Study notes, Mathematical methods courses Math 121A and Math121B at UC Berkeley.

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Contents

1 Misc. notes

1. it is OVER !! finals finished.

2. math 121A. review HW 11 done 11 PM wed. review of HW 12...finished 3:20 AM. (4 hrs per HW to go over!). getting ready to start on HW 13...6 AM, almost finished HW 13 most of the rest I know, about calculus of variations, studied that before, but need to finish HW13 (may be one more hr). Then start on the HW 10 (Fourier series). will do after wake up. now going to sleep. 1 pm Thursday.. finishing HW 13 now.... 3:00 PM finished HW 13. This contained important stuff. 8 hrs study only today

3. now midnight Thursday. 1 hr study. went over some problem from midterm2, and Lagrange equations physics problem derivations. need to finish review of midterm 2, then back to HW review. it is 4 AM now, read notes and finished midterm, starting on HW 9, getting tired, will not be able to review everything before finals, need to try to concentrate on last stuff only... 4:40 AM finished HW 10 (Fourier series, easy stuff), now starting on HW 9...5:20 AM Friday, finished HW 9. HW 8 is on Laurent series, so important... 6:40 AM, ok finished. going to eat something and sleep and wake up for the exam.

4. finals for math 121B over. I made 3 very stupid mistakes., can't believe I did those. blow away 3 fairly easy questions I could have full credit on. but I think I can pull a B in the course. keep fingers crossed.

5. Practice more chapter 7 Fourier series tricks (odd/even) stuff

6. Make sure I remember ds^2 in all coordinates

7. learn better how to evaluate this:$\left(-\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{\sin x}{x}\right)$

8. HW's for 121B went over since midterm exam: HW5 chapter 12, HW 6 done, HW 7 stop here. Saturday night.., HW 11 done, HW 12 working on..finished. Now studying probability distribution, last HW

9. write down the same space for the 2 die, with the sum, some problems use it.

questions:

1. Why did we use series method to find solution to Legendre ODE, but used generalized series method to find solution to Bessel ODE? how to know when to use which? Answer: if ODE has something like $(1-x)y''$, then at $x=0$ we'll have problem, then use the generalized power series)

2. The Legendre ODE is solved using series method, assuming l is an integer. We get one solution which is Legendre function of first kind $p_l(x)$. What if l is not an integer? A: Legendre P_l is only defined for integer l? YES? No, there are tables for non-integer, but these cases are not important.

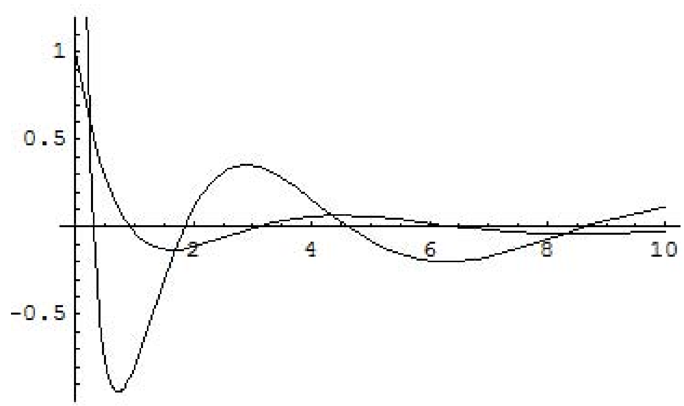
3. What if we get a legendre ODE and we want to find solution for $x > 1$? Since legendre functions are only defined for x less than one (to have convergence). Physics example? usually x is the cosine of an angle so it is ≤ 1.

4. What if l is not an integer in the legendre ODE? how to get a solution? this is special cases, not important, look up handbooks.

5. problem I solved in HW#6, chapter 12, 16.3. check my solution. I claimed that the second solution is N_p but since I found P NOT to be an integer, hence the second solution is one containing log and not a combination of J_{-p}. When I solved it in mathematica, I get this solution (notice complex number?), could this second solution be converted to log function? answer: OK, the solution I did will turn out to have log in it if I put p=integer and use L'hospital's rule to evaluate.

```
eq = x y''[x] + 2 y'[x] + 4 y[x] = 0
sol = DSolve[eq, y[x], x]
allY = y[x] /. sol;
allY = allY /. {C[1] → 1, C[2] → 2}
Plot[{Evaluate[Re[allY]], Evaluate[Im[allY]]}, {x, 0, 10}]
```

$$\left\{ y[x] \to \frac{\text{BesselJ}[1, 4\sqrt{x}]\ C[1]}{2\sqrt{x}} - \frac{i\ \text{BesselY}[1, 4\sqrt{x}]\ C[2]}{\sqrt{x}} \right\}$$



6. When solving for equation 16.1 on page 516, we seem to only take the positive root for the variables, why? see for example page 516. $b = 2$ but it is really $b = \pm 2$ answer: OK, any of these will give a good solution, just pick one.

7. on page 528, can I just set $n = 0$ always to solve for the indical equation as shown in the example? is it better to solve this using the $\sum$ directly as shown in the example instead of setting up a table? table seems more clear, but the example method seems shorter.

8. How to solve chapter 16, 4.1 part (c) using Bayes rule? I write: Let A=event first chair is empty, let B=event second chair is empty. We need to find $P(AB) = P(A)P_A(B) = (\frac{1}{10})(\frac{1}{9}) = \frac{1}{90}$, but the answer should be $\frac{1}{45}$, what is it I am doing wrong? wrong. $P(A) = 1/5$ *not* 1/10.

9. for problem HW 12, chapter 16, 4.8, part b. It says given 2 cards drawn from deck, if you know one is an ace, what it the chance the BOTH are an ace? I know how to solve by the book. but why can I not say the following: since we KNOW that one card is an ace, then the chance that both cards are an ace is just the chance the second card being an ace (since we know the first is an ace). So this should give $\frac{3}{51}$
10. random variable is defined as a function on the sample space. however, it is multivalued. for example, if x= sum of 2 die throw, then more than one event can give the same random variable. is this OK? I thought a function must be single valued? answer: I am wrong. it is NOT multivalued.
11. check that my solution for chapter 16, 5.1 MATH 121B is correct, I have solution on paper. this is the last HW

2 Table summary of topics to study

2.1 Math 121A

ch.	title	topics	Exam
1	series	infinite series, power series,def. of covergence, tests for convergence, test for alternating series, power series, binomial series	1
2	complex numbers	finding circle of convergence (limit test), Euler formula power and roots of complex numbers, log, inverse log	1
4	partial differentiation	total diffenertials, chain rule, implicit differentiation partial diff for max and minumum, Lagrange muktipliers, change of variables Leibniz rule for differnetiation of integrals	1
14	complex functions	Def. of analytic fn, Cauchy-Riemann conditions, laplace equation, contour integrals, Laurent series, Residue theorm, methods of finding residues, pole type, evaluating integrals by residue, Mapping, conformal	1
7	Fourier series	expansion of function in sin and cosin, complex form, how to find coeff, Dirichlet conditions, different intervals, even/odd, Parseval's	2

15	Laplace/Fourier transforms	Laplce transform, table, how to use Laplace to solve ODE, Methods of finding inverse laplace, partial fraction, convolution, sum of residues, Fourier transform, sin/consine transforms, Direc Delta Green method to solve ODE using impluse	F
9	Calculus of variations	Euler equation solving, Setting up Lagrange equations, KE, PE Solving Euler with constrainsts	F

2.2 Math 121B

ch.	title	topics	Exam
11	Special functions	Gamma, Debta, Error function	1
12	Series solution to ODE	Legendre, Bessel, orthogonality	1
13	PDE	separation of variables, Laplace (steady state), Heat (diffusion), Wave equation. Laplce in different coordinates, Laplacian, Wave in different coord.Poission equation	2
16	Probability	Baye's formula, how to find probability, methods of counting, Random variable concept, mean, Var, SD, distributions (Binomial, Gauss, Poisson)	final

3 Math 121 A notes

3.1 Chapter 1. Series

$a + ar + ar^2 + \cdots + ar^n + \cdots = \frac{a(1-r^n)}{1-r}$, Now, if $|r| < 1$, then the above is convergent, hence we get $S_n = \frac{a}{1-r}$. Always start by looking for a constant term a here, and then a term that is multiplied each time, r here.

3.2 Chapter 14. Complex functions

3.2.1 How to find the residue?

seek book page 598

3.3 Chapter 7. Fourier Series

Expand a periodic function (must be periodic) in sin and cos functions.

Let the function angular velocity be ω, which is defined as angles (radian) per second, i.e. $\omega = \frac{2\pi}{T}$ where T is the period in time, which is the time to make 2π angle.

$$
\begin{aligned}
f(x) = \frac{1}{2}a_0 + a_1 \cos \omega x + a_2 \cos 2\omega x + \\
\cdots + b_1 \sin \omega x + b_2 \sin 2\omega x + \cdots
\end{aligned}
$$

So, for a function whose period is 2π, i.e. $\omega = 1$, the above can be written as

$$
\begin{aligned}
f(x) = \frac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + \\
\cdots + b_1 \sin x + b_2 \sin 2x + \cdots
\end{aligned}
$$

Now, to find a_n and b_n

$$
\begin{aligned}
a_n &= \frac{2}{T} \int_T f(x) \cos \omega n x \, dx \\
b_n &= \frac{2}{T} \int_T f(x) \sin \omega n x \, dx
\end{aligned}
$$

So, I only need to remember ONE formula

note: Remember, when finding a_n, for a_0, do it separately, set $n = 0$ in the integral first and integrate that, do not set $n = 0$ in the result, leave that for $n \neq 0$. For b_n we do not need to worry about this, since for sin series it starts at $n = 1$

note: *When will this expansion converge to* $f(x)$? when the function meet the Dirichlet conditions. Basically it needs to be periodic of period 2π, single valued, has finite number of jumps. At jumps, the series converges to average of the function there.

In these kind of problems, we are given a function $f(x)$ and asked to find its F. series. So need to apply the above formulas to find the coefficients. Need to know some tricks for quickly evaluating the integrals.

Now there is a complex form of all the above equations.

$$f(x) = c_0 + c_1 e^{ix} + c_{-1} e^{-ix} + c_2 e^{2ix} + c_{-2} e^{-2ix} + \cdots = \sum_{-\infty}^{\infty}$$

$$c_n = \frac{1}{T} \int_T f(x)\, e^{-inx\omega}\, dx$$

Now, ω, is the angular velocity. i.e. $\theta = \omega t$, so for ONE period T, $\theta = 2\pi$, hence $\omega = \frac{2\pi}{T}$, so c_n can be written as

$$c_n = \frac{1}{T} \int_T f(x)\, e^{-inx\frac{2\pi}{T}}\, dx$$

Notice that in this chapters we use distance for period (i.e. wave length λ) instead of time as period T. it does not matter, they are the same, choose one. i.e. we can say that the function repeats every λ, or the function repeats every one period T.

When using λ for period, say $-l, l$ or $-\pi, \pi$ the above equation becomes

$$c_n = \frac{1}{2l} \int_{-l}^{l} f(x)\, e^{-inx\frac{2\pi}{2l}}\, dx = \frac{1}{2l} \int_{-l}^{l} f(x)\, e^{-inx\frac{\pi}{l}}\, dx$$

note: Above integral for c_n is for negative n as well as positive n. In non-complex exponential expansion, there is no negative n, only positive.

note: $c_{-n} = \bar{c}_n$

note: there is a relation between the a_n, b_n, and the c_n which is

$a_n = c_{-n} + c_n$ and $b_n = i(-c_{-n} + c_n)$

IF given $f(x)$, defined over $(0, L)$, The algorithm to find Fourier series is this:

```
IF asked to find a(n) i.e. the COSIN series, THEN
extend f(x) so that it is EVEN (this makes b(n)=0)
and period now is 2L
ELSE
IF asked to find b(n), i.e. the SIN series, THEN
extend f(x) to be ODD (this makes a(n)=0)
and period now is 2L
ELSE we want the standard SIN/COSIN
period remains L, and use the c(n) formula
(and remember to do the c(0) separatly for the DC term)
END IF
END IF
```

3.3.1 Parseval's theorem for fourier series

This theory gives a relation between the average of the square of $f(x)$ over a period and the fourier coefficients. Physically, it says that this:

the total energy of a wave is the sum of the energies of the individual harmonics it carries

Average of $\left[f(x)\right]^2 = \left(\frac{1}{2}a_0\right)^2 + \frac{1}{2}\sum_1^\infty a_n^2 + \frac{1}{2}\sum_1^\infty b_n^2$ over ONE period.

In complex form, Average of $\left|f(x)^2\right| = \sum_{-\infty}^{\infty}|c_n|^2$. Think of this like pythagoras theorem.

For example, given $f(x) = x$, then $\left[f(x)\right]^2 = \frac{1}{2}\int_{-1}^{1} x^2 dx = \frac{1}{3}$, then $\frac{1}{3} = \sum_{-\infty}^{\infty}|c_n|^2$

In the above we used the standard formula for average of a function, which is

average of $f(x) = \frac{1}{T}\int_T f(x)\,dx$, here we should need to square $f(x)$

3.4 Chapter 15. Integral transforms (Laplace and Fourier transforms)

3.4.1 Laplace and Fourier transforms definitions

$$
\begin{aligned}
F\,f(x) = F(p) &= \int_0^\infty f(x)\,e^{-px}\,dx \qquad p > 0 \\
F\,g(x) = g(\alpha) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\,e^{-i\alpha x}\,dx
\end{aligned}
$$

Associate Fourier with $\frac{1}{2\pi}$. (mind pic: Fourier=Fraction i.e.→ $\frac{1}{2\pi}$) and Fourier goes from $-\infty$ to $+\infty$ (mind pic: Fourier=whole Floor), Fourier imaginary exponent, Laplace real exponent.

Note: Laplace transform is linear operator, hence $L\left[f(t) + g(t)\right] = Lf(t) + Lg(t)$ and $L\left[c\,f(t)\right] = c\,Lf(t)$

3.4.2 Inverse Fourier and Laplace transform formulas

(We do not really use the inverse Laplace formula directly (called Bromwich integral), we find inverse Laplace using other methods, see below)

$$
\begin{aligned}
f(x) &= \frac{1}{2\pi i}\int_{c-i\,\infty}^{c+i\,\infty} F(z)\, e^{zt}\, dz \qquad t>0 \qquad \text{Inverse laplace} \\
f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\alpha)\, e^{i\alpha x}\, d\alpha \qquad\qquad \text{Inverse fourier}
\end{aligned}
$$

The Fourier transform has 2 other siblings to it (which Laplace does not), these are the sin and cos transform and inverse transform. I'll add these later but I do not think we will get these in the exam.

Note: To get the inverse Laplace transform the main methods are

1. using partial fractions to break the expression to smaller ones we can lookup in tables
2. Use Convolution. i.e. given $Y = L(f_1)\, L(f_2) \rightarrow y = \int_0^t g(t-\tau)\, f(\tau)\ d\tau = g \otimes f$ use this as an alternative to partial fraction decomposition if easier. mind pic: t one time, τ 2 times.
3. Use the above integral (Bromwich) directly (hardly done)
4. To find $f(t)$ from the Laplace transform, instead of using the above formula, we can write

   $f(t)$ = sum of residues of $F(z)e^{zt}$ at all poles. For example given $F(z)$, we multiply it by e^{zt}, and then find all the poles of the resulting function (i.e. the zeros of the denominator), then add these.

Note: To find Fourier transform , $g(\alpha)$, must carry the integration (i.e. apply the integral directly, no tables like with Laplace).

Note: we use Laplace transform as a technique to solve ODE. Why do we need Fourier transform? To represent an arbitrary function (must be periodic or extend to be period if not) as a sequence of sin/cosine functions. And why do we do this? To make it easier to analyze it and find what frequency components it has. For continuous function, use fourier transform (integral).

note: Function must satisfy Dirichlet conditions to use in fourier transform or Fourier series.

note: Fourier series expansion of a function will accurately fit the function as more terms are added. But in places where there is a jump, it will go to the average value of the function at the jump.

question: When do we use fourier series, and when to use fourier transform? Why do we need F. transform if we can use F. Series? We use F. transform for continuous frequencies. What does this really mean?

3.4.3 Using Laplace transform to solve ODE

Remember

$$
\begin{aligned}
L(y) &= Y \\
L(y') &= pY - y_0 \\
L(y'') &= p^2 Y - p y_0 - y'_0
\end{aligned}
$$

note: p has same power as order of derivative. do not mix up where the p goes in the y'' equation. remember the y_0' has no p with it. mind pic: think of the y_0 as the senior guy since coming from before so it is the one who gets the p.

note: if $y_0 = y_0' = 0$ (which most HW problem was of this sort), then the above simplifies to

$$L\left(y'\right) = pY$$
$$L\left(y''\right) = p^2 Y$$

So given an ODE such as $y'' + 4y' + y = f(t) \rightarrow \left(p^2 + 4p + 4\right)Y = L(f(t))$

i.e. just replace y'' by p^2, etc... This saves lots of time in exams. Now we get an equation with Y in terms of p, now solve to find $y(t)$ from Y using tables. Notice that solution of ODE this way gives a particular solution, since we used the boundary conditions already.

For an ODE such as

$$Ay'' + By' + Cy = h(t)$$

its Laplace transform can be written immediately as

$$Ap^2Y + BpY + CY = L\,h(t)$$
$$Y = \frac{L\,h(t)}{Ap^2 + Bp + C}$$

whenever the B.C. are $y_0' = 0$ and $y_0 = 0$

3.4.4 Partial fraction decomposition

When denominator is linear time quadratic or quadratic time quadratic PFD is probably needed.

This is how to do PFD for common cases

$$\frac{1}{(x+c)\left(x^2+x+6\right)} = \frac{A}{(x+c)} + \frac{Bx+C}{\left(x^2+x+6\right)} \quad \text{(quadratic in denominator case)}$$

$$\frac{1}{\left(x^2+3x+4\right)\left(x^2+x+6\right)} = \frac{Ax+B}{\left(x^2+3x+4\right)} + \frac{Cx+D}{\left(x^2+x+6\right)} \quad \text{(quadratic in denominator case)}$$

$$\frac{1}{(x+c)(x+d)} = \frac{A}{(x+1)} + \frac{B}{(x+d)}$$

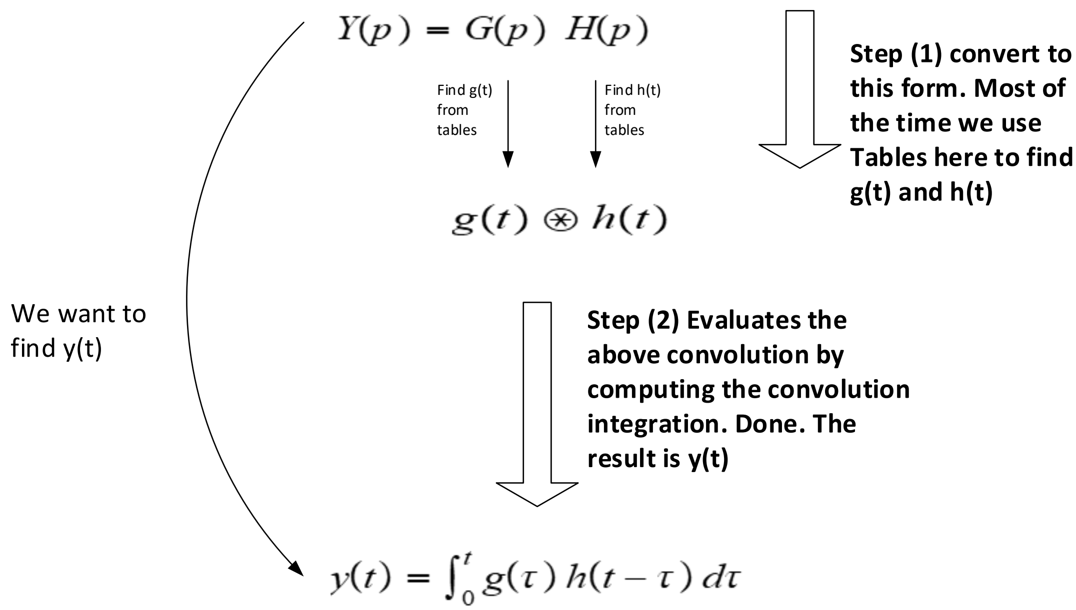
$$\frac{x^2 + x + b}{(x+c)(x-d)^2} = \frac{A}{(x+1)} + \frac{B}{(x-d)} + \frac{B}{(x-d)^2} \quad \text{(repeated roots case)}$$

we get some equations which we solve for A, B, etc... This part can be time consuming in exam.

3.4.5 convolution

Main use of convolution in this class is to find the inverse laplace transform.

If we are given the transform itself (i.e. frequency domain) function, and asked to find the inverse, i.e. the time domain function. Then look at the function given, if it made of 2 functions multiplied by each others, then good chance we use convolution.



Finding the inverse Laplace transform using convolution

Example:

Given this equation

$$Y(p) = G(p)\ H(p)$$

We first find the inverse of $G(p)$ and $H(p)$ separately. i.e. we find $g(t)$ and $h(t)$. we usually do this by looking up tables. Once we do this step, the next step is to take the convolution of these 2 time domain functions.

The result, will be $y(t)$, i.e. the inverse of $Y(z)$.

Notice that you **can NOT** just say $y(t) = g(t)\ h(t)$, DO NOT DO THIS. But we must use convolution to find $y(t)$:

$$
\begin{aligned}
y(t) &= g(t) \circledast h(t)\\
y(t) &= \int_0^t g(\tau)\, h(t-\tau)\, d\tau\\
&= \int_0^t g(t-\tau)\, h(\tau)\, d\tau
\end{aligned}
$$

Notice, choose the simpler function to put the $(t-\tau)$ in. It does not matter if it is the f or the h. remember, the τ occur 2 times in the integral, the t one time.

The above means

$$
\begin{aligned}
\mathscr{L}y = \mathscr{L}g(t)\ \mathscr{L}h(t) &= \mathscr{L}\left[g(t) \circledast h(t)\right]\\
y = g(t) \circledast h(t)
\end{aligned}
$$

The above comes when we want to solve an ODE. Usually we know $g(t)$ which is the transfer function, and $h(t)$ is given (the forcing function of the ODE).

For Fourier transform, convolution can be used as well. it is very similar equation:

$$
F\left(g(t)\right)\ \ F\left(h(t)\right) = \frac{1}{2\pi}\mathscr{F}\left[g(t) \circledast h(t)\right]
$$

So difference is the $\frac{1}{2\pi}$

3.4.6 Parseval's theorem

(total energy in a signal equal the sum of the energies in the harmonics that make up the signal).

$$
\int_{-\infty}^{\infty}\left|g(\alpha)\right|^2 d\alpha = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|f(x)\right|^2 dx
$$

3.4.7 Dirac delta and Green function for solving ODE

Dirac delta function is a function defined for t, who has an area of 1 and zero width and ∞ value at t. (not a real function). Used to represent an impulse force being applied at t.

When multiplied with any other function inside an integral will given that other function at the time the impulse was applied. i.e. $\int f(t)\ \delta(t-t_0)\ dt = f(t_0)$, here t_0 is the time the impulse is applied.

note: Fourier transform of delta function: $g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(x-x_0)e^{-i\alpha x}\,dx = e^{-i\alpha x_0}$

note: Green function $G(t,t')$ is the response of a system (solution of an ODE) when the force (input) is an impulse at time $t = t'$

How to use Green function to solve an ODE? Given $G(t,t')$, $y(t) = \int_0^\infty G(t,t')\ f(t')\,dt'$ Where $f(t)$ is the force on the system (the RHS to the ODE). Usually we are given the Green function and asked to solve the ODE. So just need to apply the above integral.

Question: ask about if the above is correct for the finals or is it possible we need to find G as well?

Solving an ODE using green method Here we are given an ODE, with a forcing function (i.e. nonhomogeneous ODE). And given 2 solutions to it, and asked to find the particular solution.

Example, $y'' - y = f(t)$ and solutions are y_1, y_2 then the particular solution is $y_p = y_2 \int \frac{y_1\, f(t)}{W} dt - y_1 \int \frac{y_2\, f(t)}{W} dt$ where $W = \begin{vmatrix} y_1' & y_2' \\ y_1 & y_2 \end{vmatrix}$

3.5 Chapter 2. Complex Numbers

note: When given a problem such as evaluate $(-2-2i)^{\frac{1}{5}}$, always start by finding the length of the complex number, then extract it out before converting to the $re^{in\theta}$ form. For example, $(-2-2i)^{\frac{1}{5}} = 2\sqrt{2}\left(\frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$, the reason is that now the stuff inside the brackets has length ONE. So we now get $2\sqrt{2}\left(\frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) = 2\sqrt{2}\, e^{-\frac{3}{4}\pi i}$ and only now apply the last raising of power to get $\left(2\sqrt{2}\, e^{-\frac{3}{4}\pi i}\right)^{\frac{1}{5}} = 2^{\frac{3}{10}} e^{\frac{\frac{3}{4}\pi i + 2n\pi}{5}}$ for $n = 0, 1, 2, 3, \cdots$ make sure not to forget the $2n\pi$, I seem to forget that.

3.6 Chapter 9. Calculus of variations

3.6.1 Euler equation

How to construct Euler equation $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$. If integrand does not depend on x then change to y. Example $\int_{x_2}^{x_1} y'^2\, y\, dx \to \int_{y_2}^{y_1} \frac{1}{x'^2}\, y \left(x'\, dy\right) \to \int_{y_2}^{y_1} \frac{1}{x'}\, y\ dy$ this is done by making the substitution $y' = \frac{1}{x'}$ and $dx = x'\, dy$. Now Euler equation changes from $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$ to $\frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) - \frac{\partial F}{\partial x} = 0$.

Normally, $\frac{\partial F}{\partial y}$ will be zero. Hence we end up with $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$ and this means $\frac{\partial F}{\partial y'} = c$, and so we only need to do ONE integral (i.e. solve a first order ODE). If I find myself with a 2 order ODE (for this course!) , I have done something wrong since all problems we had are of this sort.

3.6.2 Lagrange equations

are just Euler equations, but one for each dimension.

F is now called L. where $L = T - V$ where $T = K.E.$ and $V = P.E.$, $T = \frac{1}{2}mv^2$, $V = mgh$

So given a problem, need to construct L ourselves. Then solve the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=0$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right)-\frac{\partial L}{\partial y}=0$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right)-\frac{\partial L}{\partial z}=0$$

The tricky part is finding v^2 for different coordinates. This is easy if you know ds^2, so just remember those

$$\begin{aligned} ds^2 &= dr^2 + r^2 d\theta^2 && \text{(polar)} \\ ds^2 &= dr^2 + r^2 d\theta^2 + dz^2 && \text{(cylindrical)} \\ ds^2 &= dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 && \text{(spherical)} \end{aligned}$$

So to find v^2 just divide by dt^2 and it follows right away the following

$$\begin{aligned} v^2 &= \dot{r}^2 + r^2\dot{\theta}^2 && \text{(polar)} \\ v^2 &= \dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2 && \text{(cylindrical)} \\ v^2 &= \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2 && \text{(spherical)} \end{aligned}$$

To help remember these: Note ds^2 all start with $dr^2 + r^2 d\theta^2$ for each coordinates system. So just need to remember the third terms. (think of polar as subset to the other two). Also see that each variable is squared. So the only hard think is to remember the last term for the spherical.

Remember that in a system with particles, need to find the KE and PE for each particle, and then sum these to find the whole system KE and PE, and this will give one L for the whole system before we start using the Lagrange equations.

3.6.3 Solving Euler-Lagrange with constraints

The last thing to know is this chapter is how to solve constraint problems. This is just like solving for Euler, expect now we have an additional integral to deal with.

So in these problems we are given 2 integrals instead of one. One of these will be equal to some number say l.

So we need to minimize $I = \int_{x_2}^{x_1} F(x,y',y)\,dx$ subject to constraint that $g = \int_{x_2}^{x_1} G(x,y',y)\,dx = l$

Follow the same method as Euler, but now we write

$$\frac{d}{dx}\left(\frac{\partial}{\partial y'}(F+\lambda G)\right)-\frac{\partial}{\partial y}(F+\lambda G)=0$$

So replace F by $F + \lambda G$

This will give as equation with 3 unknowns, 2 for integration constants, and one with λ, we solve for these given the Boundary conditions, and l but we do not have to do this, just need to derive the equations themselves.

Some integrals useful to know in solving the final integrals for the Euler problems are these

$$\int \frac{c}{\sqrt{y^2 - c^2}} dy = c \cosh^{-1}\left(\frac{y}{c}\right) + k$$

$$\int \frac{c}{\sqrt{1 - c^2\, y^2}} dy = c \sin^{-1}\left(c\, y\right) + k$$

$$\int \frac{c}{y\sqrt{y^2 - c^2}} dy = \frac{1}{c} \cos^{-1}\left(\frac{c}{y}\right) + k$$

4 MATH 121B Notes

4.1 Chapter 12. Series solution of ODE and special functions

Bessel ODE	$x^2\, y'' + x\, y' + \left(x^2 - p^2\right)y = 0$ defined for INTEGER and NON integer P
first solution	$y_1 = J_p(x) = \sum \frac{-1^n}{\Gamma(n+1)\Gamma\left(n+p+1\right)}\left(\frac{x}{2}\right)^{2n+p}$ (for p an integer or not)
second solution	$y_2 = N_p(x) = Y_p(x) = \frac{\cos\left(\pi p\right)J_p(x) - J_{-p}}{\sin \pi p}$ (note: p here is NOT an integer)
second solution	y_x contains a log function. note: p here IS an integer.
Orthogonality	$\int_0^1 x\, J_p(ax)\, J_p(bx)\, dx = \begin{cases} 0 & if\ a \neq b \\ \frac{1}{2}J_{p+1}^2(a) = \frac{1}{2}J_{p-1}^2(a) = \frac{1}{2}J_p'^2(a) & if\ a = b \end{cases}$ a, b are zeros of J_p
recursive formula	$\frac{d}{dx}\left[x^p J_p\right] = x^p J_{p-1},\quad \frac{d}{dx}\left[\frac{1}{x^p}J_p\right] = -\frac{1}{x^p}J_{p+1},\quad J_{p-1} + J_{p+1} = \frac{2p}{x}J_p,\quad J_{p-1} - J_{p+1} = 2J_p'$
	$J_p' = -\frac{p}{x}J_p + J_{p-1} = \frac{p}{x}J_p - J_{p+1}$ NOTICE: No Rodrigues formula for Bessel func, since not polyn.
notes:	We used a *generalized power series* method to find the solutions. IF p is NOT an integer, then J_p and J_{-p} (or N_p) are two independent solutions IF p is an integer, then J_p and J_{-p} are NOT two independent solutions, use log for y_2 J_p is called Bessel function of first kind, and Y_p is called second kind. p is called the ORDER. IF $p = n + \frac{1}{2}$, a special case, we get spherical bessel functions $j_n(x)$ and $y_n(x)$ $j_n(x) = \sqrt{\frac{\pi}{2x}}\, J_{n+\frac{1}{2}}(x) = x^n\left(-\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{\sin x}{x}\right)$

Legendre ODE	$\left(1-x^2\right)y'' - 2x\,y' + l(l+1)y = 0$ defined for INTEGER l only
first solution	$y_1 = P_l(x)$ examples: $P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}\left(3x^2-1\right), P_3(x) = \frac{1}{2}\left(5x^3-3x\right)$ $P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right),\ P_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right)$
second sol.	We do not use this. Called Legendre polynomials of second kind $Q_l(x)$
Orthogonality	$\int_{-1}^{1} P_l(x)\ P_m(x)\,dx = 0$ if $m \neq n$, also $\int_{-1}^{1} P_l(x)\ \times \left(\text{any poly degree} < l\right)dx = 0$
Normalization	$\int_{-1}^{1}[P_l(x)]^2dx = \frac{2}{2l+1}$
Generating function	$\Phi(x,h) = \frac{1}{\sqrt{(1-2xh+h^2)}},\ \lvert h\rvert < 1,\ \Phi(x,h) = P_0(x) + hP_1(x) + h^2P_2(x) + \cdots = \sum_{l=0}^{\infty} h^l P_l(x)$
recursive formula	later, see book page 491
Rodrigues	$P_l = \frac{1}{2^l l!}\frac{d^l}{dx^l}\left(x^2-1\right)^l$
notes:	we used series method to find the solution (not generalized series method). x must be less than 1, this is needed to have convergence. Hence Legendre solution only defined over $-1,1$. Also, l is assumed to be a non-negative integer.l is called the ORDER of legendre poly.

Associated Legendre	$\left(1-x^2\right)y'' - 2x\,y' + \left[l(l+1) - \frac{m^2}{1-x^2}\right]y = 0$
first solution	$y_1 = P_l^m(x) = \left(1-x^2\right)^{\frac{m}{2}}\ \frac{d^m}{dx^m}\left(p_l(x)\right)$
second solution	do not use
Orthogonality	did not cover, but should be the same as Legendre polynomials P_l
Normalization	$\int_{-1}^{1}\left[P_l^m(x)\right]^2dx = \frac{2}{2l+1}\left(\frac{l+m!}{l-m!}\right)$

example using recursive formula for Legendre: $\Phi(x,h) = \frac{1}{\sqrt{(1-2xh+h^2)}}$, let $y = 2xh - h^2$ then $\Phi(x,h) = \frac{1}{\sqrt{(1-y)}} = 1 + \frac{1}{2}y + \frac{\frac{1}{2}\frac{3}{2}}{2!}y^2 + \cdots$, then sub back for y, and simplify we get $\Phi = 1 + xh + h^2\left(\frac{3}{2}x^2 - \frac{1}{2}\right) + \cdots = P_0 + hP_1 + hP_2 + \cdots$, hence $P_0 = 1$, $P_1 = x, P_2 = \left(\frac{3}{2}x^2 - \frac{1}{2}\right)$,etc..

Series solution: $y = a_0 + a_1 x + a_2 x^2 + \cdots$

Generalized series solution: $y = a_0 x^s + a_1 x^{s+1} + a_2 x^{s+2} + \cdots$ solve for s, we get indicial eq. for each s we solve again to find the a_0 and the a_1 solutions. Final solution is the sum of the solutions for both s values. Will only get 2 solutions in total (for second order ODE).

4.1.1 Leibniz Rule for differentiation of product

$$\begin{aligned}
\frac{d^n}{dx^n}\left(fg\right) &= \begin{pmatrix} n \\ 0 \end{pmatrix}\frac{d^0}{dx^0}f\frac{d^n}{dx^n}g + \begin{pmatrix} n \\ 1 \end{pmatrix}\frac{d}{dx}f\ \frac{d^{n-1}}{dx^{n-1}}g + \begin{pmatrix} n \\ 2 \end{pmatrix}\frac{d^2}{dx^2}f\ \frac{d^{n-2}}{dx^{n-2}}g + \cdots + \begin{pmatrix} n \\ n \end{pmatrix}\frac{d^n}{dx^n}f\frac{d^0}{dx^0}g \\
&= \frac{d^0}{dx^0}f\frac{d^n}{dx^n}g + n\ \frac{d}{dx}f\ \frac{d^{n-1}}{dx^{n-1}}g + \frac{n \times n-1}{2!}\frac{d^2}{dx^2}f\ \frac{d^{n-2}}{dx^{n-2}}g + \cdots + \frac{d^n}{dx^n}f\frac{d^0}{dx^0}g
\end{aligned}$$

For example $\frac{d^9}{dx^9}(x\ \sin x) = x\ \frac{d^9}{dx^9}\sin x + 9 \times \frac{d}{dx}x\ \frac{d^3}{dx^3}\sin x +$ rest is ZERO terms, so we get $\frac{d^9}{dx^9}(x\ \sin x) = x\ \cos x + 9\sin x$ this is much faster than actually differentiating 9 times !

This can be remembered since it is the same form as the binomial expansion

$$\begin{aligned}
\left(f+g\right)^n &= \begin{pmatrix} n \\ 0 \end{pmatrix} f^0\ g^n + \begin{pmatrix} n \\ 1 \end{pmatrix} f^1\ g^{n-1} + \begin{pmatrix} n \\ 2 \end{pmatrix} f^2\ \ g^{n-1} + \cdots + \begin{pmatrix} n \\ n \end{pmatrix} f^n\ g^0 \\
\left(f+g\right)^9 &= \ g^9 +\ 9\ f\ g^8 + \ \frac{9 \times 8}{2!}\ f^2\ \ g^7 + \cdots +\ f^9
\end{aligned}$$

4.1.2 Finding second solution for ODE

when the indicial equation gives only one value for s (from the generalized power series method), we can find the second solution by assuming

$$y_2 = y_1 \ln(x) + \sum_{n=0}^{\infty} b_n\ x^{n+s}$$

Then find y', y'', from these, and sub back into ODE and set $n = 0$ to solve for the new indicial equation, find s from it (should get one solution), then most likely you'll find $b_n = 0$ for all $n > 0$ (for the HW's we did), and so just need to use $b_0 x^{n+s}$ and this gives the complete solution. $y = A\ y_1 + B\ y_2 = A\ y_1 + \left(y_1 \ln(x) + \sum_{n=0}^{\infty} b_n\ x^{n+s}\right)$

note: IF when solving the indicial equation, 2 values for s that differ by an integer from each others (say $4, 6$), then must use the value 6, also when we solve for the second solution, s there must come out to be the first s which we did not use for the first solution, i.e. 4 in this example (so I really do not need to solve for s again!, expect I need to find the recursive formula).

4.2 Chapter 16. Probability

Let A, B are 2 successive events.

$P_A(B)$ is the probability that B will happen KNOWING that A has already happened.

$P(AB)$ is the prob. that A and B will both happen

$$\begin{aligned} P(AB) &= P(A)P_A(B) \\ &= P(B)P_B(A) \end{aligned}$$

Or

$$P_A(B) = \frac{P(AB)}{P(A)}$$

If A and B are independent, then $P_A(B) = P(B)$

Then it follows that

$$P(AB) = P(A)\ (B) \qquad \text{If A,B independent}$$

Probability that A OR B will happen is:

$$\begin{aligned} &P(A+B) = P(A) + \ (B) - P(AB) \\ &P(A+B) = P(A) + \ (B) \qquad \text{IF A,B are mutually exclusive} \end{aligned}$$

This means that $P(AB) \ = 0$ if they are mutually exclusive (obvious)

$$P(A+B+C) = P(A) + P(B) + P(C) - \{P(AB) + P(AC) + P(BC)\} + P(ABC)$$

note: P_r^n = number of permutations (arrangements) or n things taken r at a time. $P_r^n = \frac{n!}{(n-r)!}$ Here order is important. i.e. ABC is DIFFERENT from CAB, hence this number will be larger than the one below.

$$\begin{pmatrix} n \\ r \end{pmatrix} = C_r^n = \frac{n!}{(n-1)!\ r!}$$

Number of combinations OR selections of n things r at a time. here order is NOT important. so ABC is counted the same as CAB, hence this number will be smaller.

note: In how many ways can 10 people be seated on a bench with 4 seats?

A)$\begin{pmatrix} 10 \\ 4 \end{pmatrix} 4! = \frac{10!}{6!4!} 4! = \frac{10!}{6!} = 10 \times 9 \times 8 \times 7$

To understand this: $\begin{pmatrix} 10 \\ 4 \end{pmatrix}$ is the number of ways 4 people can be selected out of 10. ONCE those 4 people have been selected, then there are 4! different ways they can be arranged on the bench. Hence the answer is we multiply these together.

note: Find number of ways of putting r particles in n boxes according to the 3 kinds of statistics.

Answer

1. For Maxwell-Blotzman (MB) it is n^r
2. For Fermi-Dirac (FM), it is ${}_nC_r$
3. For Bose-Einstein (BE) it is ${}_{n+1}C_r$

note: If asked this: there is box A which has 5 red balls and 6 black balls, and box B which has 5 red balls and 8 white balls, what is *the prob. of picking a red ball*? Answer:

P(pick box A) P(pick red ball from it) + P(pick box B) P(pick red ball from it)

note: If we get a problem such as 2 boxes A,B, and more than more try picking balls, it is easier to draw a tree diagram and pull the chances out the tree than having to calculate them directly in the exam. Tree can be drawn in 2 minutes and will have all the info I need.

note: write down the cancer chance problem.

note: random variable x is a function defined on the sample space (for the example, the sum of 2 die throw). The probability density is the probability of each random variable.

average or mean of a random variable $\mu = \sum x_i\ P_i$ where P_i is the probability of the random variable.

The Variance Var measures the spread of the random variable around the average, also called dispersion defined as

$$Var(x) = \sum \left(x_i - \mu\right)^2 P_i$$

Standard deviation is another measure of the dispersion, defined as $\sigma(x) = \sqrt{Var(x)}$

Distribution function is just a histogram of the probability density. it tells one what the probability of a random variable being less than a certain x value. see page 711.

4.3 Chapter 13. PDE

PDE	*solution*	*equation*	*notes*
Laplace	$u(x,y,z)$	$\nabla^2 u = 0$	describes steady state (no time) of region with no source for example, gravitional potential with no matter, electrostatic potential with no charge, or *steady state* Temp. distribution
Poisson	$u(x,y,z)$	$\nabla^2 u = f(x,y,z)$	Same as Laplace, i.e. sescribes steady state, howevere here the source of the field is present. $f(x,y,z)$ is called the source density. i.e. it is a function that describes the density distribution of the source of the potential.
Diffusion or heat equation	$u(t,x,y,z)$	$\nabla^2 u = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}$	Here u is usually the temperature T function. Now time is involved. So this equation is alive.
Wave equation	$u(t,x,y,z)$	$\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$	Here u is the position of a point on the wave at time t. Notice the wave equation has second derivative w.r.t. time while the diffusion is first derivative w.r.t. time
Helmholtz equation	$F(x,y,z)$	$\nabla^2 F + k^2 F = 0$	The diffusion and wave equation generate this. This is the SPACE only solution of the wave and heat equations. i.e. $u = F(x,y,z)T(t)$ is the solution for both heat and wave eq.

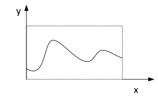
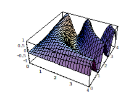
Each of these equations has a set of candidate solutions, which we start with and try to fit the boundary and initial condition into to eliminate some solution of this set that do not fit until we are left with the one candidate solution. We then use this candidate solution to find the general solution, which is a linear combination of it. We use fourier series expansion in this part of the solution.

In table below I show for each equation what the set of candidate solutions are. Use these to start the solution with unless the question asks to start at an earlier stage, which is the separation of variables.

So the algorithm for solving these PDE is

```
Select THE PDE to use ---->
Obtain set of candidate solution ---->
Eliminate those that do not fit ----->
obtain the general solution by linear combination
(use orthogonality principle here)
```

PDE	candidate solutions	notes
$\nabla^2 u = 0$	$u(x,y) = \begin{cases} e^{ky} \cos kx \\ e^{ky} \sin kx \\ e^{-ky} \cos kx \\ e^{-ky} \sin kx \end{cases}$	for 2 dimensions
$\nabla^2 u = f(x,y,z)$	$u(x,y,z) = -\frac{1}{4\pi} \int \int \int \frac{f(x',y',z')}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2+}} dx'\ dy'\ dz'$	$f(x',y',z')$ is a function that describes mass density distribution evaluated at point x',y',z'. The point x,y,z is where we are calculating the
		potential u itself
$\nabla^2 u = \frac{1}{\alpha^2} \frac{\partial u}{\partial t}$	$u(t,x) = \begin{cases} e^{-k^2\alpha^2 t} \cos kx \\ e^{-k^2\alpha^2 t} \sin kx \end{cases}$	for one space dimension

$\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$	$Y = XT$, where $X(x) = \begin{cases} \cos kx \\ \sin kx \end{cases}$ $T(t) = \begin{cases} \cos \omega t \\ \sin \omega t \end{cases}$ $Y(x,t) = \begin{cases} \cos kx\ \cos \omega t \\ \cos kx\ \sin \omega t \\ \sin kx\ \sin \omega t \\ \sin kx\ \cos \omega t \end{cases}$	v is the wave velosity, 1D 
	$Z = XYT$, where $X(x) = \begin{cases} \cos k_x x \\ \sin k_x x \end{cases}$ $Y(x) = \begin{cases} \cos k_y x \\ \sin k_y x \end{cases}$ $T(t) = \begin{cases} \cos \omega t \\ \sin \omega t \end{cases}$	2D case in rectangular coord 
$\nabla^2 F + k^2 F = 0$		

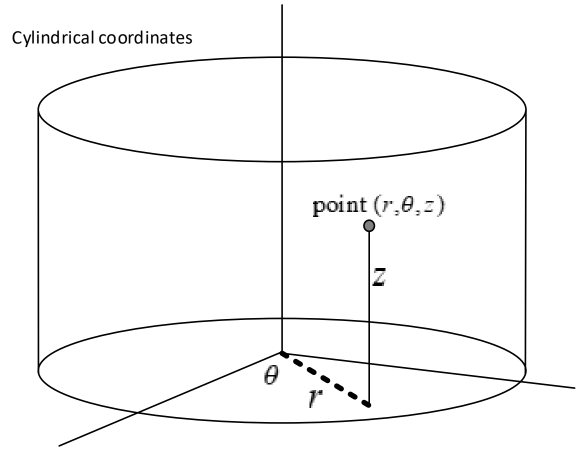
Now the solutions in different coordinates systems

$$X(x) = \begin{cases} \cos kx \\ \sin kx \end{cases}$$

$$T(t) = \begin{cases} \cos \omega t \\ \sin \omega t \end{cases}$$

4.3.1 Laplace equation in cylindrical coordinates

The Laplacian in cylindrical is $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$, the solution can be written as $u = R(r)\Theta(\theta)Z(z)$



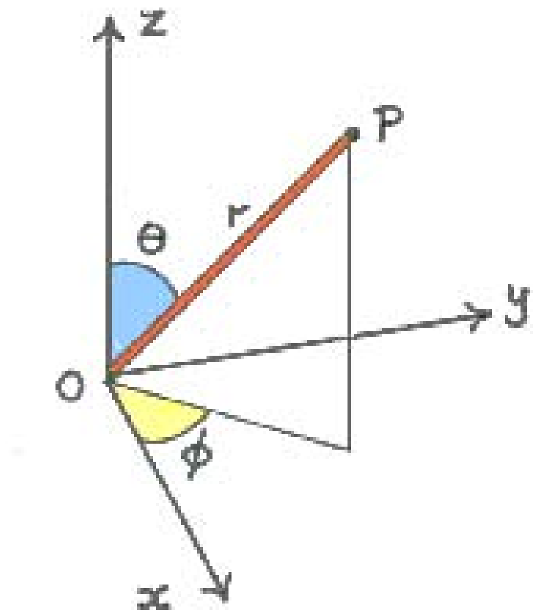
$Z(z) = \begin{cases} e^{kz} \\ e^{-kz} \end{cases}$, we quickly eliminate the e^{kz} since we do not want the potential to blow up as z becomes larger. $\Theta(\theta) = \begin{cases} \sin n\theta \\ \cos n\theta \end{cases}$, $R(r) = J_n(kr)$ where $J_n(kr)$ is Bessel function of order n, we do not use $N_n(kr)$ solutions since we origin is on base of cylinder. see book for more details, all problems we will get will be like this. We find k from boundary conditions, it will turn out to be the zeros of J_n. From above, the set of candidate solutions for Laplace on cylindrical is

$$u(r,\theta,z) = \begin{cases} J_n(kr)\ \sin n\theta\, e^{-kz} \\ \\ J_n(kr)\ \cos n\theta\, e^{-kz} \end{cases}$$

Now usually we eliminate the θ dependency if boundary condition is such that it is not dependent of angle. So we get $u(r,\theta,z) = J_0(k_m r)\ e^{-k_m z}$ and from this we need to solve $u = \sum_{m=1}^{\infty} c_m\, J_0(k_m r)\ e^{-k_m z}$, now we use boundary condition to find c_m, for example if given that base $(z = 0)$ was at temp (or potential) = 100, then we need to solve $100 = \sum_{m=1}^{\infty} c_m\, J_0(k_m r)$ and here to use orthogonality of bessel functions to expand RHS.

4.3.2 Laplace equation in spherical coordinates

The Laplacian in spherical is $\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\ \frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial^2 u}{\partial\phi^2} = 0$. Separate using $u = R(r)\Theta(\theta)\Phi\left(\phi\right)$



The solutions are $\Phi = \begin{cases} \sin m\phi \\ \cos m\phi \end{cases}$ and $\Theta = P_l^m(\cos\theta)$ and $R(r) = \begin{cases} r^l \\ r^{-l-1} \end{cases}$ here l is an integer (came from separation of constants by setting $k = l(l+1)$), Here P_l^m is the associated Legendre function. Now we quickly discard solution r^{-l-1} because we want solution inside the sphere, so our set of candidate solutions are $u = R(r)\Theta(\theta)\Phi\big(\phi\big) = r^l \quad P_l^m(\cos\theta) \begin{cases} \sin m\phi \\ \cos m\phi \end{cases}$. For symmetry w.r.t. ϕ we set $m = 0$ and solution reduces to $r^l \quad P_l(\cos\theta)$ and then the general solution is $u = \sum c_l\, r^l \quad P_l(\cos\theta)$

4.3.3 Wave equation in polar coordinates

5 General equations

$$\sin nx = \frac{e^{inx} - e^{-inx}}{2i}$$
$$\cos nx = \frac{e^{inx} + e^{-inx}}{2}$$

$$\int \frac{\sin x}{\cos x} dx = \ln(\sin x)$$
$$\csc x = \frac{1}{\sin x}$$
$$\text{average value of } f(x) \text{ over } [b,a] = \frac{\int_a^b f(x)dx}{b-a}$$

$$\cos^2 kx = \frac{1+\cos(2k)}{2}$$
$$\sin^2 kx = \frac{1-\cos(2k)}{2}$$

$$\sin A \ \sin B = \frac{1}{2}[\ \cos(A-B) - \cos(A+B)]$$
$$\cos A \ \cos B = \frac{1}{2}[\ \cos(A-B) + \cos(A+B)]$$
$$\sin A \ \cos B = \frac{1}{2}[\ \sin(A-B) + \sin(A+B)]$$

I need a geometric way to visualize these equations, but for now for the exam remember them as follows: they all start with $A - B$, and when the functions being multiplied are different on the LHS, we get sin on the RHS, else we get cos (think of cos as nicer, since even function :).

$$\int \tanh(x) \ = \ln(\cosh \ x)$$
$$\int \tan x = -\ln(\cos \ x)$$

$\int_a^b \cos^2 kx \ dx = \frac{b-a}{2}$. if $k(b-a)$ is an integer multiple of π. (the same for $\sin^2 kx$), for example $\int_{-1}^1 \cos^2 \pi x \ dx = 1$, $\int_{-1}^1 \cos^2 5\pi x \ dx = 1$, $\int_{-5}^1 \cos^2 7\pi x \ dx = 3$, $\int_{-1}^1 \sin^2 \pi x \ dx = 1$, etc... this can be very useful so remember it!

$\int_a^b \cos kx \ dx = 0$ if over a complete period. same for $\sin x$, for example $\int_{-\pi}^{\pi} \cos kx \ dx = 0$

$$\sinh x = -i \ \sin(ix)$$
$$\cosh x = \cos(ix)$$
$$\tanh x = -i \ \tan(ix)$$

$$e^{\ln z} = z$$
$$z^b = e^{b \ln z}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

$$\begin{aligned}
\arctan x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \\
\sinh x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \\
\cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - \cdots \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \qquad -1 < x \leq 1 \\
(1+x)^p &= 1 + px + \frac{p\left(p-1\right)}{2!}x^2 + \ \cdots + \qquad |x| < 1
\end{aligned}$$

Leibinz rule for differentiation of integrals

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(x,t)\,dt = f(x,v(x))\,\frac{d}{dx}v(x) - f(x,u(x))\,\frac{d}{dx}u(x) + \int_u^v \frac{\partial}{\partial x}f(x,t)\,dt$$

example:

$$\begin{aligned}
\frac{d}{dx}\int_x^{2x} \frac{e^{xt}}{t}\,dt &= \frac{e^{x(2x)}}{2x}\,\frac{d}{dx}(2x) - \frac{e^{x(x)}}{x}\,\frac{d}{dx}(x) + \int_x^{2x} \frac{\partial}{\partial x}\left(\frac{e^{xt}}{t}\right)\,dt \\
&= \frac{e^{2x^2}}{x} - \frac{e^{x^2}}{x} + \int_x^{2x} \frac{te^{xt}}{t}\,dt \\
&= \frac{e^{2x^2}}{x} - \frac{e^{x^2}}{x} + \left[\frac{e^{xt}}{x}\right]_x^{2x}
\end{aligned}$$

To help remember the above 2 formulas, notice that when $+x$ we get a $-$ shown (i.e. terms flip flop), but when we have $-x$ the series is all positive terms. These are very important to remember for problems when finding Laurent expansion of a function.

Expansion of cos **and** sin **around a point different than** 0 expand $\cos(z)$ around a, we get

$$\left(\cos(a)-\frac{\cos(a)(z-a)^2}{2!}+\frac{\cos(a)(z-a)^4}{4!}-\cdots\right)+\left(-\sin(a)(z-a)+\frac{\sin(a)(z-a)^3}{3!}\ \cdots\right)$$

For example to expand $\cos(x)$ about π we get

$$\left(\cos(\pi)-\frac{\cos(\pi)(z-\pi)^2}{2!}+\frac{\cos(\pi)(z-\pi)^4}{4!}-\cdots\right)+\overbrace{\left(-\sin(\pi)(z-\pi)+\frac{\sin(a)(z-\pi)^3}{3!}\ \cdots\right)}^{=0}$$
$$=-1+\frac{1}{2}(\pi-z)^2-\frac{1}{24}(\pi-z)^4+\ \cdots$$

so above is easy to remember. The $\cos(z)$ part is the same as around zero, but it has $\cos(a)$ multiplied to it, and the sin part is the same as the $\sin(z)$ about zero but has $\sin(a)$ multiplied to it, and the signs are reversed.

For expansion of $\sin(z)$ use

$$\left(\sin(a)-\frac{\sin(a)(z-a)^2}{2!}+\frac{\sin(a)(z-a)^4}{4!}-\cdots\right)+\left(\cos(a)(z-a)-\frac{\cos(a)(z-a)^3}{3!}\ \cdots\right)$$

This is the same as the expansion of $\cos(z)$ but the roles are reversed and notice the cos part start now with positive not negative term. SO all what I need to remember is that expansion of $\cos(z)$ starts with $\cos(a)$ terms while expansion of $\sin(z)$ start with the $\sin(a)$ term. This is faster than having to do Taylor series expansion to find these series in the exam.

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$
$$\Gamma(P+1)=P\Gamma(P)$$