac{400}{\pi} \qquad n = 0, 1, 2, 3, \ldots$$

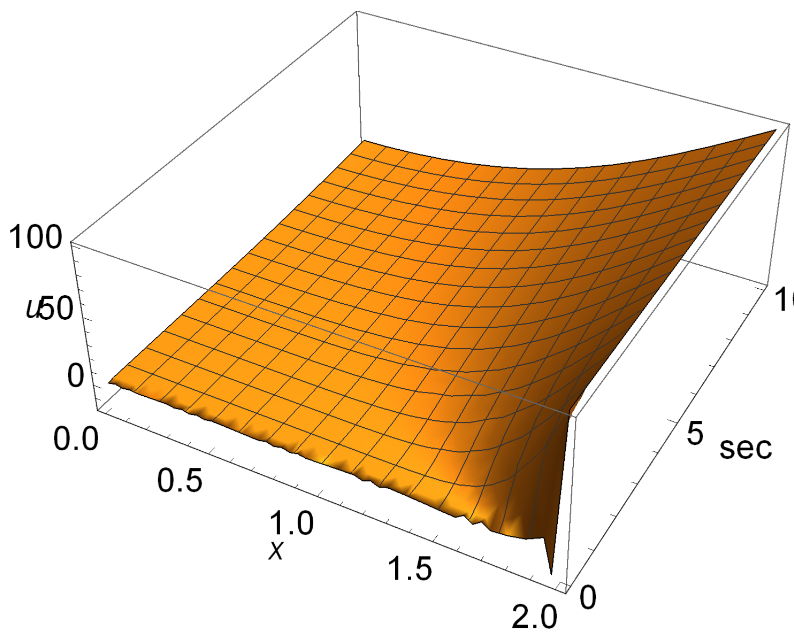
From equation (2) above we had

$$u(x,t) = 100 + \sum_{n=1}^{\infty} b_n\, e^{-\left(\alpha \frac{2n-1}{2}\frac{\pi}{L}\right)^2 t} \cos\left(\frac{2n-1}{2}\frac{\pi x}{L}\right)$$

Substituting the value for b_n and adjusting the summation index to start from $n = 0$ since this is where b_n is defined to start from, and so we need to replace $\frac{2n-1}{2}$ by $\frac{2n+1}{2}$ in the rest of the above terms. Hence

$$u(x,t) = 100 + \sum_{n=0}^{\infty} -\frac{(-1)^n}{2n+1}\frac{400}{\pi}\, e^{-\left(\alpha\frac{2n+1}{2}\frac{\pi}{L}\right)^2 t} \cos\left(\frac{2n+1}{2}\frac{\pi x}{L}\right)$$
$$= 100 - \frac{400}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\, e^{-\left(\alpha\frac{2n+1}{2}\frac{\pi}{L}\right)^2 t} \cos\left(\frac{2n+1}{2}\frac{\pi x}{L}\right)$$

```
L0 = 2; a = .2;
T0[x_, t_, m_] := 100 - 400/Pi  Sum[(-1)^n/(2 n + 1)
Exp[-(a (2 n + 1)/2 Pi/L0)^2 t] Cos[(2 n + 1)/2 Pi/L0 x], {n, 0, m, 1}];
p = Plot3D[T0[x, t, 50], {x, 0, L0}, {t, 0, 10}, PlotRange -> All,
AxesLabel -> {x, "sec", u}, BaseStyle -> 15]
```



9 Problem chapter 13, 3.10. Mary Boas book. Second edition

Separate the wave equation $\nabla^2 u = \frac{1}{v}\frac{\partial^2 u}{\partial t^2}$ into space and time equation and show that the space equation is the Helmholtz equation.

Solution

Assume that the solution is of this form $u\left(x,y,z,t\right) = F\left(x,y,z\right)T(t)$

That is, the solution of the PDE is the product of 2 functions, one that depends only on the spatial displacements and a function that depends only on time.

Substituting back in the PDE which when written in the long form is:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{v}\frac{\partial^2 u}{\partial t^2}$$

Hence,

$$\begin{aligned}
\frac{\partial u}{\partial x} &= T(t)\frac{\partial F}{\partial x} \\
\frac{\partial^2 u}{\partial x^2} &= T(t)\frac{\partial^2 F}{\partial x^2}
\end{aligned}$$

similarly we get

$$\begin{aligned}
\frac{\partial^2 u}{\partial y^2} &= T(t)\frac{\partial^2 F}{\partial y^2} \\
\frac{\partial^2 u}{\partial z^2} &= T(t)\frac{\partial^2 F}{\partial z^2} \\
\frac{\partial^2 u}{\partial t^2} &= F\left(x,y,z\right)\frac{d^2 T}{dt^2}
\end{aligned}$$

Now $\nabla^2 u = \frac{1}{v}\frac{\partial^2 u}{\partial t^2}$ can be written as

$$\begin{aligned}
T(t)\frac{\partial^2 F}{\partial x^2} + T(t)\frac{\partial^2 F}{\partial y^2} + T(t)\frac{\partial^2 F}{\partial z^2} &= \frac{1}{v}F\left(x,y,z\right)\frac{d^2 T}{dt^2} \\
T\,\nabla^2 F &= \frac{1}{v}F\,\frac{d^2 T}{dt^2}
\end{aligned}$$

dividing the equation by $T\,F$, we get

$$\frac{1}{F}\,\nabla^2 F = \frac{1}{v}\frac{1}{T}\,\frac{d^2 T}{dt^2}$$

Since the LHS is a function of space only, and RHS is a function of time only, and they equal to each others, then they must be equal to a constant, say $-k^2$

Hence we get

$$\begin{aligned}
\frac{1}{F}\,\nabla^2 F &= -k^2 \\
\frac{1}{v}\frac{1}{T}\,\frac{d^2 T}{dt^2} &= -k^2
\end{aligned}$$

Looking at the space equation only:

$$\begin{aligned}
\frac{1}{F(x,y,z)}\,\nabla^2 F(x,y,z) &= -k^2 \\
\nabla^2 F(x,y,z) &= -F(x,y,z)\,k^2 \\
\nabla^2 F(x,y,z) + F(x,y,z)\,k^2 &= 0
\end{aligned}$$

So the space equation is the Helmholtz equation.