39/40

HW # 3

Math 121 B

NASSER ABBASI

UCB extension.

Ch 12 (1)

$\boxed{1.1}$

solve DE using power series and by elementry method. Verify same solution.

$$y'' + 4y = 0$$

let $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$

$$\left.\begin{aligned}
y' &= a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots \\
y'' &= 2a_2 + 2\cdot 3 a_3 x + 3\cdot 4\, a_4 x^2 + \cdots + (n+1)(n+2)\, a_{n+2} x^n \\
4y &= 4a_0 + 4a_1 x + 4a_2 x^2 + 4a_3 x^3 + 4\, a_n x^n
\end{aligned}\right\}$$

so $\boxed{(n+1)(n+2)\, a_{n+2} = -4\, a_n}$

now I can generate few a's to see pattern for even and odd a's.

$n=0$

$$1\cdot 2\, a_2 = -4a_0 \Rightarrow a_2 = -\frac{4}{1\cdot 2} a_0$$

$n=1$

$$2\cdot 3\, a_3 = -4a_1 \Rightarrow a_3 = -\frac{4}{2\cdot 3} a_1$$

$n=2$

$$3\cdot 4\, a_4 = -4a_2 \Rightarrow a_4 = \frac{-4}{3\cdot 4} a_2 = \frac{-4}{3\cdot 4}\left(\frac{-4}{1\cdot 2}\right) a_0 = \frac{+4^2}{1\cdot 2\cdot 3\cdot 4} a_0$$

$n=3$

$$4\cdot 5\, a_5 = -4a_3 \Rightarrow a_5 = \frac{-4}{4\cdot 5} a_3 = \frac{-4}{4\cdot 5}\left(\frac{-4}{2\cdot 3}\right) a_1 = \frac{4^2}{2\cdot 3\cdot 4\cdot 5} a_1$$

$n=4$

$$5\cdot 6\, a_6 = -4a_4 \Rightarrow a_6 = \frac{-4}{5\cdot 6} a_4 = \frac{-4}{5\cdot 6}\,\frac{4^2}{1\cdot 2\cdot 3\cdot 4} a_0 = \frac{-4^3}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} a_0$$

$n=5$

$$6\cdot 7\, a_7 = -4a_5 \Rightarrow a_7 = \frac{-4}{6\cdot 7} a_5 = \frac{-4}{6\cdot 7}\,\frac{4^2}{2\cdot 3\cdot 4\cdot 5} a_1 = \frac{-4^3}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} a_1$$

from this I see that for odd n, $a_n = \dfrac{4^{\frac{(n-1)}{2}}}{n!} a_1$

and for even n, $a_n = \dfrac{4^{n/2}}{n!} a_0$ $\Rightarrow$

so $$y = a_0 \sum_{\substack{n \text{ even} \\ n \geq 0}} s(n) \frac{4^{n/2}}{n!} x^n + a_1 \sum_{\substack{n \text{ odd} \\ n \geq 1}} c(n) \frac{4^{\frac{n-1}{2}}}{n!} x^n$$

this function flips the sign.
$n=0$ it is +
at $n=2$ it is −
$n=4$ it is +
$n=6$ it is −
⋮
etc

this flips the sign
$n=1$ it is +
at $n=3$ it is −
$n=5$ it is +
$n=7$ it is −
⋮
etc.

not sure how to write this in the sum directly.

looking at few terms in y we see

$$y = a_0 \left[\underbrace{1}_{n=0} - \underbrace{\frac{4}{2} x^2}_{n=2} + \underbrace{\frac{4^2}{4!} x^4}_{n=4} - \underbrace{\frac{4^3}{6!} x^6}_{n=6} + \cdots \right]$$

This is power series of $\cos 2x$
Can be better seen by noting That $4 = 2^2$

$$+ a_1 \left[\underbrace{x}_{n=1} - \underbrace{\frac{4}{3!} x^3}_{n=3} + \underbrace{\frac{4^2}{5!} x^5}_{n=5} - \cdots \right]$$

let $a_1 = 2C$ where C is some constant. I need to do this to make second series a sin series.

so $$y = a_0 [\cos 2x] + C \left[2x - \frac{2^3}{3!} x^3 + \frac{2^5}{5!} x^5 - \cdots \right]$$

$$\boxed{y = a_0 \cos 2x + C \sin 2x}$$

now I solve using basic method to verify the series solution.

$$y'' + 4y = 0$$

let $y = Ae^{mx}$

$y' = Ame^{mx}$

$y'' = Am^2e^{mx}$

so $Am^2e^{mx} + 4Ae^{mx} = 0$

i.e $e^{mx}(Am^2 + 4A) = 0 \Rightarrow m^2 + 4 = 0 \Rightarrow m = \pm 2i$

so $y_1 = A_1 e^{2ix}$, $y_2 = A_2 e^{-2ix}$

so general solution $= y_1 + y_2 = A_1 e^{2ix} + A_2 e^{-2ix}$

$= A_1(\cos 2x + i \sin 2x) + A_2(\cos -2x + i \sin -2x)$

but $\cos -x = \cos x$

$\sin -x = -\sin x$

so $y = A_1(\cos 2x + i\sin 2x) + A_2(\cos 2x - i \sin 2x)$

$= \cos 2x\,(A_1 + A_2) + \sin 2x\,((A_1 - A_2)\,i)$

let $A_1 + A_2 = C_1$

let $i(A_1 - A_2) = C_2$

so $\boxed{y = C_1 \cos 2x + C_2 \sin 2x}$

which match series solution, where $C_1 = a_0$

and $\frac{a_1}{2} = C_2$.

Ch 12

$\boxed{1.11}$ solve by series method

$$y'' - x^2 y' - xy = 0$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

$$y'' = 2a_2 + 2\cdot 3\, a_3 x + 3\cdot 4\, a_4 x^2 + \cdots + (n+1)(n+2)\, a_{n+2}\, x^n + \cdots$$

$$xy = a_0 x + a_1 x^2 + a_2 x^3 + \cdots + a_{n-1} x^n$$

$$x^2 y' = a_1 x^2 + 2a_2 x^3 + \cdots + (n-1)\, a_{n-1}\, x^n$$

so recursive formula is

$$(n+1)(n+2)\, a_{n+2}\, x^n - (n-1)\, a_{n-1}\, x^n - a_{n-1} x^n = 0$$

$$\Rightarrow \quad (n+1)(n+2)\, a_{n+2} - (n-1)\, a_{n-1} - a_{n-1} = 0$$

$n=1$:

$$2\cdot 3\, a_3 - a_0 = 0 \Rightarrow a_3 = \frac{1}{2\cdot 3} a_0$$

$n=2$

$$3\cdot 4\, a_4 - a_1 - a_1 = 0 \Rightarrow a_4 = \frac{1}{3\cdot 4}(a_1 + a_1) = \frac{2}{3\cdot 4} a_1$$

$n=3$

$$4\cdot 5\, a_5 - 2a_2 - a_2 = 0 \Rightarrow a_5 = \frac{1}{4\cdot 5}\, 2a_2 + a_2 = \frac{3}{4\cdot 5} a_2$$

$n=4$

$$5\cdot 6\, a_6 - 3a_3 - a_3 = 0 \Rightarrow a_6 = \frac{4}{5\cdot 6} a_3 = \frac{4}{5\cdot 6}\,\frac{1}{2\cdot 3} a_0 = \frac{4}{2\cdot 3\cdot 5\cdot 6} a_0$$

$n=5$

$$6\cdot 7\, a_7 - 4a_4 - a_4 = 0 \Rightarrow a_7 = \frac{5}{6\cdot 7} a_4 = \frac{5}{6\cdot 7}\,\frac{2}{3\cdot 4} a_1$$

$n=6$

$$7\cdot 8\, a_8 - 5a_5 - a_5 = 0 \Rightarrow a_8 = \frac{6}{7\cdot 8} a_5 = \frac{6}{7\cdot 8}\,\frac{3}{4\cdot 5} a_2$$

$\longrightarrow$

now note that $a_2 = 0$ ✓ (by looking at Table of coefficients.)

good. the recursion relation is only valid for $x \geq 1$, but there is an x^0 term.

so
$$a_3 = \frac{1}{2\cdot 3} a_0$$
$$a_4 = \frac{2}{3\cdot 4} a_1$$
$$a_5 = 0$$
$$a_6 = \frac{4}{2\cdot 3\cdot 5\cdot 6} a_0$$
$$a_7 = \frac{5}{6\cdot 7}\cdot\frac{2}{3\cdot 4} a_1$$
$$a_8 = 0$$
$$\vdots$$

so, plugging in y, we get

$$y = a_0 + a_1 x + \left(\frac{1}{2\cdot 3} a_0\right) x^3 + \left(\frac{2}{3\cdot 4}\right) a_1 x^4 + \frac{4}{2\cdot 3\cdot 5\cdot 6} a_0 x^6 + \frac{5}{6\cdot 7}\frac{2}{3\cdot 4} a_1 x^7$$

$$y = a_0\left[1 + \frac{1}{2\cdot 3}x^3 + \frac{4}{2\cdot 3\cdot 5\cdot 6}x^6 + \cdots\right]$$
$$+ a_1\left[x + \frac{2}{3\cdot 4}x^4 + \frac{5}{6\cdot 7}\frac{2}{3\cdot 4}x^7 + \cdots\right]$$

To make denominators factorial expressions, I multiply numerator and denominator for each term as needed:

$$y = a_0\left[1 + \frac{x^3}{3!} + \frac{(4)(4)}{2\cdot 3\cdot 4\cdot 5\cdot 6}x^6 + \cdots\right]$$
$$+ a_1\left[x + \frac{2(2)}{2\cdot 3\cdot 4}x^4 + \frac{5(5)}{2\cdot 3\cdot 4}\frac{2(2)}{5\cdot 6\cdot 7}x^7 + \cdots\right]$$

$$\boxed{y = a_0\left[1 + \frac{x^3}{3!} + \frac{4^2}{6!}x^6 + \cdots\right] + a_1\left[x + \frac{2^2}{4!}x^4 + \frac{(5\cdot 2)^2}{7!}x^7 + \cdots\right]}$$

this is the Series Solution.

ch 12
$\boxed{1.16}$

solve $(x^2+1)y'' - 2xy' + 2y = 0$ by series method.

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$
$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$
$$y'' = 2a_2 + 6a_3 x + 12a_4 x^2 + \cdots$$
$$2y = 2a_0 + 2a_1 x + 2a_2 x^2 + 2a_3 x^3 + \cdots$$
$$2xy' = 2xa_1 + 4a_2 x^2 + 6a_3 x^3 + \cdots$$
$$x^2 y'' = 2a_2 x^2 + 6a_3 x^3 + 12a_4 x^4 + \cdots$$

by inspection, looking at first column, $2a_2 + 2a_0 = 0$ i.e $\boxed{a_2 = -a_0}$

now I write the general recursive formula for x^n

$$x^2y'' + y'' - 2xy' + 2y \Rightarrow$$

$$n(n-1)a_n x^n + (n+1)(n+2)a_{n+2}x^n - 2n a_n x^n + 2a_n x^n$$

since D.E. equation equalls zero, then coeff. of each power of x must be zero as well. hence

$$\left(n(n-1)a_n + (n+1)(n+2)a_{n+2} - 2n a_n + 2a_n \right) x^n = 0$$

i.e

$$(n+1)(n+2)a_{n+2} = -n(n-1)a_n - 2a_n + 2n a_n$$

$$\boxed{(n+1)(n+2)a_{n+2} = a_n\left(2n - 2 - n(n-1)\right)}$$

I will now use this to generate few 'a' terms $\rightarrow$

let me simplify the recursive equation a little more

$$(n+1)(n+2)\, a_{n+2} = a_n(3n - n^2 - 2)$$

~~example~~

start with $n=1$ since I allready know a_2.

$\underline{n=1}$

$$(2)(3)\, a_3 = a_1(3-1-2) \Rightarrow a_3 = 0$$

$\underline{n=2}$

$$(3)(4)\, a_4 = a_2(6-4-2) \Rightarrow a_4 = 0$$

$\underline{n=3}$

$$(4)(5)\, a_5 = a_3(9-9-2) \rightarrow a_5 = 0$$

actually, no need to some more:
since, $a_3 = 0$ and $a_4 = 0$, and this recursive relation

finds a_{n+2} in terms of a_n, Then all a_n are

zero for $n = 3, 4, 5, \ldots$!

so $y = a_0 + a_1 x + a_2 x^2$

$$= a_0 + a_1 x - a_0 x^2$$

$$\boxed{y = a_0\left[1 - x^2\right] + a_1 x}$$

Ch 12 (6)

$\boxed{2.1}$ using 2.6 : $a_{n+2} = -\dfrac{(l-n)(l+n+1)}{(n+2)(n+1)} a_n$ and

2.7: $$y = a_0\left[1 - \frac{l(l+1)}{2!}x^2 + \frac{l(l+1)(l-2)(l+3)}{4!}x^4 - \cdots\right] + a_1\left[x - \frac{(l-1)(l+2)}{3!}x^3 + \frac{(l-1)(l+2)(l-3)(l+4)}{5!}x^5 - \cdots\right]$$

and the requirment that $P_l(1)=1$, find $P_2(x)$, $P_3(x)$ and $P_4(x)$.

Solution

If I write y as

$$y = a_0\left(1 + a_2x^2 + a_4x^4 + a_6x^6 + \cdots\right] \quad \text{The even } l \text{ series}$$

$$+ a_1\left(x + a_3x^3 + a_5x^5 + \cdots\right] \longrightarrow \text{The odd } l \text{ series.}$$

The 'a_0' series is the one that remains for $l = 0, 2, 4, 6, 8, \ldots$ and the 'a_1' series diverges in those cases and not used.

the 'a_1' series remains for $l = 1, 3, 5, 7, \ldots$ and the 'a_0' series diverges for these values and not used.

so for $P_2(x)$, this is $l=2$. hence will use the a_0 series.
for $P_3(x)$, this is $l=3$, hence use the a_1 series
for $P_4(x)$, this is $l=4$, hence use the a_0 series.

so $P_2(x) = a_0(1 + a_2x^2)$

$P_3(x) = a_1(x + a_3x^3)$

$P_4(x) = a_0(1 + a_2x^2 + a_4x^4)$.

So I just need to find the a's above to complete the solution $\Longrightarrow$

for $l=2$, $\underset{\text{i.e } n=0}{a_2} = -\dfrac{(l-0)(l+0+1)}{(0+2)(0+1)} a_0$

$$= -\frac{(l)(l+1)}{2} a_0 = -\frac{(2)(3)}{2} a_0 = -\frac{3}{1} a_0.$$

hence $P_2(x) = a_0(1-3x^2)$

a_0 is found by using the restriction that y must be 1 when $x=1$.

so $1 = a_0(1-3(1)^2) = a_0(1-3)$

so $a_0 = -\frac{1}{2}$

so $P_2(x) = -\frac{1}{2}(1-3x^2) = \boxed{\frac{1}{2}(3x^2-1)}$

for $l=3$.

$P_3(x) = a_1(x + a_3 x^3)$

$a_3 \equiv a_{n+2}$ so $n=1$

so $a_3 = -\dfrac{(l-1)(l+1+1)}{(1+2)(1+1)} a_1 = -\dfrac{(l-1)(l+2)}{3\cdot 2} a_1$

let $l=3$, $a_3 = -\dfrac{(3-1)(3+2)}{3\cdot 2} a_1 = -\dfrac{(2)(5)}{3\cdot 2} a_1 = -\frac{5}{3} a_1$

so $P_3(x) = a_1\left(x - \frac{5}{3}x^3\right)$. apply the boundary restriction:

$1 = a_1\left(1-\frac{5}{3}\right) \Rightarrow 1 = a_1\left(\frac{-2}{3}\right) \Rightarrow a_1 = -\frac{3}{2}$

so $P_3(x) = -\frac{3}{2}\left(x - \frac{5}{3}x^3\right) = \frac{5}{2}x^3 - \frac{3}{2}x = \boxed{\frac{1}{2}(5x^3-3x)}$

$\Longrightarrow$

for $l = 4$

$$P_4(x) = a_0(1 + a_2 x^2 + a_4 x^4)$$

find a_2, and use to find a_4.

a_2, i.e. $n=0 \Rightarrow$ $$a_2 = -\frac{(l-0)(l+0+1)}{(0+2)(0+1)} a_0$$

$$= -\frac{l(l+1)}{2} a_0 . \xrightarrow{l=4} -\frac{4(5)}{2} a_0 = -10\, a_0$$

a_4, i.e. $n=2 \Rightarrow$ $$a_4 = -\frac{(l-2)(l+2+1)}{(2+2)(2+1)} a_2$$

$$\xrightarrow{l=4} \quad a_4 = -\frac{(4-2)(4+2+1)}{(4)(3)} a_2 = -\frac{(2)(7)}{(4)(3)} \overbrace{(-10\, a_0)}^{a_2}$$

$$a_4 = +\frac{70}{6} a_0 = +\frac{35}{3} a_0$$

so $P_4(x) = a_0\left(1 - 10x^2 + \frac{35}{3} x^4\right)$.

now apply boundary condition to find a_0

$$1 = a_0\left(1 - 10 + \frac{35}{3}\right) \Rightarrow 1 = a_0\left(\frac{3-30+35}{3}\right) = a_0\left(\frac{8}{3}\right).$$

so $a_0 = \frac{3}{8}$

so $$P_4(x) = \frac{3}{8}\left(1 - 10x^2 + \frac{35}{3}x^4\right) = \frac{3}{8} - \frac{30x^2}{8} + \frac{35}{8}x^4$$

$$\boxed{P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)}$$

ch 12

$\boxed{2.2}$ show that $P_\ell(-1) = (-1)^\ell$

$$P_\ell(x) = a_0\underbrace{\left[1 + a_2 x^2 + a_4 x^4 + \cdots\right]}_{\text{sum of even functions in } x} + a_1\underbrace{\left[x + a_3 x^3 + \cdots\right]}_{\text{sum of odd functions in } x.}$$

There are 2 cases to consider. when ℓ is even, and odd.

<u>when ℓ is even</u>

then $P_\ell(x)$ is the sum of even functions $(x^2, x^4, x^6, \ldots)$

but sum of even functions is an even function.

so $P_\ell(-x) = P_\ell(x)$.

for $x=1$, we get $P_\ell(-1) = P_\ell(1)$

but $P_\ell(1) = 1$ by definition, since this is the boundary condition; we want to solve for.

so $P_\ell(-1) = 1$. now since ℓ is even, then $1 = (-1)^\ell$ i.e $1 = -1^0$, $1 = -1^2$ or $1 = -1^4$ or $1 = -1^6$ $\vdots$

so $\boxed{P_\ell(-1) = (-1)^\ell}$ (1)

now for the <u>Case ℓ is odd:</u>

here $P_\ell(x)$ is sum of odd function of x, $(x, x^3, x^5, \ldots)$

so $P_\ell(x)$ is an odd function.

i.e $P_\ell(-x) = -P_\ell(x)$

for $x=1$, we have $P_\ell(-1) = -P_\ell(1) = -1$

again, since ℓ is odd, then -1 is the same as $(-1)^\ell$

hence $\boxed{P_\ell(-1) = (-1)^\ell}$ (2). from (1),(2), Then $\boxed{P_\ell(-1) = -1^\ell \text{ for all } \ell}$

Ch 12

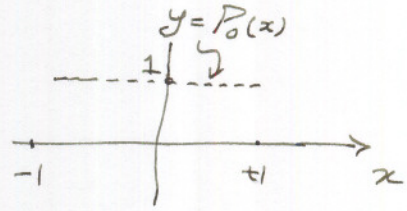
$\boxed{2.3}$ sketch graph of $P_0(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$ from $x=-1$ to $x=1$.

in all graphs we must have $P_\ell(1) = 1$ since this is the boundary condition on the solution of the D.E. we used to obtain the legendre polynomials.

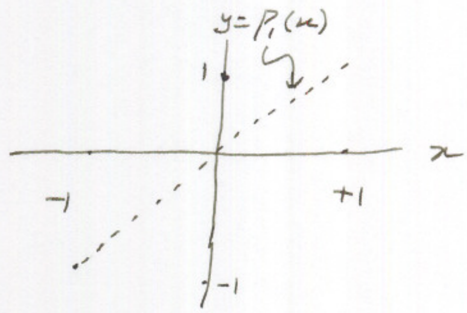
in addition $P_\ell(0) = 0$ for odd ℓ.

also, $P_\ell(-1) = (-1)^\ell$, so $P_\ell(-1) = 1$ for even ℓ, and $P_\ell(-1) = -1$ for odd ℓ.

$P_0(x) = 1$, plot is



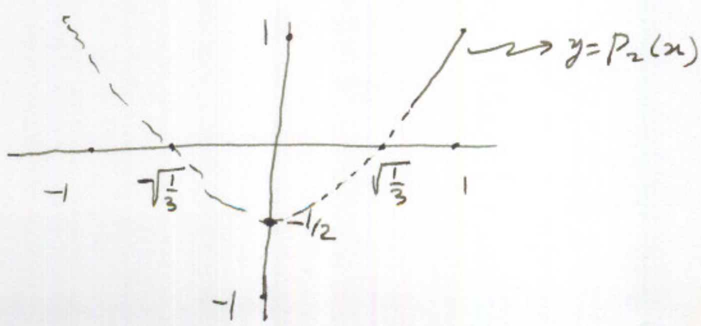
$P_1(x) = x$, plot is



$P_2(x) = \frac{1}{2}(3x^2-1)$

when $y=0 \Rightarrow 3x^2-1=0$ i.e $x = \pm\sqrt{\frac{1}{3}}$ are the roots.

when $x=0 \Rightarrow P_2(x) = -\frac{1}{2}$. So plot



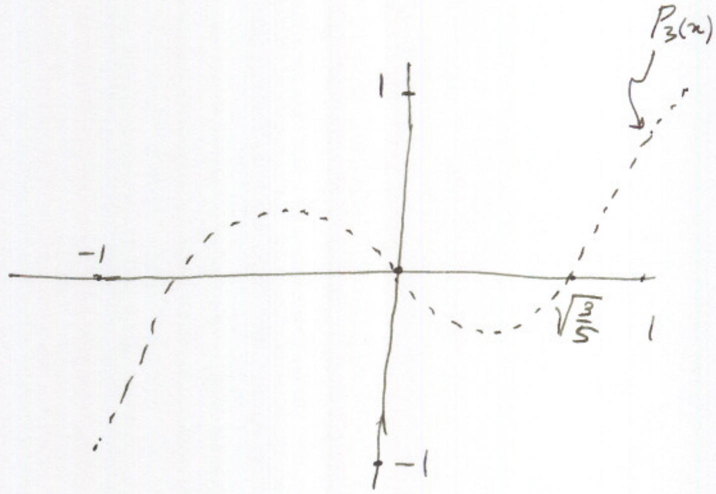
$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

when $y=0 \Rightarrow \quad 5x^3 - 3x = 0 \quad$ or $\quad x(5x^2-3) = 0$.

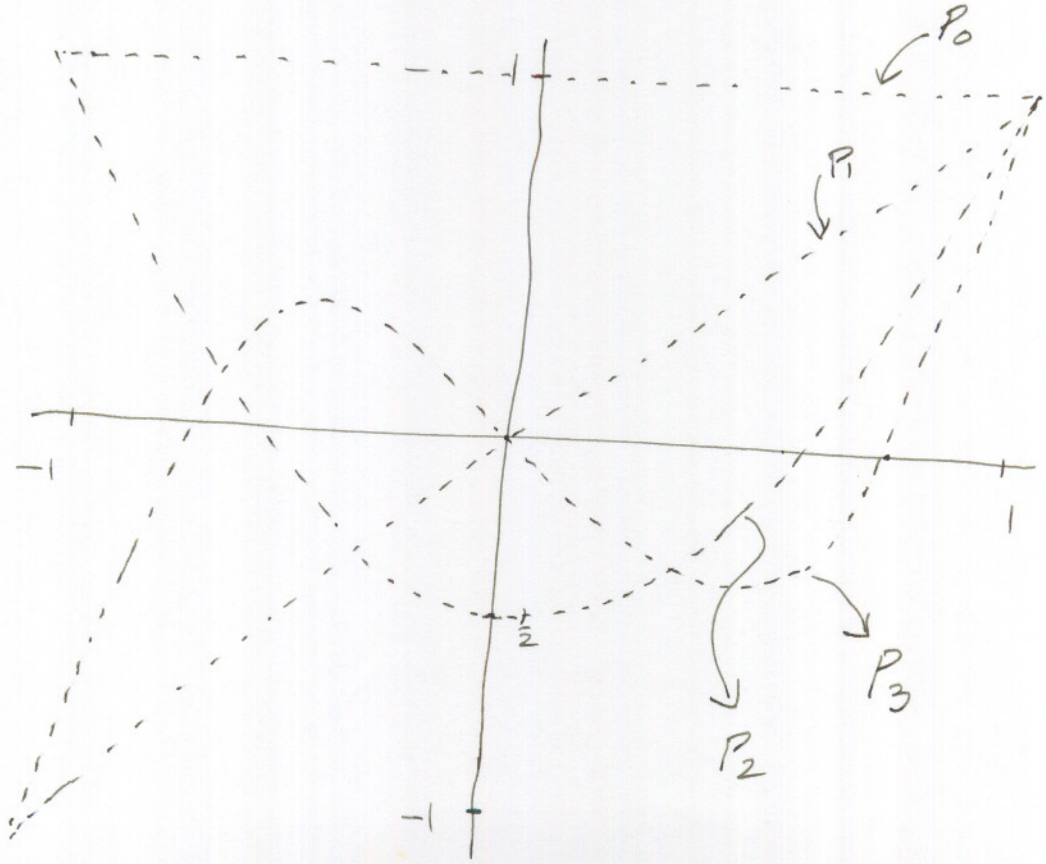
i.e $\quad x = 0 \quad$ or $\quad 5x^2 - 3 = 0 \quad$ i.e $\quad x^2 = \frac{3}{5} \quad$ or $\quad x = \pm\sqrt{\frac{3}{5}}$

so roots are $\quad 0, +\sqrt{\frac{3}{5}}, -\sqrt{\frac{3}{5}}$

so, in the plot we have



putting The plots all on one diagram we have



Ch 12

$\boxed{3.5}$ Solve $\dfrac{d^{100}}{dx^{100}} x^2 e^{-x}$ using Leibniz rule.

Leibniz rule is used for differentiation of products

it says
$$\frac{d^n}{dx^n} uv = \frac{d^0}{dx^0}u \frac{d^n}{dx^n}v + n \frac{d^1}{dx^1}u \frac{d^{n-1}}{dx^{n-1}}v + \frac{n(n-1)}{2!} \frac{d^2}{dx^2}u \frac{d^{n-2}}{dx^{n-2}}v + \cdots$$

taking $u = x^2$ and $v = e^{-x}$, we get

$$\frac{d^{100}}{dx^{100}} x^2 e^{-x} = \frac{d^0}{dx^0}x^2 \frac{d^{100}}{dx^{100}}e^{-x} + 100 \frac{d^1}{dx^1}x^2 \frac{d^{99}}{dx^{99}}e^{-x} + \frac{(100)(99)}{2!} \frac{d^2 x^2}{dx^2} \frac{d^{98}}{dx^{98}} e^{-x} + \frac{(100)(99)(98)}{3!} \frac{d^3 x^2}{dx^3} \frac{d^{97}}{dx^{97}} e^{-x} + \cdots$$

but $\dfrac{d^n}{dx^n} x^m = 0$ for $n > m$. so all terms from here and the rest are zero.

So $$\frac{d^{100}}{dx^{100}} x^2 e^{-x} = x^2 \frac{d^{100}}{dx^{100}} e^{-x} + 100(2x) \frac{d^{99}}{dx^{99}} e^{-x} + \frac{(100)(99)}{2}(2) \frac{d^{98}}{dx^{98}} e^{-x}$$

now need to find $\dfrac{d^m}{dx^n} e^{-x}$. by trying few terms I see

$$\left.\begin{aligned} \frac{d}{dx} e^{-x} &= -e^{-x} \\ \frac{d^2}{dx^2} e^{-x} &= e^{-x} \\ \frac{d^3}{dx^3} e^{-x} &= -e^{-x} \end{aligned}\right\} \text{ so } \frac{d^m}{dx^m} e^{-x} = -e^{-x} \text{ when } m \text{ is even}$$
$$= e^{-x} \text{ when } m \text{ is odd.}$$

hence Result $= x^2(+e^{-x}) + 200x(\cancel{e^{-x}}) + 9900(+e^{-x}) = \boxed{e^{-x}(+9900 + x^2) - e^{-x}(200x)}$

Ch 12

$\boxed{4.3}$ Find $P_0(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$ and $P_4(x)$ from Rodrigues formula (4.1). Compare your solution with (2-8) and problem 2.1.

Rodrigues formula generates Legendre's polynomials for different l values and given by:

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2-1)^l$$

$$P_0(x) = \frac{1}{2^0 \; 0!} \frac{d^0}{dx^0}(x^2-1)^0 = \boxed{1}$$

$$P_1(x) = \frac{1}{2 \; 1!} \frac{d}{dx}(x^2-1)^1 = \frac{1}{2}(2x) = \boxed{x}$$

$$P_2(x) = \frac{1}{2^2 2!} \frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8}\frac{d}{dx}\left(\frac{d}{dx}(x^2-1)^2\right)$$

$$= \frac{1}{8}\frac{d}{dx}\left(2(x^2-1)\cdot 2x\right) = \frac{1}{8}\frac{d}{dx}\left(4x^3-4x\right) = \frac{1}{8}\left(12x^2-4\right)$$

$$= \boxed{\frac{1}{2}(x^2-1)}$$

$$P_3(x) = \frac{1}{2^3 \; 3!}\frac{d^3}{dx^3}(x^2-1)^3 = \frac{1}{8\cdot 6}\frac{d^2}{dx^2}\left(\frac{d}{dx}(x^2-1)^3\right)$$

$$= \frac{1}{48}\frac{d^2}{dx^2}\left(3(x^2-1)^2\cdot 2x\right) = \frac{1}{48}\frac{d^2}{dx^2}\left(6x\left(x^4-2x^2+1\right)\right)$$

$$= \frac{1}{48}\frac{d^2}{dx^2}\left(6x^5-12x^3+6x\right) = \frac{1}{48}\frac{d}{dx}\left(\frac{d}{dx}\left(6x^5-12x^3+6x\right)\right)$$

$$\cancel{= \frac{1}{48}\frac{d}{dx}\left(18x^2-24x+6\right) = \frac{1}{48}\left(36x-24\right)}$$

$$= \frac{1}{48}\frac{d}{dx}\left(30x^4-36x^2+6\right) = \frac{1}{48}\left(120x^3-72x\right) = \boxed{\frac{1}{2}\left(5x^3-3x\right)}$$

→ back

$$\frac{d^2}{dx^2}(x^2-1)^3 = \frac{d}{dx}\left(3(x^2-1)^2\,2x\right) = \frac{d}{dx}\left(6x^5-12x^3+6x\right)$$

$$= 30x^4 - 36x^2 + 6$$

$$\frac{d^3}{dx^3}(x^2-1)^3 = \frac{d}{dx}\left(\qquad \right) = 120x^3 - 72x$$

$$\frac{d^4}{dx^4}(x^2-1)^3 = \frac{d}{dx}\left(\qquad \right) = \overset{360}{\cancel{720}}x^2 - 72$$

so now plug above into ① we get:

$$=(x^2-1)(360x^2-72) + 8x(120x^3-72x) + 12(30x^4-36x^2+6)$$

$$= 144 - 1440x^2 + \cancel{\ldots} + 1680x^4$$

$$\text{so } P_4(x) = \frac{1}{2^4 4!}\left(\qquad \right)$$

$$= \frac{1}{384}\left(\qquad \right)$$

$$\text{so } P_4(x) = \frac{1}{384}\left(1680x^4 - 1440x^2 + 144\right)$$

$$\boxed{P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)}$$

this result agrees with result obtained in 2.1

(15)

ch 12

$\boxed{4.4}$ show that $\int_{-1}^{1} x^m P_\ell(x)\, dx = 0$ if $m<\ell$.

4/5

use Rodrigues formula, write $P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell}(x^2-1)^\ell$

let $K = \frac{1}{2^\ell \ell!}$ so $P_\ell(x) = K \frac{d^\ell}{dx^\ell}(x^2-1)^\ell$.

hence integral is $K\int_{-1}^{1} x^m \frac{d}{dx}\left(\frac{d^{\ell-1}}{dx^{\ell-1}}(x^2-1)^\ell\right) dx = K\int_{-1}^{1} x^m\, d\left(\frac{d^{\ell-1}}{dx^{\ell-1}}(x^2-1)^\ell\right)$

apply integration by parts: $\int u\, dv = uv - \int v\, du$.

$u = x^m \Rightarrow du = m x^{m-1}$

$dv = d\left(\frac{d^{\ell-1}}{dx^{\ell-1}}(x^2-1)^\ell\right) \Rightarrow v = \frac{d^{\ell-1}}{dx^{\ell-1}}(x^2-1)^\ell$

hence integral $= \underbrace{K\left[x^m \frac{d^{\ell-1}}{dx^{\ell-1}}(x^2-1)^\ell\right]_{-1}^{1}} - Km\int_{-1}^{1} x^{m-1}\frac{d^{\ell-1}}{dx^{\ell-1}}(x^2-1)^\ell$ (1)

now I show that $\rightarrow$ is zero.

looking at $\frac{d^{\ell-1}}{dx^{\ell-1}}(x^2-1)^\ell \Rightarrow$ look at $(x^2-1)^\ell$. write as $(x-1)^\ell(x+1)^\ell$

~~differentiate this we get $\ell(x^2-1)^\ell$~~

~~... twice we get~~

so $\frac{d^{\ell-1}}{dx^{\ell-1}}(x^2-1)^\ell = \frac{d^{\ell-1}}{dx^{\ell-1}}(x-1)^\ell(x+1)^\ell$

apply leibniz rule for differentiation of products. $\frac{d^n}{dx^n}ab = a\frac{d^n}{dx^n}b + n\frac{d}{dx}a\frac{d^{n-1}}{dx^{n-1}}b + \cdots$

$= (x+1)^\ell \frac{d^{\ell-1}}{dx^{\ell-1}}(x-1)^\ell + (\ell-1)\frac{d}{dx}(x+1)^\ell \frac{d^{\ell-2}}{dx^{\ell-2}}(x-1)^\ell + \binom{\ell-2}{2}\frac{d^2}{dx^2}(x+1)^\ell \frac{d^{\ell-3}}{dx^3}(x-1)^\ell$

$+ \cdots$

I'm not sure how the result follows.... I think the idea is that $\frac{d^k}{dx^k}(x+1)^\ell \sim (x+1)^{\ell-k} + \cdots$ which is zero when $x=-1$. Similar result for $\frac{d^k}{dx^k}(x-1)^\ell$ (for $k<\ell$)

now $\frac{d^2}{dx^2}(x+1)^\ell = \frac{d}{dx}\left(\frac{d}{dx}(x+1)^\ell\right) = \frac{d}{dx}(\ell) = 0$ hence in the above we are left with

every term in the expansion above is a product of such terms, hence vanishes at both $x=1$ and $x=-1$. $\longrightarrow$

$\frac{d^{\ell-1}}{dx^{\ell-1}}(x^2-1)^\ell = (x+1)^\ell \frac{d^{\ell-1}}{dx^{\ell-1}}(x-1)^\ell + (\ell-1)\ell \frac{d^{\ell-2}}{dx^{\ell-2}}(x-1)^\ell$

now going back to equation ①, we have

$$\int_{-1}^{1} x^m P_\ell(x)\,dx = \underbrace{K\left[x^m \frac{d^{\ell-1}}{dx^{\ell-1}}(x^2-1)^\ell\right]_{-1}^{1}}_{\text{this is zero as shown.}} - Km\int_{-1}^{1} x^{m-1}\frac{d^{\ell-1}}{dx^{\ell-1}}(x^2-1)^\ell\,dx$$

so $\displaystyle\int_{-1}^{1} x^m P_\ell(x)\,dx = -Km\int_{-1}^{1} x^{m-1}\frac{d^{\ell-1}}{dx^{\ell-1}}(x^2-1)^\ell\,dx$

Now, apply integration by parts again to this

$$= -Km\int_{-1}^{1} x^{m-1}\frac{d}{dx}\left(\frac{d^{\ell-2}}{dx^{\ell-2}}(x^2-1)^\ell\right)dx = -Km\int_{-1}^{1} x^{m-1}\,d\left(\frac{d^{\ell-2}}{dx^{\ell-2}}(x^2-1)^\ell\right)$$

as before, we get the $[uv]-\int v\,du$, and as before, the $[uv]$ term reduced to zero.

hence each time we apply integration by parts, $x^m \to x^{m-1}$ and

$$\frac{d^\ell}{dx^\ell}(x^2-1)^\ell \to \frac{d^{\ell-1}}{dx^{\ell-1}}(x^2-1)^\ell.$$

This is a race between m an ℓ.

if $m<\ell$, then we can terminate integration by parts with $\displaystyle\int_{-1}^{1}(\text{some constant})\frac{d^n}{dx^n}(x^2-1)^\ell\,dx$

but $\displaystyle\int_{-1}^{1}\frac{d}{dx}\left(\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^\ell\right)dx = \left[\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^\ell\right]_{-1}^{1}$

by the fundamental theory of calculus.

but the expression we have shown to be zero. Hence this completes the proof.

so $\int_{-1}^{1} x^m P_\ell(x)\,dx = 0$ if $m<\ell$.

ch 12

$\boxed{5.1}$ Find $P_3(x)$ by getting one more term in the generating function expansion 5.3.

$$\phi(x,h) = \frac{1}{(1-2xh+h^2)^{1/2}} \qquad |h|<1 \qquad (1)$$

$$\phi(x,h) = P_0(x) + h\,P_1(x) + h^2 P_2(x) + h^3 P_3(x) + \cdots \; h^l P_l(x) + \cdots$$

expand (1) in power series. let $y = 2xh - h^2$, then (1) can be written as

$\phi(y) = \phi(x,h) = (1-y)^{-1/2}$. expand $\phi(y)$ as taylor series around $y=0$

$$\phi(y) = (1-y)^{-1/2} \Rightarrow 1 \quad \text{at } y=0$$

$$\phi'(y) = -\tfrac{1}{2}(1-y)^{-3/2}(-1) \Rightarrow +\tfrac{1}{2} \quad \text{at } y=0$$

$$\phi''(y) = +\tfrac{1}{2}(-\tfrac{3}{2})(1-y)^{-5/2}(-1) \Rightarrow +\frac{3}{2^2} \quad \text{at } y=0$$

$$\phi'''(y) = +\tfrac{1}{2}(+\tfrac{3}{2})(-5/2)(1-y)^{-7/2}(-1) \Rightarrow +\frac{3\cdot 5}{2^3} \quad \text{at } y=0$$

$$\text{so } \phi(y) = \phi(\bar{y}) + \phi'(\bar{y})\,y + \frac{\phi''(\bar{y})\,y^2}{2!} + \frac{\phi'''(\bar{y})\,y^3}{3!} + \cdots$$

$$= 1 + (+\tfrac{1}{2})y + \frac{3}{4}\frac{1}{2!}y^2 + \left(+\frac{3\cdot 5}{8}\right)\frac{1}{3!}y^3 + \cdots$$

$$\phi(y) = 1 + \tfrac{1}{2}y + \frac{3}{8}y^2 + \frac{15}{48}y^3 + \cdots$$

now replace y with $2xh - h^2$ we get

$$\phi(x,h) = 1 + \tfrac{1}{2}(2xh-h^2) + \tfrac{3}{8}(2xh-h^2)^2 + \frac{15}{48}(2xh-h^2)^3 + \cdots$$

$$= 1 + xh - \frac{h^2}{2} + \frac{3}{8}\left(4x^2h^2 - 4xh^3 + h^4\right) + \frac{15}{48}\left((2xh-h^2)^2(2xh-h^2)\right)$$

$$= 1 + xh - \frac{h^2}{2} + \frac{12}{8}x^2h^2 - \frac{3}{2}xh^3 + \frac{3}{8}h^4 + \frac{15}{48}\left((4x^2h^2 - 4xh^3 + h^4)(2xh-h^2)\right)$$

$\longrightarrow$

Ch 12

$\boxed{5.3}$ use recursion relation $l P_l(x) = (2l-1)x\, P_{l-1}(x) - (l-1)P_{l-2}(x)$

and the values P_0 and P_1 to find P_2, P_3, P_4, P_5, P_6.

$P_0 = 1$

$P_1 = x$

so for P_2, $l=2$. hence from the recursion formula

$$2P_2 = (4-1)x\,P_1 - P_0 = 3x(x) - 1 = 3x^2 - 1$$

i.e. $P_2 = \boxed{\frac{1}{2}(3x^2-1)}$

now set $l=3$

so $$3P_3 = 5x\,P_2 - P_1 = 5x\left(\tfrac{1}{2}(3x^2-1)\right) - 2x$$

$$= 5x\left(\tfrac{3}{2}x^2 - \tfrac{1}{2}\right) - 2x = \frac{15x^3}{2} - \frac{5x}{2} - 2x = \frac{15x^3}{2} - \frac{(5+4)x}{2}$$

$$3P_3 = \frac{15x^3}{2} - \frac{9}{2}x = \frac{1}{2}\left(15x^3 - 9x\right)$$

so $\boxed{P_3 = \frac{1}{2}\left(5x^3 - 3x\right)}$

For P_4, $l = 4$.

so $$4P_4 = 7x\,P_3 - 3P_2 = 7x\left(\tfrac{5}{2}x^3 - \tfrac{3}{2}x\right) - 3\left(\tfrac{3x^2}{2} - \tfrac{1}{2}\right)$$

$$= \frac{35}{2}x^4 - \frac{21}{2}x^2 - \frac{9x^2}{2} + \frac{3}{2}$$

$$= \frac{35}{2}x^4 - \frac{30}{2}x^2 + \frac{3}{2}$$

so $$P_4 = \frac{35}{8}x^4 - \frac{30}{8}x^2 + \frac{3}{8}$$

$$= \boxed{\frac{1}{8}\left(35x^4 - 30x^2 + 3\right)}$$

$$= 1 + \underbrace{xh}_{} + h^2\underbrace{\left(-\frac{1}{2} + \frac{3}{2}x^2\right)}_{} - \frac{3}{2}xh^3 + \frac{3}{8}h^4 + \frac{15}{48}\left(8x^3h^3 - 4x^2h^4 - 8x^2h^4 + 4xh^5 + 2xh^5 - h^6\right)$$

do not care for P_3

$$= P_0 + hP_1 + h^2 P_2 + h^3\left(-\frac{3}{2}x + \frac{15}{48}x^3\, 8\right) + h^4\left(\cdots\cdots\right) + \cdots$$

$$= P_0 + hP_1 + h^2 P_2 + h^3\left(-\frac{3}{2}x + \frac{15}{6}x^3\right) + \cdots$$

$$= P_0 + hP_1 + h^2 P_2 + h^3\ \frac{1}{2}\left(\frac{15}{3}x^3 - 3x\right) + \cdots$$

$$= P_0 + hP_1 + h^2 P_2 + h^3\ \underbrace{\frac{1}{2}\left(5x^3 - 3x\right)}_{\text{but this is } P_3(x).} + \cdots$$

hence $= P_0(x) + h\, P_1(x) + h^2\, P_2(x) + h^3\, P_3(x) + \cdots$

for P_5, $l=5$

so $5P_5 = 9xP_4 - 4P_3$

$$= 9x\left(\frac{35}{8}x^4 - \frac{30}{8}x^2 + \frac{3}{8}\right) - 4\left(\frac{5}{2}x^3 - \frac{3}{2}x\right)$$

$$= \frac{315x^5}{8} - \frac{270x^3}{8} + \frac{27x}{8} - \frac{20x^3}{2} + \frac{12x}{2}$$

$$\cancel{= \frac{315x^5}{8} - 240x^3}$$

$$= \frac{315}{8}x^5 - \frac{270x^3}{8} - \frac{80x^3}{8} + \frac{27x}{8} + \frac{48x}{8}$$

$$5P_5 = \frac{315}{8}x^5 - \frac{350x^3}{8} + \frac{75x}{8}$$

so $P_5 = \frac{315x^5}{5\cdot 8} - \frac{350x^3}{5\cdot 8} + \frac{75x}{5\cdot 8} = \frac{63x^5}{8} - \frac{70}{8}x^3 + \frac{15x}{8}$

$$\boxed{P_5 = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right)}$$

for P_6, $l=6$

so

$6P_6 = 11xP_5 - 5P_4$

$$= 11x\left(\frac{63}{8}x^5 - \frac{70}{8}x^3 + \frac{15}{8}x\right) - 5\left(\frac{35}{8}x^4 - \frac{30}{8}x^2 + \frac{3}{8}\right)$$

$$= \left(\frac{693x^6}{8} - \frac{770}{8}x^4 + \frac{165}{8}x\right) - \frac{175}{8}x^4 + \frac{150x^2}{8} - \frac{15}{8}$$

$$\Rightarrow P_6 = \frac{1}{16}\left(231x^6 - 315x^4 + 105x^2 - 5\right)$$

ch 12

$\boxed{5.5}$ Differentiate (5.8a) and use recursion relation 5.8b with l replaced by $l-1$ to prove 5.8c.

5.8a is given by $l P_l(x) = (2l-1)xP_{l-1} - (l-1)P_{l-2}$

5.8b is
$$xP_l' - P_{l-1}' = lP_l$$

5.8c is
$$P_l' - xP_{l-1}' = l P_{l-1}$$

differentiate 5.8a, we get

$$lP_l' = (2l-1)xP_{l-1}' + (2l-1)P_{l-1} - (l-1)P_{l-2}' \quad \text{—} ①$$

from 5.8b, replace l by $l-1$, we get

$$xP_{l-1}' - P_{l-2}' = (l-1)P_{l-1}$$

or

$$P_{l-2}' = xP_{l-1}' - (l-1)P_{l-1} \quad \text{———} ②$$

Plug ② into ① to remove P_{l-2}' term in ①, we get

$$lP_l' = (2l-1)xP_{l-1}' + (2l-1)P_{l-1} - (l-1)\left[xP_{l-1}' - (l-1)P_{l-1}\right]$$

expand and simplify:

$$lP_l' = 2lxP_{l-1}' - xP_{l-1}' + 2lP_{l-1} - P_{l-1} - lxP_{l-1}' + l(l-1)P_{l-1} + xP_{l-1}' - (l-1)P_{l-1}$$

$\longrightarrow$

$$\ell P_\ell' = 2\ell x P_{\ell-1}' \cancel{- x P_{\ell-1}'} \cancel{+ 2\ell P_{\ell-1}} \cancel{- P_{\ell-1}} - \ell x P_{\ell-1}' + \ell^2 P_{\ell-1} \cancel{- \ell P_{\ell-1}} \cancel{+ x P_{\ell-1}'} \cancel{- \ell P_{\ell-1}} \cancel{+ P_{\ell-1}}$$

$$\ell P_\ell' = 2\ell x P_{\ell-1}' - \ell x P_{\ell-1}' + \ell^2 P_{\ell-1}$$

$$P_\ell' = x P_{\ell-1}' + \ell P_{\ell-1}$$

or $$\boxed{P_\ell' - x P_{\ell-1}' = \ell P_{\ell-1}}$$

ch 12

$\boxed{5.6}$ From 5.8b and 5.8c obtain 5.8d. Then differentiate 5.8d and eliminate P'_{l-1} using 5.8b. your result should be the legendre equation.

5.8b : $x P'_l - P'_{l-1} = l P_l$

5.8c : $P'_l - x P'_{l-1} = l P_{l-1}$

5.8d: $(1-x^2) P'_l = l P_{l-1} - l x P_l$

multiply 5.8b by x and 5.8c − 5.8b leads to

$$x^2 P'_l - x P'_{l-1} = x l P_l$$
$$P'_l - x P'_{l-1} = l P_{l-1}$$

$$\boxed{(1-x^2) P'_l = l P_{l-1} - x l P_l}$$ which is 5.8d.

differentiate 5.8d, we get

$$(1-x^2) P''_l + P'_l(-2x) = l P'_{l-1} - \left[x l P'_l + l P_l \right]$$

$$(1-x^2) P''_l - 2x P'_l = l P'_{l-1} - x l P'_l - l P_l$$

eliminate P'_{l-1} in above equation by using 5.8b

from 5.8b, $P'_{l-1} = x P'_l - l P_l$. hence subtitute in to get.

$$(1-x^2) P''_l - 2x P'_l = l \left[x P'_l - l P_l \right] - x l P'_l - l P_l$$

$$(1-x^2) P''_l - 2x P'_l = \cancel{l x P'_l} - l^2 P_l - \cancel{x l P'_l} - l P_l$$

$$\boxed{(1-x^2) P''_l - 2x P'_l + l(l+1) P_l = 0}$$ which is the legendre equation.

ch 12

$\boxed{5.7}$ write 5.8c with l replaced by $l+1$ and use it to eliminate the xP_l' term in 5.8b. you should get 5.8e.

5.8c: $P_l' - xP_{l-1}' = l\,P_{l-1}$

5.8b: $xP_l' - P_{l-1}' = lP_l$

5.8e: $(2l+1)\,P_l = P_{l+1}' - P_{l-1}'$

replace l with $l+1$ in 5.8c, we get

$$P_{l+1}' - xP_l' = (l+1)\,P_l$$

$$\Rightarrow xP_l' = P_{l+1}' - (l+1)\,P_l \qquad (1)$$

sub (1) into 5.8b

$$\left[P_{l+1}' - (l+1)\,P_l\right] - P_{l-1}' = lP_l$$

$$P_{l+1}' - (l+1)P_l - P_{l-1}' = lP_l$$

$$P_{l+1}' - P_{l-1}' = lP_l + (l+1)P_l$$

$$= lP_l + lP_l + P_l$$

$$= 2lP_l + P_l$$

$$\boxed{P_{l+1}' - P_{l-1}' = P_l\,(2l+1)}$$

which is 5.8e.

ch 12

$\boxed{5.11}$ express $x - x^3$ as a linear combination of legendre polynomials.

$$f(x) = x - x^3$$

$$P_3 = \frac{1}{2}(5x^3 - 3x)$$

so $P_3 = \frac{5}{2}x^3 - \frac{3}{2}x$

$$\frac{5}{2}x^3 = P_3 + \frac{3}{2}x$$

$$\boxed{x^3 = \frac{2}{5}P_3 + \frac{3}{5}x}$$

so $f(x) = x - \left[\frac{2}{5}P_3 + \frac{3}{5}x\right] = -\frac{2}{5}P_3 + x - \frac{3}{5}x = -\frac{2}{5}P_3 + \frac{2}{5}x$

$f(x) = \frac{2}{5}(x - P_3)$ ①

~~now $P_1 = 1 - x$ or $x = 1 - P_1$~~

~~hence $f(x) = \frac{2}{5}((1-P_1) - P_3) = \frac{2}{5}(1 - P_1 - P_3)$~~

opps.

~~i.e $\boxed{x - x^3 = \frac{2}{5}(1 - P_1 - P_3)}$~~

now $P_1 = x$, hence from ① we set

so $f(x) = \frac{2}{5}(P_1 - P_3)$

i.e $\boxed{x - x^3 = \frac{2}{5}(P_1 - P_3)}$