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HW # 5

Math 121 B

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UCB extension

ch 12

$\boxed{11.2}$

Solve using method of Frobenius (generalized power series).

$$x^2 y'' + xy' - 9y = 0.$$

let $y = a_0 x^s + a_1 x^{s+1} + a_2 x^{s+2} + a_3 x^{s+3} + \cdots$

$$y' = s a_0 x^{s-1} + a_1(s+1)x^s + a_2(s+2)x^{s+1} + \cdots$$

$$y'' = s(s-1)a_0 x^{s-2} + a_1(s+1)(s)x^{s-1} + a_2(s+2)(s+1)x^s + \cdots$$

$\rightarrow$ $x^2y'' = s(s-1)a_0 x^s + a_1(s+1)(s)x^{s+1} + a_2(s+2)(s+1)x^{s+2} + \cdots$

$\rightarrow$ $xy' = s a_0 x^s + a_1(s+1)x^{s+1} + a_2(s+2)x^{s+2} + \cdots$

$\rightarrow$ $9y = 9a_0 x^s + 9a_1 x^{s+1} + 9a_2 x^{s+2} + \cdots$

now arrange Table:

	x^s	x^{s+1}	x^{s+2}	x^{s+n}
x^2y''	$s(s-1)a_0$	$a_1(s+1)(s)$	$a_2(s+2)(s+1)$	$a_n(s+n)(s+n-1)$
xy'	sa_0	$a_1(s+1)$	$a_2(s+2)$	$a_n(s+n)$
$-9y$	$-9a_0$	$-9a_1$	$-9a_2$	$-9a_n$

total coeff. of each power $= 0$.

from coeff for x^s: $s(s-1)a_0 + s a_0 - 9a_0 = 0$

$$(s(s-1) + s - 9)\, a_0 = 0$$

since $a_0 \neq 0$ by hypothesis,

$$s(s-1) + s - 9 = 0 \quad \text{or} \quad s^2 - s + s - 9 = 0$$

or $s^2 - 9 = 0 \Rightarrow \boxed{s = \pm 3}$

$\rightarrow$

now looking at each column shows that each

$a_n = 0$ for $n > 0$. since no recursive formula.

(and each column adds to zero. so we get $K a_n = 0$ for some $K(n)$.

$\Rightarrow a_n = 0$

so solutions are only

$$\boxed{y = a_0 x^s}$$

i.e $y_1 = a_0 x^3$ or $y_2 = a_0 x^{-3}$

so general solution is $\boxed{y = a_0 x^3 + a_0 x^{-3}}$

or can be written as

$$\boxed{y = A x^3 + B x^{-3}}$$

A, B to be found from initial conditions.

Ch 12

$\boxed{11.6}$ use method of generalized power series to solve

$$3xy'' + (3x+1)y' + y = 0$$

write as $3xy'' + 3xy' + y' + y = 0$.

let $y = a_0 x^s + a_1 x^{s+1} + a_2 x^{s+2} + a_3 x^{s+3} + \cdots$

$y' = s a_0 x^{s-1} + a_1 (s+1) x^s + a_2 (s+2) x^{s+1} + a_3 (s+3) x^{s+2} + \cdots$

$3xy' = 3 s a_0 x^s + 3 a_1 (s+1) x^{s+1} + 3 a_2 (s+2) x^{s} + \cdots$

$y'' = s(s-1) a_0 x^{s-2} + a_1 (s+1) s x^{s-1} + a_2 (s+2)(s+1) x^s + \cdots$

$3xy'' = 3s(s-1) a_0 x^{s-1} + 3 a_1 (s+1) s x^s + 3 a_2 (s+2)(s+1) x^{s+1} + \cdots$

Now set up the Table

	x^{s-1}	x^s	x^{s+1}	x^{s+n}
$3xy''$	$3s(s-1)a_0$	$3s(s+1)a_1$	$3(s+1)(s+2)a_2$	$3(s+n)(s+n+1)a_{n+1}$
$3xy'$	0	$3s a_0$	$3(s+1)a_1$	$3(s+n)a_n$
y'	$s a_0$	$(s+1)a_1$	$(s+2)a_2$	$(s+n+1)a_{n+1}$
y	0	a_0	a_1	a_n

First Solve the indicial equation.

From first column, we set $3s(s-1)a_0 + s a_0 = 0$

or $a_0 \left(s + 3s(s-1) \right) = 0$

but $a_0 \neq 0$ by hypothesis. $\Rightarrow$ $s + 3s^2 - 3s = 0$

$3s^2 - 2s = 0 \longrightarrow$

$$S = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-(-2) \pm \sqrt{4-4(0)}}{2(3)}$$

$$= \frac{2 \pm 2}{6} = \frac{4}{6} \text{ or } 0$$

i.e. $\boxed{S_1 = \frac{2}{3} \quad , \quad S_2 = 0}$

for $S_1 = \frac{2}{3}$

Recursive equation is

$$3(s+n)(s+n+1)\, a_{n+1} + 3(s+n)\, a_n + (s+n+1)\, a_{n+1} + a_n = 0$$

$$a_{n+1}\left[3(s+n)(s+n+1) + (s+n+1)\right] = -a_n\left[1 + 3(s+n)\right]$$

$$a_{n+1}\left[3\left(\tfrac{2}{3}+n\right)\left(\tfrac{2}{3}+n+1\right) + \left(\tfrac{2}{3}+n+1\right)\right] = -a_n\left[1+3\left(\tfrac{2}{3}+n\right)\right]$$

$$a_{n+1}\left[(2+3n)\left(\tfrac{5}{3}+n\right) + \left(\tfrac{5}{3}+n\right)\right] = -a_n\left[1+(2+3n)\right]$$

$$a_{n+1}\left[\tfrac{10}{3}+2n+5n+3n^2+\tfrac{5}{3}+n\right] = -a_n\left[3+3n\right]$$

$$a_{n+1}\left[\tfrac{15}{3}+8n+3n^2\right] = -a_n\left[3+3n\right]$$

$$a_{n+1}\left(5+8n+3n^2\right) = -3a_n(1+n)$$

$$\boxed{a_{n+1} = -\frac{3a_n(1+n)}{5+8n+3n^2}}$$

$\longrightarrow$

for $n=0$

$$a_1 = \frac{-3a_0(1)}{5} = \boxed{-\frac{3}{5}a_0}$$

for $n=1$

$$a_2 = \frac{-3a_1(2)}{5+8+3} = \frac{-6}{16}\left(-\frac{3}{5}a_0\right) = \frac{18}{80}a_0 = \boxed{\frac{9}{40}a_0}$$

for $n=2$

$$a_3 = \frac{-3a_2(3)}{5+8(2)+3(2^2)} = \frac{-9}{33}\left(\frac{9}{40}\right)a_0 = \frac{-162}{2640}a_0 = \frac{-81}{1320}a_0 = \boxed{\frac{-27}{440}}$$

so $y = a_0x^s + a_1x^{s+1} + a_2x^{s+2} + \cdots$

$$y = a_0x^{2/3} - \frac{3}{5}a_0x^{5/3} + \frac{9}{40}a_0x^{8/3} - \frac{27}{440}a_0x^{11/3}$$

$$= a_0\left(x^{2/3} - \frac{3}{5}x^{5/3} + \frac{9}{40}x^{8/3} - \frac{27}{440}x^{11/3}\right)$$

$$\boxed{y_1 = a_0x^{2/3}\left(1 - \frac{3}{5}x + \frac{9}{40}x^2 - \frac{27}{440}x^3 + \cdots\right)}$$

The above solution is for $s = 2/3$.

now I need to find second solution for $s=0$.

from Recursive equation

$$a_{n+1}\left(3(s+n)(s+n+1) + (s+n+1)\right) = -a_n\left(1+3(s+n)\right)$$

$$s=0 \Rightarrow a_{n+1}\left(3(n)(n+1) + (n+1)\right) = -a_n(1+3n)$$

$$a_{n+1}\left(3(n^2+n) + (n+1)\right) = -a_n(1+3n)$$

$$a_{n+1}\left(3n^2+4n+1\right) = -a_n(1+3n)$$

$$\Rightarrow \boxed{a_{n+1} = -a_n\frac{(1+3n)}{3n^2+4n+1}} \rightarrow$$

$n=0$

$$a_1 = \frac{-a_0(1)}{1} = \boxed{-a_0}$$

$n=1$

$$a_2 = -a_1\left(\frac{1+3}{3+4+1}\right) = -\frac{4}{8}a_1 = -\frac{1}{2}(-a_0) = \boxed{\frac{a_0}{2}}$$

$n=2$

$$a_3 = -a_2\left(\frac{1+6}{3\times 2^2 + 4\times 2 + 1}\right) = \frac{7}{12+8+1}(-a_2)$$

$$= -\frac{7}{21}\left(\frac{a_0}{2}\right) = -\frac{7}{42}a_0 = \boxed{-\frac{1}{6}a_0}$$

$n=3$

$$a_4 = -a_3\left(\frac{1+9}{3\times 3^2 + 4\times 3 + 1}\right) = \frac{-10}{27+12+1}\left(-\frac{7}{42}a_0\right)$$

$$= \frac{70}{1680}a_0 = \frac{7}{168}a_0 = \boxed{\frac{1}{24}a_0}$$

so $y_2 = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$

$$= a_0 - a_0 x + \frac{a_0}{2}x^2 - \frac{1}{6}a_0 x^3 + \frac{1}{24}a_0 x^4 + \cdots$$

$$\boxed{y_2 = a_0\left(1 - x + \frac{x^2}{2} - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \cdots\right)}$$

So general solution is $y = Ay_1 + By_2$

$$\boxed{\begin{aligned} y = {} & Ax^{2/3}\left(1 - \frac{3}{5}x + \frac{9}{40}x^2 - \frac{27}{440}x^3 + \cdots\right) \\ & + B\left(1 - x + \frac{x^2}{2} - \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots\right) \end{aligned}}$$

where A, B are constants found from initial conditions.

ch 12

$\boxed{12.5}$

show that $\frac{d}{dx}\left(x J_1(x)\right) = x J_0(x)$.

use power series expansion $J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+p}$

for $P=1$, $J_1(x) = \sum \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+2)} \left(\frac{x}{2}\right)^{2n+1}$

so $\frac{d}{dx}\left(x J_1(x)\right) = \frac{d}{dx}\left(x \sum \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+2)} \left(\frac{x}{2}\right)^{2n+1}\right)$

$$= \frac{d}{dx}\left(\sum \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+2)} \; x \, (x)^{2n+1} \left(\frac{1}{2}\right)^{2n+1}\right)$$

$$= \frac{d}{dx}\left(\sum \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+2)} \; x^{(2n+2)} \left(\frac{1}{2}\right)^{2n+1}\right)$$

$$= \sum \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+2)} (2n+2)\, x^{(2n+1)} \left(\frac{1}{2}\right)^{2n+1}$$

$$= \sum \frac{(-1)^n (2n+2)}{\Gamma(n+1)\Gamma(n+2)} \left(\frac{x}{2}\right)^{2n+1}$$

but $\Gamma(n+2) = (n+1)\Gamma(n+1)$ from definition.

so

$$= \sum \frac{(-1)^n \, 2\cancel{(n+1)}}{\Gamma(n+1)\Gamma(n+1)\cancel{(n+1)}} \left(\frac{x}{2}\right)^{2n+1} = \sum \frac{(-1)^n \cancel{2}}{\Gamma(n+1)\Gamma(n+1)} \left(\frac{x}{2}\right)^{2n} \left(\frac{x}{\cancel{2}}\right)$$

$$= \sum \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1)} \left(\frac{x}{2}\right)^{2n} x \qquad \text{——— (1)}$$

but $x J_0(x)$ is just the above, because

$x J_0(x) = x \sum \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1)} \left(\frac{x}{2}\right)^{2n}$ $\Rightarrow$ move x into the sum gives (1).

QED

hence $\frac{d}{dx}\left(x J_1(x)\right) = x J_0(x)$

ch 12

$\boxed{12.6}$ show that $J_0(x) - J_2(x) = 2\frac{d}{dx}J_1(x)$.

here I will write few terms of $J_0 - J_2$ and compare to few terms of $2\frac{d}{dx}J_1$.

$$J_p = \sum \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p+1)}\left(\frac{x}{2}\right)^{2n+p}$$

$$J_0 = \sum \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1)}\left(\frac{x}{2}\right)^{2n}$$

$$J_1 = \sum \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+2)}\left(\frac{x}{2}\right)^{2n+1}$$

$$J_2 = \sum \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+3)}\left(\frac{x}{2}\right)^{2n+2}$$

$$J_0 - J_2 = \left(\frac{1}{\Gamma(1)\Gamma(1)} - \frac{1}{\Gamma(2)\Gamma(2)}\left(\frac{x}{2}\right)^2 + \frac{1}{\Gamma(3)\Gamma(3)}\left(\frac{x}{2}\right)^4 - \cdots\right) - \left(\frac{1}{\Gamma(1)\Gamma(3)}\left(\frac{x}{2}\right)^2 - \frac{1}{\Gamma(2)\Gamma(4)}\left(\frac{x}{2}\right)^4 + \cdots\right)$$

$$= \frac{1}{\Gamma(1)\Gamma(1)} + \left(\frac{x}{2}\right)^2\left[-\frac{1}{\Gamma(2)\Gamma(2)} - \frac{1}{\Gamma(1)\Gamma(3)}\right] + \left(\frac{x}{2}\right)^4\left[\frac{1}{\Gamma(3)\Gamma(3)} + \frac{1}{\Gamma(2)\Gamma(4)}\right] + \cdots$$

now $\Gamma(1) = \Gamma(2)$, $\Gamma(3) = 2\Gamma(2)$, $\Gamma(4) = 3\Gamma(3)$, so above becomes

$$J_0 - J_2 = \frac{1}{\Gamma(1)\Gamma(1)} + \left(\frac{x}{2}\right)^2\left[-\frac{1}{\Gamma(2)\Gamma(2)} - \frac{1}{2\Gamma(2)\Gamma(2)}\right] + \left(\frac{x}{2}\right)^4\left[\frac{1}{\Gamma(3)\Gamma(3)} + \frac{1}{\frac{1}{2}\Gamma(3)\,3\Gamma(3)}\right] + \cdots$$

$$= 1 - \left(\frac{x}{2}\right)^2\left[\frac{3}{2}\frac{1}{\Gamma(2)\Gamma(2)}\right] + \left(\frac{x}{2}\right)^4\left[\frac{1}{\Gamma(3)\Gamma(3)} + \frac{2}{3}\frac{1}{\Gamma(3)\Gamma(3)}\right]$$

$$= 1 - \left(\frac{x}{2}\right)^2\left[\frac{3}{2}\frac{1}{\Gamma(2)\Gamma(2)}\right] + \left(\frac{x}{2}\right)^4\left[\frac{5}{3}\frac{1}{\Gamma(3)\Gamma(3)}\right] + \cdots \quad \text{———} \; (1)$$

now look at $2\frac{d}{dx}J_1(x)$.

$$2\frac{d}{dx}J_1(x) = 2\frac{d}{dx}\left(\frac{1}{\Gamma(1)\Gamma(2)}\left(\frac{x}{2}\right) - \frac{1}{\Gamma(2)\Gamma(3)}\left(\frac{x}{2}\right)^3 + \frac{1}{\Gamma(3)\Gamma(4)}\left(\frac{x}{2}\right)^5 - \cdots\right)$$

$$= 2\frac{d}{dx}\left(\frac{1}{\Gamma(1)\Gamma(2)}\frac{1}{2}x - \frac{1}{\Gamma(2)\,2\Gamma(2)}\left(\frac{1}{2}\right)^3 x^3 + \frac{1}{\Gamma(3)\,3\Gamma(3)}\left(\frac{1}{2}\right)^5 x^5 - \cdots\right)$$

$$= 2\left(\frac{1}{2}\frac{1}{\Gamma(1)\Gamma(1)} - \frac{1}{2}\frac{1}{\Gamma(2)\Gamma(2)}\left(\frac{1}{2}\right)^3 3x^2 + \frac{1}{3}\frac{1}{\Gamma(3)\Gamma(3)}\left(\frac{1}{2}\right)^5 \times 5x^4 - \cdots\right)$$

$$= 1 - \frac{3}{\Gamma(2)\Gamma(2)}\left(\frac{1}{2}\right)^3 x^2 + \frac{1}{3}\frac{5}{\Gamma(3)\Gamma(3)}\left(\frac{1}{2}\right)^4 x^4 - \cdots$$

$$= 1 - \frac{3}{2}\frac{1}{\Gamma(2)\Gamma(2)}\left(\frac{x}{2}\right)^2 + \frac{5}{3}\frac{1}{\Gamma(3)\Gamma(3)}\left(\frac{x}{2}\right)^4 - \cdots \quad \text{———} \; (2)$$

looking at (1) and (2), show they are the same, i.e. coefficient of x are equal in both sequences. hence $\boxed{2\frac{d}{dx}J_1(x) = J_0 - J_2}$

ch 12

$\boxed{12.8}$

show that $\lim_{x \to 0} x^{-3/2} J_{3/2}(x) = \frac{1}{3}\sqrt{\frac{2}{\pi}}$.

$$J_p = \sum \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+p}$$

$$J_{3/2} = \sum \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+\frac{3}{2}+1)} \left(\frac{x}{2}\right)^{2n+3/2} = \left(\frac{x}{2}\right)^{3/2} \sum \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+\frac{3}{2}+1)} \left(\frac{x}{2}\right)^{2n}$$

expand:

$$J_{3/2} = \left(\frac{x}{2}\right)^{3/2} \left(\frac{1}{\Gamma(1)\Gamma(\frac{3}{2}+1)} - \text{terms with } x \text{ in numerator. I do not care about since in the limit will be all removed} \right)$$

$$\lim_{x \to 0} \cancel{\frac{1}{x^{3/2}}} \frac{\cancel{(x)^{3/2}}}{(2)^{3/2}} \left(\frac{1}{\Gamma(1)\Gamma(\frac{3}{2}+1)} - \cancel{\text{''}}^{=0} \right)$$

$$= \frac{1}{2^{3/2}} \left(\frac{1}{\Gamma(1)\Gamma(\frac{3}{2}+1)} \right) = \frac{1}{2^{3/2}} \left(\frac{1}{\Gamma(1)\frac{3}{2}\Gamma(\frac{3}{2})} \right)$$

here I used $\Gamma(p+1) = p\Gamma(p)$

$$= \frac{1}{2^{3/2}} \left(\frac{1}{\Gamma(1)\frac{3}{2} \times \frac{1}{2}\Gamma(\frac{1}{2})} \right) = \frac{1}{2^{3/2}} \left(\frac{1}{\frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi}} \right)$$

$$= \frac{1}{2^{3/2}} \frac{2^2}{3} \frac{1}{\sqrt{\pi}} = \frac{1}{3} \frac{1}{\sqrt{\pi}} 2^{2-\frac{3}{2}} = \frac{1}{3} \frac{1}{\sqrt{\pi}} 2^{1/2} \boxed{= \frac{1}{3}\sqrt{\frac{2}{\pi}}}$$

ch 12

$\boxed{12.9}$

Show that $\sqrt{\frac{\pi x}{2}}\, J_{(1/2)}^{(x)} = \sin x$

$$J_p = \sum \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+p}$$

$$J_{1/2} = \sum \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+\frac{1}{2}+1)} \left(\frac{x}{2}\right)^{2n+\frac{1}{2}}$$

$$= \frac{1}{\Gamma(1)\Gamma(\frac{1}{2}+1)}\left(\frac{x}{2}\right)^{1/2} - \frac{1}{\Gamma(2)\Gamma(1\frac{1}{2}+1)}\left(\frac{x}{2}\right)^{2\,1/2} + \frac{1}{\Gamma(3)\Gamma(2\frac{1}{2}+1)}\left(\frac{x}{2}\right)^{4\frac{1}{2}} + \cdots$$

but $\Gamma(p+1) = p\,\Gamma(p)$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

so write $\Gamma(\frac{1}{2}+1)$ as $\frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$

and $\Gamma(1\frac{1}{2}+1) = 1\frac{1}{2}\,\Gamma(1\frac{1}{2}) = \frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi}$

and $\Gamma(2\frac{1}{2}+1) = 2\frac{1}{2}\,\Gamma(2\frac{1}{2}) = \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi}$

$\Gamma(3\frac{1}{2}+1) = 3\frac{1}{2}\,\Gamma(3\frac{1}{2}) = \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi}$ etc…

so $J_{1/2} = \frac{1}{\Gamma(1)\frac{1}{2}\sqrt{\pi}}\left(\frac{x}{2}\right)^{1/2} - \frac{1}{\Gamma(2)\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}}\left(\frac{x}{2}\right)^{\frac{5}{2}} + \frac{1}{\Gamma(3)\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}}\left(\frac{x}{2}\right)^{\frac{9}{2}} - \cdots$

and $\Gamma(1) = 1$
$\Gamma(2) = 1$
$\Gamma(3) = 2!$
$\Gamma(4) = 3!$ etc…

so $J_{1/2} = \frac{1}{\frac{1}{2}\sqrt{\pi}}\left(\frac{x}{2}\right)^{1/2} - \frac{1}{\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}}\left(\frac{x}{2}\right)^{\frac{5}{2}} + \frac{1}{2!\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}}\left(\frac{x}{2}\right)^{\frac{9}{2}} - \frac{1}{3!\times\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}}\left(\frac{x}{2}\right)^{\frac{13}{2}} + \cdots$

$$= \frac{1}{\sqrt{\pi}}\left(\frac{1}{\frac{1}{2}}\,\frac{x^{1/2}}{2^{1/2}} - \frac{x^{5/2}}{\underbrace{\frac{3}{2}\times\frac{1}{2}}_{2^2}\times 2^{5/2}} + \frac{x^{9/2}}{2\times\underbrace{\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}}_{2^3}\times 2^{9/2}} - \frac{x^{14/2}}{(2\times3)\underbrace{\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}}_{2^4}\times 2^{\frac{13}{2}}} + \cdots\right)$$

$$= \frac{1}{\sqrt{\pi}}\left(\frac{x^{1/2}}{2^{-\frac{1}{2}}} - \frac{x^{5/2}}{1\times3\times2^{-2}\times2^{5/2}} + \frac{x^{9/2}}{2\times5\times3\times1\times2^{-3}\times2^{9/2}} - \frac{x^{\frac{14}{2}}}{(2\times3)\,7\times5\times3\times1\times2^{-4}\times2^{\frac{13}{2}}}\right.$$

$$= \frac{1}{\sqrt{\pi}}\left(\frac{x^{1/2}}{2^{-\frac{1}{2}}} - \frac{x^{5/2}}{2^{\frac{1}{2}}\times1\times3} + \frac{x^{9/2}}{2\times5\times3\times1\times2^{3/2}} - \frac{x^{\frac{14}{2}}}{\underbrace{(2\times3)}_{=6}\,7\times5\times3\times1\times2^{\frac{5}{2}}} + \cdots\right.$$

use this to get 4 for factorial

use this to get 2x4 for factorial

$$= \frac{1}{\sqrt{\pi}}\left(\frac{x^{\frac{1}{2}}}{2^{-\frac{1}{2}}} - \frac{x^{\frac{5}{2}}}{2^{-\frac{1}{2}} \times \underset{\text{borrow}}{1\times2\times3}} + \frac{x^{\frac{9}{2}}}{\underset{\substack{2^2\\ \text{borrowed}}}{1\times2\times3\times4\times5} \times 2^{-\frac{1}{2}}} - \frac{x^{\frac{14}{2}}}{\underset{\text{borrow}\quad\text{borrow}}{1\times2\times3\times4\times5\times6\times7} \times 2^{-\frac{1}{2}}}\right)$$

ok, almost there.

now multiply $J_{1/2}$ by $\sqrt{\frac{\pi x}{2}}$

$$= \sqrt{\frac{\pi x}{2}} \times$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$= \sin x$$

QED

ch 12

$\boxed{13.1}$ Using eq 12.9 and 13.1, write out first few terms of $J_0(x)$, $J_1(x)$, $J_{-1}(x)$, $J_2(x)$, $J_{-2}(x)$. Show that $J_{-1}(x) = -J_1(x)$ and $J_{-2}(x) = J_2(x)$.

(12.9) $$J_p(x) = \sum \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+p}$$

(13.1) $$J_{-p}(x) = \sum \frac{(-1)^n}{\Gamma(n+1)\Gamma(n-p+1)} \left(\frac{x}{2}\right)^{2n-p}$$

First few terms:

$$J_0(x) = \frac{1}{\Gamma(1)\Gamma(1)} - \frac{1}{\Gamma(2)\Gamma(2)}\left(\frac{x}{2}\right)^2 + \frac{1}{\Gamma(3)\Gamma(4)}\left(\frac{x}{2}\right)^4 - \frac{1}{\Gamma(4)\Gamma(5)}\left(\frac{x}{2}\right)^6 + \cdots$$

$$J_1(x) = \frac{1}{\Gamma(1)\Gamma(2)}\left(\frac{x}{2}\right) - \frac{1}{\Gamma(2)\Gamma(3)}\left(\frac{x}{2}\right)^3 + \frac{1}{\Gamma(3)\Gamma(4)}\left(\frac{x}{2}\right)^5 - \frac{1}{\Gamma(4)\Gamma(5)}\left(\frac{x}{2}\right)^7 + \cdots$$

$$J_{-1}(x) = \frac{1}{\Gamma(1)\underbrace{\Gamma(0)}_{\infty}}\left(\frac{x}{2}\right)^{-1} - \frac{1}{\Gamma(2)\Gamma(1)}\left(\frac{x}{2}\right)^1 + \frac{1}{\Gamma(3)\Gamma(2)}\left(\frac{x}{2}\right)^3 - \frac{1}{\Gamma(4)\Gamma(3)}\left(\frac{x}{2}\right)^5 \cdots$$

$$= 0 - \frac{1}{\Gamma(2)\Gamma(1)}\left(\frac{x}{2}\right) + \frac{1}{\Gamma(3)\Gamma(2)}\left(\frac{x}{2}\right)^3 - \frac{1}{\Gamma(4)\Gamma(3)}\left(\frac{x}{2}\right)^5 + \cdots$$

$$J_2(x) = \frac{1}{\Gamma(1)\Gamma(3)}\left(\frac{x}{2}\right)^2 - \frac{1}{\Gamma(2)\Gamma(4)}\left(\frac{x}{2}\right)^4 + \frac{1}{\Gamma(3)\Gamma(5)}\left(\frac{x}{2}\right)^6 - \cdots$$

$$J_{-2} = \frac{1}{\Gamma(1)\underbrace{\Gamma(-1)}_{\infty}}\left(\frac{x}{2}\right)^{-2} - \frac{1}{\Gamma(2)\underbrace{\Gamma(0)}_{\infty}}\left(\frac{x}{2}\right)^0 + \frac{1}{\Gamma(3)\Gamma(1)}\left(\frac{x}{2}\right)^2 - \cdots$$

$$= 0 - 0 + \frac{1}{\Gamma(3)\Gamma(1)}\left(\frac{x}{2}\right)^2 - \cdots$$

$\longrightarrow$

now need to show that $J_{-1}(x) = -J_1(x)$

From power series for $J_{-1}(x)$:

$$= -\frac{1}{0! \times 1!}\left(\frac{x}{2}\right) + \frac{1}{2! \times 1!}\left(\frac{x}{2}\right)^3 - \frac{1}{3! \times 2!}\left(\frac{x}{2}\right)^5 + \cdots$$

and from power series for $J_1(x)$

$$= \frac{1}{0! \times 1!}\left(\frac{x}{2}\right) - \frac{1}{1! \times 2!}\left(\frac{x}{2}\right)^3 + \frac{1}{2! \times 3!}\left(\frac{x}{2}\right)^5 - \cdots$$

so we see that $-J_1(x) = J_{-1}(x)$.

now show that $J_{-2}(x) = J_2(x)$.

power series for $J_{-2}(x) =$

$$\frac{1}{2! \times 0!}\left(\frac{x}{2}\right)^2 - \frac{1}{3! \times 1!}\left(\frac{x}{2}\right)^4 + \cdots$$

while power series for $J_2(x) =$

$$\frac{1}{2! \times 0!}\left(\frac{x}{2}\right)^2 - \frac{1}{1! \times 3!}\left(\frac{x}{2}\right)^4 + \cdots$$

which are the same.

ch 12

$\boxed{13.2}$ show that in general for integral n, $J_{-n} = (-1)^n J_n$

and $J_n(-x) = (-1)^n J_n$. (I'll use J_{-m}, J_m instead since 'n' already used in the sum terms)

$$J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+m)!} \left(\frac{x}{2}\right)^{2n+m} \quad \text{for } m \text{ integer.} \qquad (1)$$

write out some terms:

$$J_m(x) = \frac{1}{m!}\left(\frac{x}{2}\right)^m - \frac{1}{(1+m)!}\left(\frac{x}{2}\right)^{2+m} + \frac{1}{(2+m)!}\left(\frac{x}{2}\right)^{3+m} - \cdots$$

now if $m<0$, Then all negative factorials terms are zero, since

$\Gamma(\text{negative integer}) = \infty$. and so $\frac{1}{\Gamma(n+m)} \to 0$ for $n+m<0$

so $J_m(x)$ for negative m will have all its few terms when

$(n+m) < 0$, as zero.

so for negative m, we start sum from $\underline{n=m}$

$$\boxed{J_{-m} = \sum_{n=m}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n-m+1)} \left(\frac{x}{2}\right)^{2n-m}}$$

let $n-m=k$, so when $n=m$, $k=0$, when $n=\infty$, $k=\infty$.

so
$$J_{-m} = \sum_{k=0}^{k=\infty} \frac{(-1)^{k+m}}{\Gamma(k+m+1)\,\Gamma(k+1)} \left(\frac{x}{2}\right)^{2(k+m)-m}$$

$$J_{-m} = (-1)^m \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\,\Gamma(k+m+1)} \left(\frac{x}{2}\right)^{2k+m}$$

but This is The same sum expression for $J_m(x)$ with k replacing n.

so it can be written as
$$J_{-m} = (-1)^m \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+m-1)} \left(\frac{x}{2}\right)^{2n+m} \qquad (2)$$

compare (1) and (2), we see That $\boxed{J_{-m}(x) = (-1)^m J_m(x)}$

QED

$\boxed{13.5}$

$$N_{1/2} = \frac{\cos\left(\frac{\pi}{2}\right) J_{1/2} - J_{-1/2}}{\sin\frac{\pi}{2}}$$ using eq 13.3 page 513

$\sin\frac{\pi}{2} = 1$

$\cos\frac{\pi}{2} = 0$

so $\boxed{N_{1/2} = -J_{1/2}}$

Show that $N_{3/2} = J_{-3/2}$

$$N_{3/2} = \frac{\cos(3/2\,\pi) J_{3/2} - J_{-3/2}}{\sin(\frac{3}{2}\pi)}$$

but $\sin(\frac{3}{2}\pi) = -\sin\frac{\pi}{2} = -1$

and $\cos(\frac{3}{2}\pi) = 1$

hence $\boxed{N_{3/2}(x) = +J_{-3/2}(x)}$

ch 12

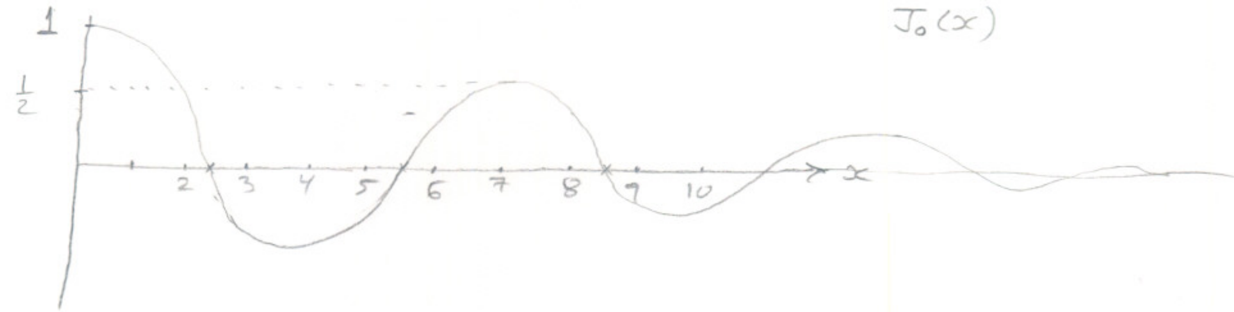
$\boxed{14.1}$

using Tables, sketch and find first 3 zeros for $J_0(x)$.

from Table 9.1, page 390, handbook of math functions, by Abramowitz:

$J_0(x)$ changes signs at $\boxed{x = 2.45,\ 5.55,\ 8.55}$

From page 359 of same book:



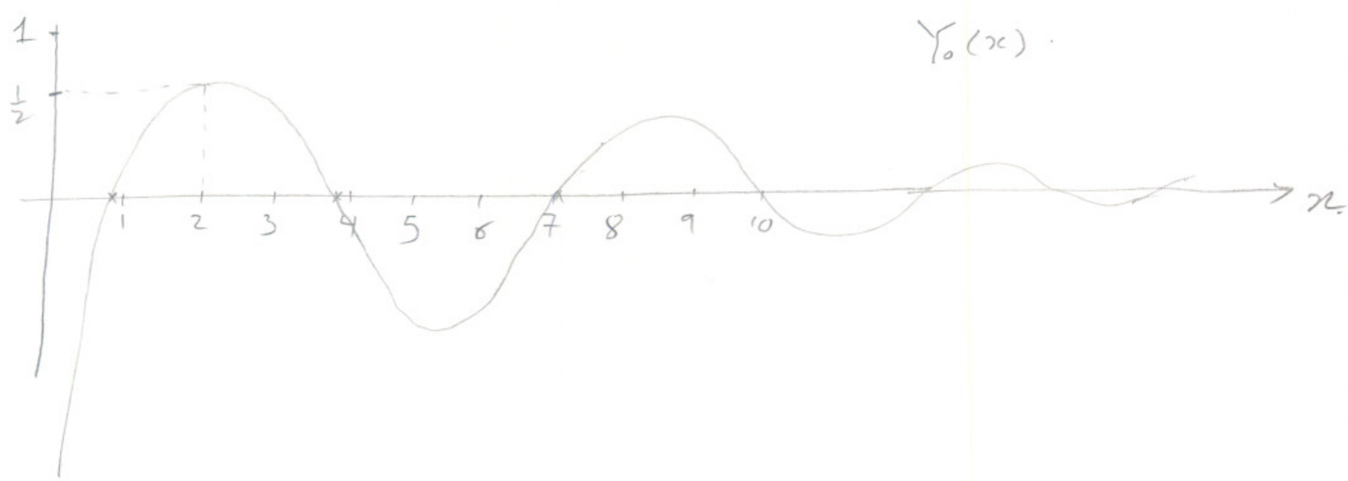
$\boxed{14.3}$ using Tables, sketch and find first 3 zeros of $N_0(x)$.

From same book, $N_0(x)$ Tables, called $Y(x)$, are given on page 391.

From these Tables, I see $Y_0(x)$ changin sign at

$\boxed{x = 0.95,\quad 3.85,\quad 7.05}$

Y_0 is very similar to J_1 for $x > 2$. from page 359



ch 12 $\boxed{15.1}$ prove 15.2 by method similar to one used to prove 15.1.

15.2 is: $\frac{d}{dx}\left[x^{-p} J_p\right] = -x^{-p} J_{p+1}$

$$J_p = \sum \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p+1)}\left(\frac{x}{2}\right)^{2n+p} \quad , \quad J_{p+1} = \sum \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p+2)}\left(\frac{x}{2}\right)^{2n+p+1}$$

multiply by x^{-p} and differentiate

$$x^{-p} J_p = \sum \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p+1)} x^{2n} \frac{1}{2^{2n+p}}$$

$$\frac{d}{dx}\left[x^{-p} J_p\right] = \sum \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p+1)}(2n)x^{2n-1} \frac{1}{2^{2n+p}}$$

$\Gamma(n+1) = n\Gamma(n)$, so $\frac{d}{dx}\left[x^{-p} J_p\right] = \sum \frac{(-1)^n}{\cancel{n}\Gamma(n)\Gamma(n+p+1)}(\cancel{2}\,\cancel{n})\, x^{2n-1} \frac{1}{\cancel{2}} \frac{1}{2^{2n+p-1}} = \sum \frac{(-1)^n}{\Gamma(n)\Gamma(n+p+1)} \frac{x^{2n-1}}{2^{2n+p-1}}$

multiply by $\frac{1}{(x)^{-p}} \Rightarrow \frac{1}{(x)^{-p}} \frac{d}{dx}\left[x^{-p} J_p\right] = \frac{1}{(x)^{-p}} \sum \frac{(-1)^n}{\Gamma(n)\Gamma(n+p+1)} \frac{x^{2n-1}}{2^{2n+p-1}} = \sum \frac{(-1)^n}{\Gamma(n)\Gamma(n+p+1)} \frac{x^{2n+p-1}}{2^{2n+p-1}}$

$$\Rightarrow \frac{1}{(x)^{-p}} \frac{d}{dx}\left[x^{-p} J_p\right] = \sum \frac{(-1)^n}{\Gamma(n)\Gamma(n+p+1)}\left(\frac{x}{2}\right)^{2n+p-1} = \boxed{\sum \frac{(-1)^n\, n}{\Gamma(n+1)\Gamma(n+p+1)}\left(\frac{x}{2}\right)^{2n+p-1}}$$

let me look at few terms in J_{p+1} and in to see pattern:

$$\sum \frac{(-1)^n\, n}{\Gamma(n+1)\Gamma(n+p+1)}\left(\frac{x}{2}\right)^{2n+p-1} = 0 - \frac{1}{\Gamma(2)\Gamma(p+2)}\left(\frac{x}{2}\right)^{p+1} + \frac{1\times 2}{\Gamma(3)\Gamma(p+3)}\left(\frac{x}{2}\right)^{p+3} - \frac{1\times 3}{\Gamma(4)\Gamma(p+4)}\left(\frac{x}{2}\right)^{p+5} + \cdots \quad \text{—— (1)}$$

$$J_{p+1} = \frac{1}{\Gamma(1)\Gamma(p+2)}\left(\frac{x}{2}\right)^{p+1} - \frac{1}{\Gamma(2)\Gamma(p+3)}\left(\frac{x}{2}\right)^{p+3} + \frac{1}{\Gamma(3)\Gamma(p+4)}\left(\frac{x}{2}\right)^{p+5} + \cdots \quad \text{—— (2)}$$

so if I rewrite (1) as

$$\sum \frac{(-1)^n\, n}{\Gamma(n+1)\Gamma(n+p+1)}\left(\frac{x}{2}\right)^{2n+p-1} = 0 - \frac{1}{1\times\Gamma(1)\Gamma(p+2)}\left(\frac{x}{2}\right)^{p+1} + \frac{1\times \cancel{2}}{\cancel{2}\Gamma(2)\Gamma(p+3)}\left(\frac{x}{2}\right)^{p+3} - \frac{1\times \cancel{3}}{\cancel{3}\Gamma(3)\Gamma(p+4)}\left(\frac{x}{2}\right)^{p+5} + \cdots$$

$$= 0 - \frac{1}{\Gamma(1)\Gamma(p+2)}\left(\frac{x}{2}\right)^{p+1} + \frac{1}{\Gamma(2)\Gamma(p+3)}\left(\frac{x}{2}\right)^{p+3} - \frac{1}{\Gamma(3)\Gamma(p+4)}\left(\frac{x}{2}\right)^{p+5} + \cdots$$

but $J_{p+1} = + \frac{1}{\Gamma(1)\Gamma(p+2)}\left(\frac{x}{2}\right)^{p+1} - \frac{1}{\Gamma(2)\Gamma(p+3)}\left(\frac{x}{2}\right)^{p+3} + \frac{1}{\Gamma(3)\Gamma(p+4)}\left(\frac{x}{2}\right)^{p+5} + \cdots$

So we see the sign shift!

i.e. $$\boxed{J_{p+1} = -\sum \frac{(-1)^n\, n}{\Gamma(n+1)\Gamma(n+p+1)}\left(\frac{x}{2}\right)^{2n+p-1} = -\frac{1}{x^{-p}} \frac{d}{dx}\left[x^{-p} J_p\right]}$$

Ch 12

$\boxed{15.2}$ solve 15.1 and 15.2 for J_{p+1} and J_{p-1}. Add and subtract these two equations to get 15.3 and 15.4.

$$(15.1) \quad \frac{d}{dx}\left(x^p J_p\right) = x^p J_{p-1}$$

$$(15.2) \quad \frac{d}{dx}\left(x^{-p} J_p\right) = -x^{-p} J_{p+1}$$

From (15.1) $\quad J_{p-1} = x^{-p}\frac{d}{dx}\left(x^p J_p\right)$

From (15.2) $\quad J_{p+1} = -x^{p}\frac{d}{dx}\left(x^{-p} J_p\right)$

so $J_{p-1} + J_{p+1} = x^{-p}\frac{d}{dx}\left(x^p J_p\right) - x^{p}\frac{d}{dx}\left(x^{-p} J_p\right) = x^{-p}\left[x^p J_p' + p x^{p-1} J_p\right] - x^p\left[x^{-p} J_p' + (-p) x^{-p-1} J_p\right]$

$$= x^{-p} x^p J_p' + x^{-p} p x^{p-1} J_p - x^p x^{-p} J_p' + p x^p x^{-p-1} J_p$$

$$= J_p' + x^{-1} p J_p - J_p' + x^{-1} p J_p$$

$$= x^{-1} p J_p + x^{-1} p J_p$$

$J_{p-1} + J_{p+1} = \boxed{2\frac{p}{x} J_p}$ which is 15.3

now $J_{p-1} - J_{p+1} = x^{-p}\frac{d}{dx}\left(x^p J_p\right) + x^{p}\frac{d}{dx}\left(x^{-p} J_p\right)$

$$= x^{-p}\left[x^p J_p' + p x^{p-1} J_p\right] + x^p\left[x^{-p} J_p' + (-p) x^{-p-1} J_p\right]$$

$$= J_p' + p x^{-1} J_p + J_p' - p x^{-1} J_p$$

$= \boxed{2 J_p'(x)}$ which is 15.4.

Ch 12

$\boxed{15.3}$ Carry out the differentiation in 15.1 and 15.2 to get 15.5

From (15.1) $\frac{d}{dx}[x^p J_p] = x^p J_{p-1}$

15.2 $\frac{d}{dx}[x^{-p} J_p] = -x^{-p} J_{p+1}$

differentiate 15.1 $\Rightarrow$ $x^p J_p' + p x^{p-1} J_p = x^p J_{p-1}$

so $J_p' = \frac{x^p J_{p-1} - p x^{p-1} J_p}{x^p} = J_{p-1} - p x^{-1} J_p = \boxed{J_{p-1} - \frac{p}{x} J_p}$ — (1)

differentiate 15.2 $\Rightarrow$

$x^{-p} J_p' + (-p) x^{-p-1} J_p = -x^{-p} J_{p+1}$

so $J_p' = \frac{-x^{-p} J_{p+1} + p x^{-p-1} J_p}{x^{-p}} = -J_{p+1} + p x^{-1} J_p$

$= \boxed{-J_{p+1} + \frac{p}{x} J_p}$ — (2)

from (1) and (2) we see that

$J_p' = J_{p-1} - \frac{p}{x} J_p = \frac{p}{x} J_p - J_{p+1}$

which is 15.5

ch 12

$\boxed{15.5}$ using 15.4 and 15.5 show that $J_0 = J_2$ at every min or max of J_1. and $J_0 = -J_2 = -J_1'$ at every position zero of J_1. sketch J_0, J_1, J_2 on same axes.

(15.4): $J_{p-1} - J_{p+1} = 2J_p'$

(15.5) $J_p' = -\frac{p}{x} J_p + J_{p-1} = \frac{p}{x} J_p - J_{p+1}$

$J_p'(x) = 0$ at min or max of J_p by definition.

From 15.4, set $p=1$ and $J_p' = 0$ we get

$J_0 - J_2 = 0 \Rightarrow \boxed{J_0 = J_2}$ at each minmax of J_1

From (15.5), set $p=1$, we get

$J_1'(x) = -\frac{1}{x} J_1 + J_0$

and $J_1'(x) = \frac{1}{x} J_1 - J_2$

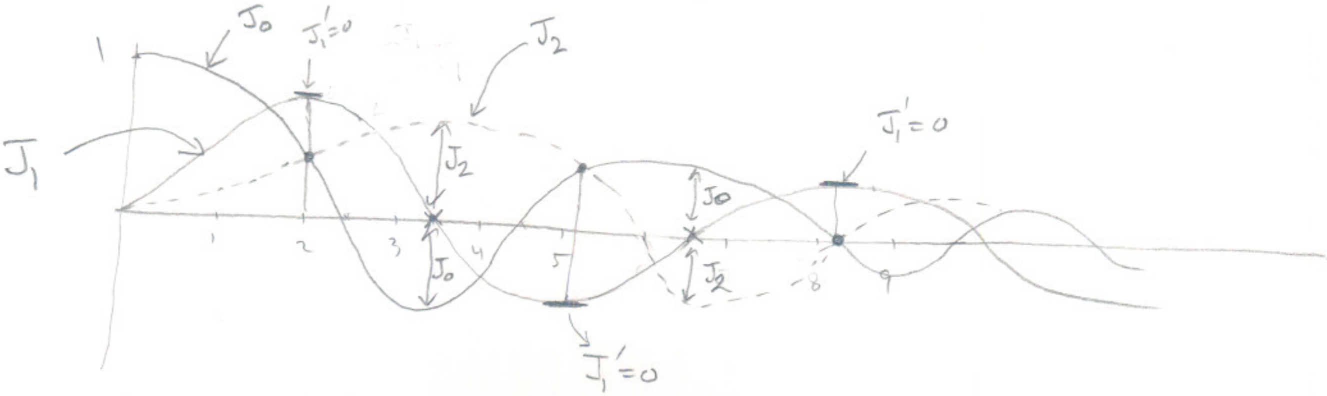
so for $x>0$, $J_1' = -\frac{J_1}{x} + J_0$ ——— ①

$J_1' = \frac{J_1}{x} - J_2$ ——— ②

at a zero of $J_1(x)$, this means $J_1(x) = 0$ by definition.

i.e

①,② $\Rightarrow$ $\left.\begin{aligned} J_1' &= J_0 \\ J_1' &= -J_2 \end{aligned}\right\}$ or $\boxed{J_0 = -J_2 = J_1'}$ at each zero of J_1



at each zero of J_1, $J_0 = -J_1$

at $J_1' = 0$, $J_0 = J_2$

ch 12

$\boxed{15.7}$

using 15.2 show that $\int_0^\infty J_1(x)\,dx = -J_0(x)\Big|_0^\infty = 1$

(15.2) $\frac{d}{dx}\left\{x^{-p} J_p\right\} = -x^{-p} J_{p+1}(x)$.

let $p=0 \Rightarrow \frac{d}{dx}\left[x^{-0} J_0\right] = -x^{-0} J_1$

or $\frac{d}{dx}(J_0) = -J_1$

From Fundamental theorm of Calculus, $\frac{d}{dx}(f(x)) = g(x) \Rightarrow \int g(x)\,dx = f(x)$

or $\int_a^b g(x)\,dx = f(x)\Big|_b - f(x)\Big|_a$

so $-\int_a^b J_1\,dx = -\left[J_0\Big|_b - J_0\Big|_a\right]$

i.e $\int_0^\infty J_1(x)\,dx = -\left[\left[J_0\right]_0^\infty\right] = -\left[J_0(\infty) - J_0(0)\right]$

From $J_p = \sum \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p+1)}\left(\frac{x}{2}\right)^{2n+p}$

when $p=0$, $x=0$ we get $\sum \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1)}\left(\frac{x}{2}\right)^{2n}$

so $J_0(x) = \frac{1}{\Gamma(1)\Gamma(1)} - \underbrace{\frac{1}{\Gamma(2)\Gamma(2)}\left(\frac{x}{2}\right)^2 + \cdots}_{=0} = 1$

when $p=0$, $J_0(\infty) = 0$ since graph of J_0

$\int_0^\infty J_1(x)\,dx = -\left[J_0(\infty) - J_0(0)\right]$

$= -\left[0 - 1\right] = \boxed{1}$