HW 12

MATH 121B

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Nasser M. Abbasi

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Contents

1 chapter 16, problem 3.3, Mary Boas, second edition

What is the probability of getting the sequence *hhhttt* in six tosses of a coin? If you know the first 3 are heads, what is the probability that the last 3 are tails? If you do not know anything about the first three, what is the probability. that the last three are tails?

Solution

First part:Looking at the sequence pattern we can get out of a six tosses of a coin, we see that there are a total of 2^6 different sequences (since there are 2 choices at each position, and there are 6 positions), this and the fact that each output of a toss of a coin is independent of the output of the previous toss, means the chance of any one specific sequence is the same as any other. Hence the chance of getting *hhhttt* will be $\frac{1}{2^6} = \frac{1}{64}$

Second part: Now, if we know the first 3 are heads, we can solve this in 2 ways.

The first way: Since the first 3 positions are now known, then the total number of different sequences we have to look at is reduced from 2^6 to 2^3, hence the chance of getting a *ttt* is $\frac{1}{2^3} = \frac{1}{8}$

The second way: Let A be the event of getting 3 heads in the first 3 tosses. Let B be the event of getting 3 tails in the last 3 tosses. Hence we want to find $P_A(B)$

But since A and B are independent events, $P_A(B) = P(B)$

So $P_A(B) = P(B) = \frac{1}{2^3} = \frac{1}{8}$

Last part, here we do not know anything about the first 3 tosses. So the first 3 positions in the sequence of length 6 are unknown. Only the last 3 positions of the sequence are known which are *ttt*. This means again that there is a chance of $\frac{1}{2^3} = \frac{1}{8}$ that the last 3 are *ttt*

2 chapter 16, problem 3.5, Mary Boas, second edition

What is the probability that a number $n, 1 \leq n \leq 99$ is divisible by both 6 and 10? By either 6 or 10 or both?

Solution

Let A=event that a number is divisible by 6. Hence $P(A) = \frac{16}{99}$ since there are 16 numbers between 1 and 99 that are divisible by 6.

Let B=event that a number is divisible by10. Hence $P(B) = \frac{9}{99}$ since there are 9 numbers between 1 and 99 that are divisible by 10.

First part: We want $P(AB)$ since these 2 events are dependent on each others, then

$$P(AB) = P(A)\ \ P_A(B)$$

Now to find $P_A(B)$, this is the event a number is divisible by 10 given it is divisible by 6. There are 3 numbers divisible by 10 out of the 16 numbers that are divisible by 6, and they are $30, 60, 90$. Hence $P_A(B) = \frac{3}{16}$

So

$$P(AB) = P(A)\ \ P_A(B) = \left(\frac{16}{99}\right)\left(\frac{3}{16}\right) = \frac{3}{99} = \frac{1}{33}$$

For the second part:

Here we want to find

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(AB) \\ &= \left(\frac{16}{99}\right) + \left(\frac{9}{99}\right) - \left(\frac{1}{33}\right) \\ &= \frac{2}{9} \end{aligned}$$

3 chapter 16, problem 3.10, Mary Boas, second edition

3 letters and their envelopes are piled on a desk. If someone puts the letters in the envelopes at random, what is the probability that each letter gets into its own envelop?

Solution

(a) Set up the sample space. Let envelopes be A,B,C, let letters be a,b,c.

Each sample point is one row in the following table.

A	B	C
a	**b**	**c**
a	c	b
b	a	c
b	c	a
c	a	b
c	b	a

From this table, we see that the probability that each letter gets into its own envelop is $\frac{1}{6}$, which means one row above meets this condition out of 6 rows. (The first row)

Another way to solve this: There is $\frac{1}{3}$ chance of a letter getting into the correct envelop. This leaves 2 letters and 2 envelopes, now we have a chance of $\frac{1}{2}$ of one of the 2 remaining letters going into one of the 2 remaining envelopes. After this, we have one letter and one envelope, which have a 1 chance of getting into the right envelop. Hence the total probability is $\frac{1}{3} \times \frac{1}{2} \times 1 = \frac{1}{6}$ the same as stated above.

(b) What is the probability that at least one letter gets into its own envelope?

From looking at the table we see that this probability is $\frac{4}{6}$

This could be solved using probability calculus as well like this: Let n be the number of envelopes or letters. Then the probability P of of all letters going to the wrong envelopes will be

$$P = 1 - \left[n\frac{(n-1)!}{n!} - \begin{pmatrix} n \\ 2 \end{pmatrix} \frac{(n-2)!}{n!} + \begin{pmatrix} n \\ 3 \end{pmatrix} \frac{(n-3)!}{n!} \right]$$

$$= 1 - \left[3\frac{(3-1)!}{3!} - \begin{pmatrix} 3 \\ 2 \end{pmatrix} \frac{(3-2)!}{3!} + \begin{pmatrix} 3 \\ 3 \end{pmatrix} \frac{(3-3)!}{3!} \right]$$

$$= 1 - \left[3 \times \frac{2}{6} - \frac{6}{2}\frac{1}{6} + \frac{1}{6} \right]$$

$$= 1 - \left[1 - \frac{1}{2} + \frac{1}{6} \right]$$

$$= 1 - \left[\frac{4}{6} \right]$$

$$= \frac{2}{6}$$

Hence the probability of at least one letter going to the correct envelop is $1 - P = 1 - \frac{2}{6} = \frac{4}{6}$ which agrees with the answer above obtained by direct counting from the table.

(c) Let A mean that a got into envelop A and so on. Find the probability $P(A)$ (i.e. that a got into A.) Find $P(B)$ and $P(C)$. Find $P(A+B)$ (means a or b got into correct envelopes). Find $P(AB)$ (meaning both a and b got into correct envelopes. Verify equation 3.6

Since there are envelopes, then $P(A) = \frac{1}{3}$

Similarly for $P(B)$ and $P(C)$

Now, $P(A+B) = \frac{1}{2}$ by looking at table above, we see that rows 1, 2 and 6 meet this criteria. so 3 sample points out of 6.

To find $P(AB)$: There are $n!$ ways to arrange n envelops, and there are $(n-2)!$ ways to arrange the remaining letters (after 2 letters got into the correct envelopes). Hence the chance of 2 letters getting into the correct envelopes is $\frac{(n-2)!}{n!} = \frac{(3-2)!}{3!} = \frac{1}{6}$

from table, this is verified by seeing that only one row out of 6 meets this condition. Hence $P(AB) = \frac{1}{6}$

To verify 3.6 which says $P(A+B) = P(A) + P(B) - P(AB)$, then substitute values found, we get for LHS $\frac{1}{2}$, and for RHS we get $\frac{1}{3} + \frac{1}{3} - \frac{1}{6} = \frac{1}{2}$, hence equation verified.

4 chapter 16, problem 3.14, Mary Boas, second edition

A player succeeds in making a basket 3 tries out of 4. How many tries are needed to have a probability of larger than 0.99 of at least one basket?

Solution

Let P be the probability of scoring at each try (in this example, $P = \frac{3}{4}$).

Let E_1=event of scoring in the first try.

Let E_2=event of scoring in the second try.

To make notations shorts, let me call P_n as the probability of even E_n occurring.

Hence $P(E_1)$ will be written as P_1

So chance of scoring after 2 tries =

$$\begin{aligned} P(E_1 + E_2) &= P(E_1) + P(E_2) - P(E_1 E_2) \\ &= P_1 + P_2 - P(E_1 E_2) \end{aligned} \tag{1}$$

But events E_1, E_2 here are independent of each others, hence $P(E_1 E_2) = P(E_1)P(E_2) = P_1 P_2$

So (1), the chance of scoring after 2 tries, can now be written as:

$$P(E_1 + E_2) = P_1 + P_2 - P_1 P_2 \tag{2}$$

Similarly, the chance of scoring after 3 tries is

$$\begin{aligned} P((E_1 + E_2) + E_3) &= P(E_1 + E_2) + P(E_3) - P((E_1 + E_2)E_3) \\ &= P(E_1 + E_2) + P(E_3) - P(E_1 + E_2)\, P(E_3) \end{aligned} \tag{3}$$

Substitute (2) into (3) we get

$$\begin{aligned} P((E_1 + E_2) + E_3) &= P_1 + P_2 - P_1 P_2 + P_3 - [(P_1 + P_2 - P_1 P_2) P_3] \\ &= P_1 + P_2 - P_1 P_2 + P_3 - [P_1 P_3 + P_2 P_3 - P_1 P_2 P_3] \\ &= P_1 + P_2 + P_3 - P_1 P_2 - P_1 P_3 - P_2 P_3 + P_1 P_2 P_3 \\ &= P_1 + P_2 + P_3 - (P_1 P_2 + P_1 P_3 + P_2 P_3) + P_1 P_2 P_3 \end{aligned}$$

Now since each P_i is the same, which is $\frac{3}{4}$ in this example, I can write the above as

$$
\begin{aligned}
P((E_1+E_2)+E_3) &= P_1+P_2+P_3-(P_1P_2+P_1P_3+P_2P_3)+P_1P_2P_3\\
&= P+P+P-(PP+PP+PP)+PPP\\
&= 3P-3P^2+P^3
\end{aligned}
$$

The above is the probability of at least one score after 3 tries. We can continue this process, getting the probability of at least one score after 4 tries. This will result in the following formula.

$$
P((E_1+E_2+E_3)+E_4) = 4P-6P^2+4P^3-P^4
$$

So, the pattern is clear, in general, after n tries, the chance of at least one score is

$$
nP-\begin{pmatrix} n\\ 2\end{pmatrix}P^2+\begin{pmatrix} n\\ 3\end{pmatrix}P^3-\begin{pmatrix} n\\ 4\end{pmatrix}P^4+\cdots+\begin{pmatrix} n\\ n-1\end{pmatrix}P^{n-1}-\ P^n
$$

Now I need to find n which will cause the above chance of getting a value larger than 0.99

So need to solve the above for $n = .99$ and then take the next n after that.

i.e. need to solve

$$
nP-\begin{pmatrix} n\\ 2\end{pmatrix}P^2+\begin{pmatrix} n\\ 3\end{pmatrix}P^3-\begin{pmatrix} n\\ 4\end{pmatrix}P^4+\cdots+\begin{pmatrix} n\\ n-1\end{pmatrix}P^{n-1}-\ P^n = .99
$$

I do not know how to solve the above as is, so I'll just make trial and error.

Will try $n = 2$, for $n = 2$, the LHS of the above equation is

$$
\begin{aligned}
2P-\begin{pmatrix} 2\\ 2\end{pmatrix}P^2 &= 2\times\frac{3}{4}-\left(\frac{3}{4}\right)^2\\
&= 0.9375
\end{aligned}
$$

This is still less than 0.99. So try for larger n, for $n = 3$

$$
\begin{aligned}
3P-\begin{pmatrix} 3\\ 2\end{pmatrix}P^2+\begin{pmatrix} 3\\ 3\end{pmatrix}P^3 &= 3\left(\frac{3}{4}\right)-3\left(\frac{3}{4}\right)^2+\left(\frac{3}{4}\right)^3\\
&= 0.98438
\end{aligned}
$$

This is still less than 0.99 so try for $n = 4$

$$4P - \begin{pmatrix} 4 \\ 2 \end{pmatrix} P^2 + \begin{pmatrix} 4 \\ 3 \end{pmatrix} P^3 - \begin{pmatrix} 4 \\ 4 \end{pmatrix} P^4 = 4\left(\frac{3}{4}\right) - 6\left(\frac{3}{4}\right)^2 + 4\left(\frac{3}{4}\right)^3 - \left(\frac{3}{4}\right)^4$$
$$= 0.99609$$

This is larger than 0.99, hence the player needs to try 4 tries

Another way to solve the problem is

Let P be the probability of scoring at each try (in this example, $P = \frac{3}{4}$).

So probability of not scoring is $1 - P = \frac{1}{4}$

So probability of not scoring after n tries is $\left(\frac{1}{4}\right)^n$

So need to solve the equation

$$\left(\frac{1}{4}\right)^n = .01$$
$$n \log\left(\frac{1}{4}\right) = \log(.01)$$
$$n = \frac{\log(.01)}{\log(.25)}$$
$$= 3.3219$$

Hence $n = 4$ (the next higher integer value).

5 chapter 16, problem 3.15, Mary Boas, second edition

Use Baye's formula 3.8 to repeat these simple problems previously done using reduced sample space method

(a) In a family of children, what is the chance that both are girls if one is girl?

(b) What is the chance of all heads in a 3 tosses of a coin if you know that at least one is head?

Solution

The sample space here is $\{gg, gb, bg, bb\}$

Let A=event that both are girls

Let B=event that at least one is a girl

(a) We want to find $P_B(A)$

$$P_B(A) = \frac{P(AB)}{P(B)} \tag{1}$$

Now $P(B) = \frac{3}{4}$

To find $P(AB)$, I can not write $P(AB) = P(A)P(B)$ since here these 2 events are not independent. Here the probability of A implies B, hence $P(AB) = P(A) = \frac{1}{4}$

So substitute into (1)

$$\begin{aligned} P_B(A) &= \frac{\frac{1}{4}}{\frac{3}{4}} \\ &= \frac{1}{3} \end{aligned}$$

(b) What is the chance of all heads in a 3 tosses of a coin if you know that at least one is head?

Let A=event of 3 heads.

Let B=event of at least one head.

$\{hhh, hth, hht, htt, thh, tht, tth, ttt\}$

$$P_B(A) = \frac{P(BA)}{P(B)} \tag{1}$$

But $P(B) = \frac{7}{8}$

But $P(BA)$ is the probability. of 3 heads and at least one head. So events are not independent. So $P(BA)$ is the same as $P(A)$ which is $\frac{1}{8}$

Substitute into (1) we get

$$\begin{aligned} P_B(A) &= \frac{\frac{1}{8}}{\frac{7}{8}} \\ &= \frac{1}{7} \end{aligned}$$

A long way to solve the above is

The sample space here is $\{gg, gb, bg, bb\}$

Let A=event that both are girls

Let B=event that at least one is a girl

(a)We want to find $P(A|B)$

$$P(A|B) = \frac{P(A)\ P(B|A)}{P(A)\ P(B|A)\ + P(not\ A\ )\ P(B|not\ A)} \tag{1}$$

Now $P(A) = \frac{1}{4} \rightarrow P(not\ A) = \frac{3}{4}$

$P(B|A) = 1$ since given both are girl, then there is a 100% chance that one is a girl.

$P(B|not\ A) = \frac{2}{3}$, since when not both are girls, the sample space is $\{gb, bg, bb\}$, and from this, there is 2 sample points with at least a girl, hence $\frac{2}{3}$

Substitute into (1) we get

$$
\begin{aligned}
P(A|B) &= \frac{\frac{1}{4}\times 1}{\frac{1}{4}\times 1 + \frac{3}{4}\times\frac{2}{3}} \\
&= \frac{\frac{1}{4}}{\frac{1}{4} + \frac{2}{4}} \\
&= \frac{\frac{1}{4}}{\frac{3}{4}} \\
&= \frac{1}{3}
\end{aligned}
$$

(b)What is the chance of all heads in a 3 tosses of a coin if you know that at least one is head?

Let A=event of 3 heads.

Let B=event of at least one head.

$\{hhh, hth, hht, htt, thh, tht, tth, ttt\}$

$$
P(A|B) = \frac{P(A)\,P(B|A)}{P(A)\,P(B|A) + P(not\ A\,)\,P(B|not\ A)} \tag{2}
$$

so $P(A) = \frac{1}{7} \rightarrow P(not\ A) = \frac{6}{7}$

$P(B|A) = 1$

$P(B|not\ A) = \frac{6}{7}$

Substitute into (2) we get

$$
\begin{aligned}
P(A|B) &= \frac{\frac{1}{7}\times\frac{6}{7}}{\frac{1}{7}\times\frac{6}{7} + \frac{6}{7}\times\frac{6}{7}} \\
&= \frac{\frac{6}{49}}{\frac{6}{49} + \frac{36}{49}} \\
&= \frac{6}{42} \\
&= \frac{1}{7}
\end{aligned}
$$

6 chapter 16, problem 3.16, Mary Boas, second edition

Suppose you have 3 nickels and 4 dimes in your right pocket and 2 nickels and a quarter in your left pocket. You pick a pocket at random and from it select a coin at random. If it is a nickel, what it the probability it came from the right pocket?

Solution

Use Baye's formula.

Let A = event that the coin picked is a nickel.

Let B = event that the pocket selected was the right pocket.

We want to find

$$P_A(B) = \frac{P(BA)}{P(A)} \tag{1}$$

So need to find $P(A)$ and $P(BA)$

$P(A)$ is the probability of picking a nickel. But there are 2 pockets, so this probability is the probability of picking the nickel from the left pocket or the probability of picking the nickel from the right pocket.

Now the probability. of picking a nickel from the left pocket is the probability. of picking the left pocket and then picking a nickel from the left pocket. This is $\frac{1}{2} \times \frac{2}{3}$

Similarly, the probability. of picking a nickel from the right pocket is the probability. of picking the right pocket and then picking a nickel from the right pocket. This is $\frac{1}{2} \times \frac{3}{7}$
Hence $P(A) = \left(\frac{1}{2} \times \frac{2}{3}\right) + \left(\frac{1}{2} \times \frac{3}{7}\right) = \frac{23}{42}$

Now, need to find $P(BA)$ which is the probability of picking the right pocket and then a nickel was selected from the right pocket in this case. This is $\frac{1}{2} \times \frac{3}{7} = \frac{3}{14}$

Substitute these values in (1) we get

$$\begin{aligned} P_A(B) &= \frac{P(BA)}{P(A)} \\ &= \frac{\frac{3}{14}}{\frac{23}{42}} \\ &= \frac{9}{23} \end{aligned}$$

7 chapter 16, problem 3.17, Mary Boas , second edition

(a)There are 3 red and 5 black balls in one box and 6 red and 4 white balls in another. If you pick a box at random, and then pick a ball from it at random, what is the probability it is red? black? white? That it is either a red or a white?

Solution

Let E=Event of selecting the first box (one with 3 red and 5 black balls)

Let M=Event of selecting the second box.

Let R=Event of selecting a red ball.

Let B=Event of selecting a black ball.

Let W=Event of selecting a white ball.

(a) The probability of picking a red is the probability of picking the A box and then the probability selecting a red ball from it, OR the probability of picking the B box and then the probability of selecting a red ball from it.

Hence

$$P(R) = P(ER) + P(MR) \tag{1}$$

But $P(ER) = \frac{1}{2} \times \frac{3}{8}$ (since there are 3 red balls out of 8 in the E box)

and $P(MR) = \frac{1}{2} \times \frac{6}{10}$ since there are 6 red balls out of 10 in the M box

Hence (1) becomes

$$\begin{aligned} P(R) &= \frac{1}{2} \times \frac{3}{8} + \frac{1}{2} \times \frac{6}{10} \\ &= \frac{3}{16} + \frac{3}{10} \\ &= \frac{39}{80} \end{aligned}$$

Now to find $P(B)$ the probability of selecting a black ball.

Using similar logic as above, we get

$$P(B) = P(EB) + P(MB) \tag{2}$$

But $P(EB) = \frac{1}{2} \times \frac{5}{8}$ (since there are 5 black balls out of 8 in the E box)

and $P(MB) = \frac{1}{2} \times 0 = 0$ since there are zero black balls out of 10 in the M box

Hence (2) becomes

$$P(B) = \frac{1}{2} \times \frac{5}{8}$$
$$= \frac{5}{16}$$

Now to find $P(W)$ the probability of selecting a white ball.

Using similar logic as above, we get

$$P(W) = P(EW) + P(MW) \tag{3}$$

But $P(EW) = \frac{1}{2} \times 0$ (since there are zero white balls out of 8 in the E box)

and $P(MW) = \frac{1}{2} \times \frac{4}{10} = \frac{1}{5}$ since there are 4 white balls out of 10 in the B box

Hence (3) becomes

$$P(W) = 0 + \frac{1}{5}$$
$$= \frac{1}{5}$$

Now to find the probability that the ball selected is either a red or a white, we need to find $P(R) + P(W)$ but from above, this is $\frac{39}{80} + \frac{1}{5} = \frac{11}{16}$

(b)What is the probability of selecting a red on the second try given that we selected a red on the first try (without placing it back into the box?)

Let A=Event that a red ball was selected on first try

Let B=Event that a red ball was selected on second try.

Then we are asked to find $P_A(B)$

But by Bayes rule,

$$P_A(B) = \frac{P(AB)}{P(A)} \tag{1}$$

Where $P(AB)$ is the probability of selecting a red ball on the first try and a red ball on the second try (without replacement).

$P(A)$ was found above in part (a) which is $\frac{39}{80}$

Now I need to find $P(AB)$. To do this, I used a tree diagram which is shown below.

From this I find that $P(AB) = \frac{187}{840}$

Another method to find P(AB) is to say: it is the probability of selecting and red ball from first box and then the probability of selecting a red ball from the same box OR it is the probability of selecting and red ball from first box and then the probability of selecting a red ball from the second box OR it is the probability of selecting and red ball from second box and then the probability of selecting a red ball from the same box OR it is the probability of selecting and red ball from second box and then the probability of selecting a red ball from the first box. This will result in the following computation:

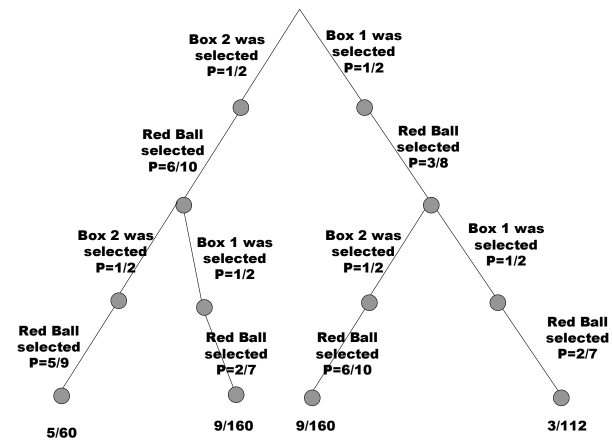
$$\left[\frac{1}{2}\left(\frac{3}{8}\right)\times\frac{1}{2}\left(\frac{2}{7}\right)\right]+\left[\frac{1}{2}\left(\frac{3}{8}\right)\times\frac{1}{2}\left(\frac{6}{10}\right)\right]+\left[\frac{1}{2}\left(\frac{6}{10}\right)\times\frac{1}{2}\left(\frac{5}{9}\right)\right]+\left[\frac{1}{2}\left(\frac{6}{10}\right)\times\frac{1}{2}\left(\frac{3}{8}\right)\right]=\frac{187}{840}$$

which agrees with the number I obtained from the tree diagram.

Another method to find P(AB) is to say: Assume the red ball came from Box A, now do the calculations to find the chance of picking a red ball on second try. Now multiply this by the chance of the assumption being true. We get a number, say x. Next, assume the red ball came from the second box B, now do the calculation to find the chance of picking a red ball on second try. Now multiply this by the chance of the assumption being true. call this number y. Now add $x + y$. and this is $P(AB)$. This is really the exact same thing as I did in the above alternative method, just expressed differently.

Hence, from (1) we finally get the conditional probability

$$P_A(B) = \frac{\frac{187}{840}}{\frac{39}{80}} = \boxed{\frac{374}{819}}$$



Tree diagram to find the probability of selecting a red on first try and a red on second try. The result is the sum of all the numbers at the leaves of the tree above. Hence the result is

5/60 + 9/160 + 9/160 + 3/112 = 187/840

(c) If both balls are red, what is the probability that they both came from the same box?

Let A=Event that both the first and second balls are red

Let B=Event that they both came from the same box

Hence we want to find

$$P_A(B) = \frac{P(AB)}{P(A)} \tag{1}$$

$P(A)$ is the probability of the first ball being red and then the second ball being red. From part(b) we found this probability to be $\frac{187}{840}$

Now $P(AB)$ is the probability that the first ball and the second ball both came from the same box and both balls are red.

Looking at the tree diagram above, I see that the 2 leaves that leads to this have the probability sum of $\frac{5}{60} + \frac{3}{112} = \frac{37}{336}$ (These are the right-most branch, and the left-most branch).

Hence (1) becomes

$$
\begin{aligned}
P_A(B) &= \frac{\frac{37}{336}}{\frac{187}{840}} \\
&= \frac{185}{374} \\
&= 0.49465
\end{aligned}
$$

8 chapter 16, problem 3.19, Mary Boas , second edition

Suppose it is known that 1% of the population have a certain kind of cancer. It is also known that a test of this kind of cancer is positive in 99% of the people who have it but is also positive in 2% of the people who do not have it. What is the probability that a person who tests positive has cancer of this type?

Solution

Let A=Event that a person has cancer

Let B=Event that a test is positive.

We want to find $P_B(A)$

Using Baye's rule

$$\begin{aligned} P_B(A) &= \frac{P(BA)}{P(A)} \\ &= \frac{P(A)P_A(B)}{P(B)} \end{aligned}$$

$P(A)$ is the probability a person has cancer. This is given as 1% or 0.01

$P(B)$ is the probability that a test is positive, this is calculated as follows

$P(B) = 99\% \times 1\% \ + \ 2\% \times (100\% - 1\%) = 0.99 \times 0.01 + 0.02 \times (0.99) = 0.0297$

$P_A(B)$ is the probability that test is positive given the person has cancer= 99% = 0.99

Hence

$$\begin{aligned} P_B(A) &= \frac{P(A)P_A(B)}{P(B)} \\ &= \frac{0.01 \times 0.99}{0.0297} \\ &= \frac{0.0099}{0.0297} \\ &= 0.33333 \end{aligned}$$

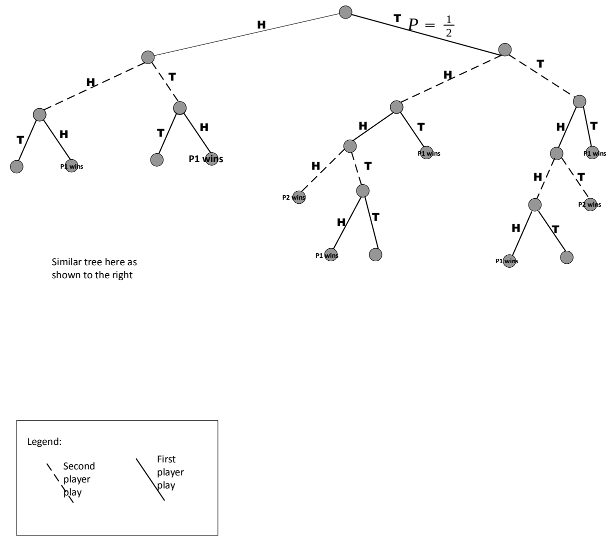
So the chance a person has cancer given the test is positive is only 33%

9 chapter 16, problem 3.21, Mary Boas , second edition

2 people take turns tossing a pair of coins. The first who gets two tosses alike wins. What is the probability for winning for the first player and for the second player?

Solution

I make a tree diagram. From this I find the needed probability sequence.



From diagram we see that the probability of player one winning after total of 3 tosses (by both players) is $4 \times \left(\frac{1}{2}\right)^3 = \frac{1}{2}$, (since there are 4 branches which leads to a win for player one), and after 5 tosses $4 \times \left(\frac{1}{2}\right)^5 = \frac{1}{8}$ and so after 7 tosses probability of first player winning is $4 \times \left(\frac{1}{2}\right)^7 = \frac{1}{32}$, after 9 tosses, it is $4 \times \left(\frac{1}{2}\right)^9 = \frac{1}{128}$

So the probability of first player winning is

$$
\begin{aligned}
&\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \frac{1}{128} + \cdots \\
&= \frac{1}{2}\left(1 + \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots\right) \\
&= \frac{1}{2}\left(1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^6 + \cdots\right) \\
&= \frac{1}{2}\,\frac{1}{1 - \left(\frac{1}{2}\right)^2} \\
&= \frac{1}{2}\,\frac{4}{3} \\
&= \frac{2}{3}
\end{aligned}
$$

Hence the probability of the second player winning is $1 - \frac{2}{3} = \frac{1}{3}$

10 chapter 16, problem 4.1, Mary Boas , second edition

(a)There are 10 chairs in a row and 8 people to be seated, in how many way can you choose them?

There are 8! ways to arrange the 8 people.

For each one of these arrangements, there is $\begin{pmatrix} 10 \\ 8 \end{pmatrix}$ ways to select 8 chairs out of the 10 chairs to seat each arrangement of the 8 people on.

Hence by the principle of counting, the final answer is

$8!\times \begin{pmatrix} 10 \\ 8 \end{pmatrix} = 8!\frac{10!}{(10-8)!\times 8!} = \frac{10!}{(10-8)!}$ which is the same as saying $\boxed{P_r^n}$

So $P_8^{10} = 1,814,400$

(b)There are 10 questions on a test and you are to do 8 of them, in how many ways can we choose them?

There are $\begin{pmatrix} 10 \\ 8 \end{pmatrix}$ ways to choose the 8 questions out of 10 without duplication.

Hence $\begin{pmatrix} 10 \\ 8 \end{pmatrix} = \frac{10!}{(10-8)! \quad 8!} = \frac{10!}{(10-8)! \quad 8!} = 45$

(c) In part (a) what is the probability that the first 2 chairs in a row are vacant.

Here we want to find number of ways 2 chairs can be empty out of 10 chairs. This is $\begin{pmatrix} 10 \\ 2 \end{pmatrix} = 45.$

Hence the probability that the first 2 chairs are the empty pairs is just $\frac{1}{45}$ (one chance out of total possible 45).

(d) In part(b), what is the probability you omit the first 2 problems in the test?

First we find the number of ways not to select 2 questions out of 10. This is given by $\begin{pmatrix} 10 \\ 2 \end{pmatrix} = 45$, so the probability of not selecting any 2 questions is $\frac{1}{45}$, and since any 2 questions are equally likely not to be selected (we are not given any extra information to suggest otherwise), then the probability of not selecting the first is also $\frac{1}{45}$

(e) Explain why the answers to (a) and (b) are different, but the answers to (c) and (d) are the same.

The answers to (a) and (b) are different, because in (b) we are looking for one set of 8 questions, and the order how the questions are arranged within the set is not important. In (a), the order was important. That is why answer for (a) is much larger than (b).

Answer to (c) and (d) is the same since in both cases the order is not important.

11 chapter 16, problem 4.4, Mary Boas , second edition

5 cards are dealt from a shuffled deck. What is the probability that they are all of the same suite?

Answer

There are a total of $\begin{pmatrix} 52 \\ 5 \end{pmatrix}$ ways to select 5 cards out of 52. (where the order of the 5 cards is not important).

There are $\begin{pmatrix} 13 \\ 5 \end{pmatrix}$ ways to select 5 cards from one suite. and there are 4 ways to select one suite. Hence number of ways to select 5 cards from the same suite is

$$4\begin{pmatrix} 13 \\ 5 \end{pmatrix}$$

So probability of selecting this will be $\dfrac{4\begin{pmatrix} 13 \\ 5 \end{pmatrix}}{\begin{pmatrix} 52 \\ 5 \end{pmatrix}} = \boxed{1.98 \times 10^{-3}}$

5 cards are dealt from a shuffled deck. What is the probability that they are all diamond?

Answer

This is just $\frac{1}{4}$ of the above probability, since there is one out of 4 chance it is a diamond. Hence the answer is $\frac{1}{4} \times 1.98 \times 10^{-3} = \boxed{4.95 \times 10^{-4}}$

5 cards are dealt from a shuffled deck. What is the probability that they are all face cards?

Answer

There are $3 \times 4 = 12$ face cards in a whole deck of cards (Jack, Queen, King).

The number of ways 5 cards can be selected from these is $\begin{pmatrix} 12 \\ 5 \end{pmatrix}$

Hence the probability they are all face cards is $\dfrac{\begin{pmatrix} 12 \\ 5 \end{pmatrix}}{\begin{pmatrix} 52 \\ 5 \end{pmatrix}} = 3.047 \times 10^{-4}$

5 cards are dealt from a shuffled deck. What is the probability that the 5 cards are in sequence in the same suite?

Answer

This is the probability of being in the same suite and then of being in sequence.

Let A=event of being in sequence

Let B=Event of being from same suite

So want to find $P(AB)$

$P(BA) = P_B(A) \times P(B)$

To find $P_B(A)$, this is the probability of being in sequence given the hand is already from one suite, we need to find number of ways 5 cards in sequence can be selected out of one suite. That is $\{1,2,3,4,5\}, \{2,3,4,5,6\}, \cdots, \{9,10,J,Q,K\}$

so there are 9 ways this could happen. But there $\begin{pmatrix} 13 \\ 5 \end{pmatrix}$ ways to select 5 cards from one suite. Hence the probability of straight 5 cards given one suite is $\frac{9}{\begin{pmatrix} 13 \\ 5 \end{pmatrix}} = 7 \times 10^{-3}$

Now $P(B)$ was found first part of this problem, and it is 1.98×10^{-3}

Hence

$$\begin{aligned} P(BA) &= P_B(A) \times P(B) = 7 \times 10^{-3} \times 1.98 \times 10^{-3} \\ &= 1.386 \times 10^{-5} \end{aligned}$$

Another way to solve this: We want to find

$$\frac{\text{number of ways to select 5 cards all from same suite in sequence}}{\text{number of ways to select 5 card}} = \frac{4 \times \binom{13}{5} \times \text{ probability of those 5}}{\binom{52}{5}}$$

$$= \frac{4 \times \binom{13}{5} \times \frac{9}{\binom{13}{5}}}{\binom{52}{5}}$$

$$= 1.39 \times 10^{-5}$$

12 chapter 16, problem 4.5, Mary Boas , second edition

In a family of 5 children, what is the probability that there are 2 boys and 3 girls?

Answer

Looking at any sequence of 5 children, such as $\{bbgbg\}$, there are 2^5 different sequences since for each position we can have either a boy or a girl (this is like looking at tail/head sequence generated from flipping a coin 5 times).

So the probability of any one sequence is $\frac{1}{2^5}$

Now the number of sequences with only 3 boys in them is $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$ which is the number of ways 3 positions can be selected out of 5 positions.

Hence the probability of having only 3 boys (and hence 2 girls) is $\frac{1}{2^5} \times \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \frac{5\times 4}{2^5 \times 2} = \frac{5}{16}$

In a family of 5 children, what is the probability that the 2 oldest are boys and the others are girls?

Answer

Let A=Event the first 2 born children are boys

Let B=Event that the last 3 born children are girls.

We want to find $P(AB)$ the probability of A and B together.

$$P(AB) = P(A)\,P_A(B) \tag{1}$$

$P(A)$ is $\left(\frac{1}{2}\right)^2$ (since the chance of each child being a boy is $\frac{1}{2}$).

Now $P_A(B)$ is the probability of the last 3 children being girls given the first 2 children are boys. Since A and B are independent, then $P_A(B) = P(B)$ which is $\left(\frac{1}{2}\right)^3$

Hence from (1)

$$P(AB) = \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^3 = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$$

13 chapter 16, problem 4.7, Mary Boas , second edition

What is the probability that the 2 and 3 of clubs are next to each others in a shuffled deck?

Answer

Let E_1= event of having the 2 card as the top of the deck

Let E_2= event of having the 2 card as the bottom of the deck

Let E_3= event having the 2 card somewhere in the middle of the deck.

Let E = event we are looking for. (i.e. the 2 and 3 cards next to each others)

So we want to find

$$P(E) = P(E_1)P_{E_1}(E) + P(E_2)P_{E_2}(E) + P(E_3)P_{E_3}(E) \tag{1}$$

But $P(E_1) = \frac{1}{52}$ since there is one position (the top) and there are 52 possible positions the card can be in.

Similarly, $P(E_2) = \frac{1}{52}$ since there is one position (the bottom) and there are 52 position.

$P(E_3) = \frac{50}{52}$ since there are 50 possible positions (not counting the top and bottom) out of 52 the card can be in.

Now need to find the conditional probabilities.

$P_{E_1}(E)$ this is the probability of having the 3 card below the 2 card, given the 2 card is in the top position. Clearly this is $\frac{1}{51}$ since there is 51 positions.

Similarly, $P_{E_2}(E)$ this is the probability of having the 3 card above the 2 card, given the 2 card is in the bottom position. Clearly this is $\frac{1}{51}$also.

Now $P_{E_3}(E)$ is the probability of having the 3 card next to the 2 card given that the 2 card is somewhere in the middle. Now the 3 card can be above or below the 2 card. Hence the probability now is $\frac{2}{51}$

Now substitute all these values in (1) gives

$$\begin{aligned} P(E) &= P(E_1)P_{E_1}(E) + P(E_2)P_{E_2}(E) + P(E_3)P_{E_3}(E) \\ &= \frac{1}{52} \times \frac{1}{51} + \frac{1}{52} \times \frac{1}{51} + \frac{50}{52} \times \frac{2}{51} \\ &= \frac{1}{26} \end{aligned}$$

14 chapter 16, problem 4.8, Mary Boas , second edition

2 cards are drawn from a shuffled deck. What is the probability that both are aces?

Answer

Let A = event first card is an ace.

Let B = event the second card is an ace.

We want to find $P(AB) = P(A)\,P_A(B)$

$P(A)$ =The probability of first card being an ace is $\frac{4}{52}$

Now, Given the first card is an ace, the probability of the second card is an ace is $\frac{3}{51}$

Hence $P_A(B) = \frac{3}{51}$

So

$$P(AB) = \frac{4}{52} \times \frac{3}{51} = \frac{1}{221}$$

2 cards are drawn from a shuffled deck. If you know one is an ace, what is the probability that both are aces?

Answer

Let A = event one of the two cards is an ace.

Let B = event both are an ace

Let C=event first is an ace

Let D=event second is an ace.

We want to find $P_A(B) = \frac{P(AB)}{P(A)} = \frac{P(B)P_B(A)}{P(A)} = \frac{P(B)}{P(A)}$ since $P_B(A) = 1$ (because given both are an ace, it is certain that one is an ace).

$P(A) = P(C) + P(D) - P(CD)$

But $P(CD)$ is the probability of both being an ace, which is the same as $P(B)$ which was found in part(a) to be $\frac{1}{221}$

And $P(C) = P(D) = \frac{4}{52}$

Hence $P(A) = \frac{4}{52} + \frac{4}{52} - \frac{1}{221} = \frac{33}{221}$

Now $P(AB) = P(B)P_B(A)$, but $P(B) = \frac{1}{221}$ from part(a).

Hence

$$P_A(B) = \frac{\left(\frac{1}{221}\right)}{\left(\frac{33}{221}\right)} = \frac{1}{33}$$

2 cards are drawn from a shuffled deck. If you know that one is an ace of spades, what is the probability that both are aces?

Answer

Let A = event one of the two cards is an ace of spades.

Let B = event both are an ace

Let C=event first card is an ace of spades

Let D=event second card is an ace of spades

We want to find

$$P_A(B) = \frac{P(AB)}{P(A)} = \frac{P(B)P_B(A)}{P(A)} \tag{1}$$

Now $P_B(A) = P_B(C) + P_B(D) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

Hence (1) becomes

$$P_A(B) = \frac{P(B)P_B(A)}{P(A)} = \frac{1}{2}\frac{P(B)}{P(A)} \tag{2}$$

Now we need to find $P(A)$ and $P(B)$.

$$P(A) = P(C) + P(D) - P(CD)$$

But $P(CD)$ is the probability of both cards being an ace of spades, this is zero since there is only ONE card which is an ace of spades and so we can not have both being an ace of spades.

Now $P(C) = P(D) = \frac{1}{52}$

Hence $P(A) = \frac{1}{52} + \frac{1}{52} = \frac{1}{26}$

To find $P(B)$. this was was found in part(a) to be $\frac{1}{221}$

Hence (2) becomes

$$
\begin{aligned}
P_A(B) &= \frac{1}{2}\frac{\left(\frac{1}{221}\right)}{\left(\frac{1}{26}\right)} \\
&= \frac{1}{17}
\end{aligned}
$$

15 chapter 16, problem 4.10, Mary Boas , second edition

What is the probability that you and your friend have different birthdays? (assume year is 365 days). What is the probability that 3 people have different birthdays? show that the probability that n people have all different birthdays is given by

$P=\left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right)\left(1-\frac{3}{365}\right)\cdots\left(1-\frac{n-1}{365}\right)$

Estimate p for $n \ll 365$ by calculating $\ln p$. Find the smallest n for which $p<\frac{1}{2}$ hence show that for a group of 23 people or more the probability is greater than $\frac{1}{2}$ that 2 of them will have the same birthday.

Answer

Let A=event that second person have different birthday from the first

Let B=event that 3rd person have different birthday from the second person.

Let C=event that all 3 have different birthdays

$P(A)=\frac{364}{365}$

So $P(C)=P(AB)=P(A)\,P_A(B)$

But $P_A(B)$ is the probability that 3rd person have different birthday from second, given that the second have a different birthday from the first. This leaves only 363 days to select a birthday from. Hence $P_A(B)=\frac{363}{365}$

Hence

$$\begin{aligned} P(C) &= \left(\frac{364}{365}\right)\left(\frac{363}{365}\right) \\ &= \left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right) \end{aligned}$$

So when we add a 4th person we will get the probability that they have different birthdays $P=\left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right)\left(1-\frac{3}{365}\right)$

Continue this for n people we get,

$$p=\overbrace{\left(1-\frac{1}{365}\right)}^{2}\overbrace{\left(1-\frac{2}{365}\right)}^{3}\cdots\overbrace{\left(1-\frac{n-1}{365}\right)}^{n}$$

Take the log of both sides we get

$$\begin{aligned}\ln p &= \ln\left[\left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right)\cdots\left(1-\frac{n-1}{365}\right)\right]\\ &= \ln\left(1-\frac{1}{365}\right)+\ln\left(1-\frac{2}{365}\right)\cdots+\ln\left(1-\frac{n-1}{365}\right)\end{aligned} \tag{1}$$

But $\ln(1+x) = x$ for small x (i.e. for $n \ll 365$)

Hence (1) becomes

$$\begin{aligned}\ln p &= -\frac{1}{365}-\frac{2}{365}\cdots-\frac{n-1}{365}\\ &= -\frac{1+2+3+\cdots+(n-1)}{365}\end{aligned}$$

But $1+2+3+\cdots+(n-1) = [(n-1)+1]\times\frac{(n-1)}{2}$ (by Gauss summation formula)

So we get

$$\begin{aligned}\ln p &= -\frac{n\times\frac{(n-1)}{2}}{365}\\ &= -\frac{n\times(n-1)}{2\times 365}\end{aligned} \tag{2}$$

The above is an estimate of $\ln p$ for $n \ll 365$

Now need to Find the smallest n for which $p < \frac{1}{2}$

$p < \frac{1}{2}$ implies $\ln p < \ln\frac{1}{2}$

Hence $\ln p < -\ln 2$

So $\ln p < -0.69315$

so from (2)

$$\begin{aligned}\frac{n\times(n-1)}{2\times 365} &> 0.69315\\ n\times(n-1) &> 506.00\\ n^2-n-506 &> 0\end{aligned}$$

Hence $n = 23$ or $n = -22$

since n is number of people, we select positive value. Hence for $n > 23$ there is a chance just less than $\frac{1}{2}$ that no 2 people will have the same birthday, or there is a chance just over 50% that 2 people will have the same birthday.

16 chapter 16, problem 4.11 , Mary Boas , second edition

The following game was being played on a busy street: Observe the last 2 digits on each license plate. What is the probability of observing at least 2 cars with the same last 2 digits among the first 5 cars? 10 cars? 15 cars? How many cars must you observe in order for the probability to be greater than 50% of observing 2 cars with the same last 2 digits?

Answer

This is similar to problem 4.10.

Replace the number of days in a year by the number of numbers, which is 100 numbers (2 digits, $00, 01, \ldots, 99$ is 100 numbers).

Hence I can use the formula obtained in 4.10

$$P = \left(1 - \frac{1}{100}\right)\left(1 - \frac{2}{100}\right)\left(1 - \frac{3}{100}\right)\cdots\left(1 - \frac{n-1}{100}\right)$$

and for small n compared to 100

$$\ln p = -\frac{n \times (n-1)}{2 \times 100}$$

So for $n = 5$ we get

$$\begin{aligned} \ln p &= -\frac{5 \times (5-1)}{2 \times 100} \\ \ln p &= -\frac{1}{10} \end{aligned}$$

Solution is: $p = 0.90484$

The above is the probability of all 5 cars having different 2 digits numbers. Hence Probability of observing at least 2 cars with same last 2 digits is

$$1 - p = 1 - 0.90484 = 0.09516$$

Notice that is was based on the approximation formula. To get an exact number, I would write

$$\begin{aligned} P &= \overbrace{\left(1 - \frac{1}{100}\right)}^{2}\left(1 - \frac{2}{100}\right)\left(1 - \frac{3}{100}\right)\overbrace{\left(1 - \frac{4}{100}\right)}^{5} \\ &= 0.903\,45 \end{aligned}$$

Hence Probability of observing at least 2 cars with same last 2 digits is

$$1 - 0.90345 = 0.09655$$

To solve this for $n = 10$

From $\ln p = -\frac{n\times(n-1)}{2\times100}$

$$\begin{aligned}\ln p &= -\frac{10\times(10-1)}{2\times100}\\ \ln p &= -0.45\end{aligned}$$

Solution is: $p = 0.63763$

Hence Probability of observing at least 2 cars with same last 2 digits is

$$1 - 0.63763 = 0.36237$$

For $n = 15$

$$\begin{aligned}\ln p &= -\frac{15\times(15-1)}{2\times100}\\ \ln p &= -1.05\end{aligned}$$

Solution is: $p = 0.349\,94$

Hence Probability of observing at least 2 cars with same last 2 digits is

$$1 - 0.34994 = 0.65006$$

To find how many cars one must observe to get a probability of more than 50% of having at least 2 cars with same last 2 digits, we solve for $p = \frac{1}{2}$

$$\begin{aligned}\ln p &= -\frac{n\times(n-1)}{2\times100}\\ \ln\frac{1}{2} &= -\frac{n\times(n-1)}{2\times100}\\ -\ln 2 &= -\frac{n\times(n-1)}{2\times100}\end{aligned}$$

Solution is: $\{[n = -11.285], [n = 12.285]\}$

Hence $n = 13$

17 chapter 16, problem 4.15 , Mary Boas , second edition

Problem 4.15 Chapter 16 (Mary Boas second edition)

By Nasser Abbasi

Box 1	Box 2	Box 3
A	B	
A		B
A B		
	A	B
B	A	
	A B	
	B	A
B		A
		B A

Maxwell-Blotzman distribution of 2 balls into 3 boxes. Balls are distinguishable from each others, one ball labeled A and the other B.
Total number of ways pf putting the 2 balls into the 3 boxes is given by 3^2=9

Box 1	Box 2	Box 3
o	o	
o		o
	o	o

Fermi-Dirac distribution of 2 balls into 3 boxes. Balls are NOT distinguishable, hence each is a circle. In addition, each box can have only ONE ball in it at a time. Number of ways of putting 2 balls into 3 boxes is C(3,2)=3

Box 1	Box 2	Box 3
o	o	
o		o
o o		
	o o	
o		o
		o o

Bose-Einstein distribution of 2 balls into 3 boxes. Balls are NOT distinguishable (just like with the Fermi-Dirac), however, here we can put more than one ball in the same box at one time (unlike the Fermi-Dirac).
Number of ways of putting 2 balls into 3 boxes is C(3+1,2)=6 and the probability of each one permutation is 1/6 (each is equally likely).

18 chapter 16, problem 4.17 , Mary Boas , second edition

Find number of ways of putting 2 particles in 4 boxes according to the 3 kinds of statistics.

Answer

Let n be the number of Boxes, and N be the number of balls.

For Maxwell-Boltzmann (MB) it is n^N . Hence the answer is $4^2 = 16$

For Fermi-Dirac (FM), it is ${}_nC_N = {}_4C_2 = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \boxed{6}$

For Bose-Einstein (BE) it is ${}_{n-1+N}C_N = \begin{pmatrix} 4-1+2 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} = 10$

19 chapter 16, problem 4.21 , Mary Boas , second edition

Find the number or ordered triplets of non-negative integers a, b, c whose sum adds to a given positive integer n

Answer

If we imagine the number n written as $11111\cdots111$ and then imagine we put one vertical bar to the left most and to the right most like this: $|11111\cdots111\ \ |$, then the problem becomes on how many unique partitions we can create in-between these 2 vertical bars by using 2 new vertical bars. A partition here is the same as a box.

For example, the following is an example of 2 different partitions created for $n = 8$

$$|111|1111\ \ |\ 1\ \ |$$

$$|1|11111|11|$$

Note that We can also create an empty partition, as follows

$$|1|\ \ |1111111|$$

When we create an empty partition between the 2 vertical bars, it is as if there is a 0 in there.

Hence the problem becomes a question of how many way can we insert the 2 vertical bar among n different objects.

To count this, we start by putting the first vertical bar before the first object:

$$||11111111\ \ |$$

So now the second bar can go before or after the second object, or after the third object, or after the 4th object, etc... until we get to the nth object, where it can go after it. Hence there are n choices for the second bar.

Now, the first bar can be put after the first object as this:

$$|1\ \ |1111111\ \ |$$

So now the second bar can go into any of $n-1$ positions.

We continue this way, until we get to the last object, where we can put the first bar after it, as this:

$$| 11111111 | |$$

Now the second vertical bar have only one position to go which is after the first vertical bar.

So, the first vertical bar has been placed in $m = n + 1$ positions, and for each one of these positions, the second vertical bar has been put in $k + 1$ positions where k indicates the number of object to the right of the first vertical bar at the time.

So, for example, for $n = 3$, we have $n + 1$ possible positions for the second vertical bar when the first vertical bar at the left of the first object. Then we have n possible positions for the second vertical bar when the first vertical bar to the right of the first object, then we have $n - 1$ possible positions for the second vertical bar when the first vertical bar to the right of the second object, and we have $n - 2$ or 1 possible positions for the second vertical bar when the first vertical bar to the right of the third and final object.

This is the same as the number of ways to choose r object at a time from n objects.

Hence for $n = 3$ we have $(n + 1) + (n) + (n - 1) + (n - 2) = 4n - 2 = 10$

So in general, we have $(n + 1) + (n) + \cdots + 1$ possible ordered triples.

This is, using Gauss summation trick, is the same as

$$(n + 2)\frac{(n + 1)}{2}$$

But this is the same as $\frac{(n+2)!}{n!\ 2!}$ which is the same as $\begin{pmatrix} n + 2 \\ n \end{pmatrix}$

But note that the Bose-Einstein statistics with the number of boxes being fixed at 3 gives $\begin{pmatrix} 3 - 1 + n \\ n \end{pmatrix} = \begin{pmatrix} 2 + n \\ n \end{pmatrix}$

This is the same as the Bose-Einstein statistics with the number of boxes being fixed at 3.