HW 8, Math 121 A
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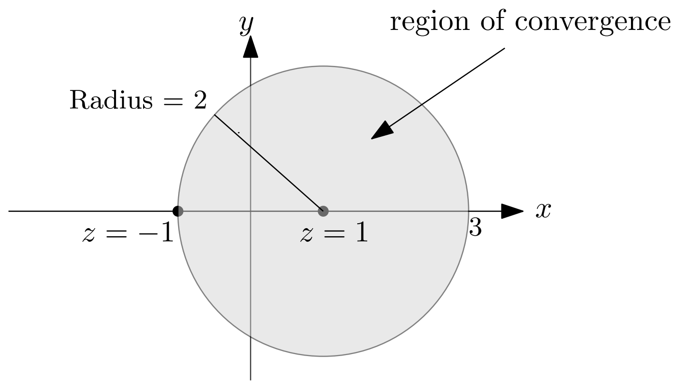
Contents

1 chapter 14, problem 6.5

Problem Find Laurent series for $f(z) = \frac{e^z}{z^2-1}$ around $z = 1$
Solution
There are two poles $z = \pm 1$. Hence the expansion around $z = 1$ will extend around $z = 1$ up to the next pole, at $z = -1$. So it will make a circle centered at $z = 1$ and radius 2.



<u>Method one</u>
Let

$$\begin{aligned} g(z) &= (z-1) f(z) \\ &= (z-1)\frac{e^z}{(z-1)(z+1)} \\ &= \frac{e^z}{(z+1)} \end{aligned}$$

Now $g(z)$ is can be expanded around $z = 1$ since it is analytic at $z = 1$. Using Taylor series.

$$\begin{aligned} g'(z) &= \frac{e^z(z+1) - e^z}{(z+1)^2} \\ g''(z) &= \frac{(e^z(z+1) + e^z - e^z)(z+1)^2 - (e^z(z+1) - e^z)2(z+1)}{(z+1)^4} \\ &\vdots \end{aligned}$$

Hence

$$\begin{aligned} g'(1) &= \frac{2e - e}{4} = \frac{e}{4} \\ g''(1) &= \frac{(2e + e - e)(4) - (2e - e)2(2)}{(2)^4} = \frac{4e}{16} = \frac{e}{4} \\ &\vdots \end{aligned}$$

Therefore, Taylor series for $g(z)$ around $z = 1$ is

$$\begin{aligned} g(z) &= g(1) + g'(1)(z-1) + \frac{g''(1)(z-1)^2}{2!} + \cdots \\ &= \frac{e}{2} + \frac{e}{4}(z-1) + \frac{e}{8}(z-1)^2 + \cdots \end{aligned}$$

But $f(z) = \frac{g(z)}{(z-1)}$, hence

$$\begin{aligned} f(z) &= \frac{1}{(z-1)}\left(\frac{e}{2} + \frac{e}{4}(z-1) + \frac{e}{8}(z-1)^2 + \cdots\right) \\ &= \frac{e}{2(z-1)} + \frac{e}{4} - \frac{e}{8}(z-1) + \cdots \end{aligned}$$

<u>Method two</u>
In this method, and when the expansion is about a point z_0 which is not zero, it is easiest to use the substitution $u = z - z_0$ first. Hence $u = z - 1$ or $z = u + 1$ and now $f(z)$ becomes

$$\begin{aligned} f(z) &= \frac{e^{u+1}}{(u+1)^2 - 1} \\ &= \frac{e^{u+1}}{u^2 + 2u} \\ &= \frac{e^{u+1}}{u(u+2)} \\ &= \frac{e^{u+1}}{u}\frac{1}{u+2} \\ &= \frac{e^{u+1}}{2u}\frac{1}{\left(1 + \frac{u}{2}\right)} \end{aligned} \tag{1}$$

And now we can expand $\frac{1}{\left(1+\frac{u}{2}\right)}$ using Binomial series for $\left|\frac{u}{2}\right| < 1$ or $-1 < \frac{u}{2} < 1$ or $-2 < u < 2$ or $-2 < z - 1 < 2$ or $-1 < z < 3$. Hence (1) becomes

$$f(z) = \frac{e^{u+1}}{2u}\left(1 - \frac{u}{2} + \left(\frac{u}{2}\right)^2 - \left(\frac{u}{2}\right)^3 + \cdots\right)$$

But $e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots$, hence the above can be written as

$$\begin{aligned} f(z) &= \frac{e}{2u}\left(1 - \frac{u}{2} + \left(\frac{u}{2}\right)^2 - \left(\frac{u}{2}\right)^3 + \cdots\right)\left(1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots\right) \\ &= \frac{e}{2u}\left[\left(1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots\right) - \frac{u}{2}\left(1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots\right) + \left(\frac{u}{2}\right)^2\left(1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots\right) + \cdots\right] \\ &= \frac{e}{2u}\left[\left(1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots\right) - \left(\frac{u}{2} + \frac{u^2}{2} + \frac{u^3}{4} + \frac{u^4}{(2)\,3!} + \cdots\right) + \left(\frac{u^2}{4} + \frac{u^3}{4} + \frac{u^4}{8} + \frac{u^5}{24} + \cdots\right) + \cdots\right] \end{aligned}$$

The above simplifies to

$$\begin{aligned} f(z) &= \frac{e}{2u}\left(1 + \frac{u}{2} + \frac{u^2}{4} + \frac{u^3}{24} + \cdots\right) \\ &= e\left(\frac{1}{2u} + \frac{1}{4} + \frac{u}{8} + \frac{u^2}{48} + \cdots\right) \end{aligned}$$

Replacing u back by $z - 1$ gives

$$f(z) = \frac{e}{2(z-1)} + \frac{e}{4} + \frac{e(z-1)}{8} + \frac{e(z-1)^2}{48} + \cdots$$

Hence residue is $\frac{e}{2}$. The above is valid for $-1 < z < 3$. Or $|z-1| < 2$. The above is the same answer found using method one. Method one is more direct, but requires lots of differentiations to find the Taylor series for $g(z)$.

2 chapter 14, problem 6.6

Problem Find Laurent series for $f(z) = \sin\left(\frac{1}{z}\right)$ around $z = 0$
Solution
Since expansion about zero is

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Then

$$\sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{(3!)\,z^3} + \frac{1}{(5!)\,z^5} - \frac{1}{(7!)\,z^7} + \cdots$$

Residue is 1. Since the series contains only the principal part and no analytical part and then number of terms with negative powers is infinite, then $z = 0$ is an essential singularity.

3 chapter 14, problem 6.7

Problem Find Laurent series for $f(z) = \frac{\sin \pi z}{4z^2 - 1}$ around $z = \frac{1}{2}$
Solution

$$\begin{aligned} f(z) &= \frac{1}{4}\frac{\sin \pi z}{z^2 - \frac{1}{4}} \\ &= \frac{1}{4}\frac{\sin \pi z}{\left(z - \frac{1}{2}\right)\left(z + \frac{1}{2}\right)} \end{aligned} \tag{1}$$

Let us first consider $g(z) = \frac{1}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{2}\right)}$. When we have the expansion about point z_0 not zero, it is easiest to use the substitution $u = z - z_0$ first. Hence $u = z - \frac{1}{2}$ or $z = u + \frac{1}{2}$ and now $g(z)$ becomes

$$\begin{aligned} g(z) &= \frac{1}{u(1+u)} \\ &= \frac{1}{u}\left(1 - u + u^2 - u^3 + \cdots\right) \\ &= \frac{1}{u} - 1 + u - u^2 + \cdots \end{aligned}$$

Replacing back to z the above becomes

$$g(z) = \frac{1}{z - \frac{1}{2}} - 1 + \left(z - \frac{1}{2}\right) - \left(z - \frac{1}{2}\right)^2 + \cdots$$

Hence (1) becomes

$$
\begin{aligned}
f(z) &= \frac{1}{4}\sin(\pi z)\, g(z) \\
&= \frac{1}{4}\sin(\pi z)\left(\frac{1}{z-\frac{1}{2}} - 1 + \left(z-\frac{1}{2}\right) - \left(z-\frac{1}{2}\right)^2 + \cdots\right)
\end{aligned}
$$

Now we expand $\sin(\pi z)$ but remember to expand it around $z = \frac{1}{2}$. The above becomes

$$
\begin{aligned}
f(z) &= \frac{1}{4}\left(1 - \frac{1}{2}\pi^2\left(z-\frac{1}{2}\right)^2 + \frac{1}{24}\pi^4\left(z-\frac{1}{2}\right)^4 - \cdots\right)\left(\frac{1}{z-\frac{1}{2}} - 1 + \left(z-\frac{1}{2}\right) - \left(z-\frac{1}{2}\right)^2 + \cdots\right) \\
&= \frac{1}{4}\frac{1}{z-\frac{1}{2}} - \frac{1}{4} + \left(\frac{1}{4} - \frac{\pi^2}{8}\right)\left(z-\frac{1}{2}\right) + \cdots
\end{aligned}
$$

Hence residue is $\frac{1}{4}$.

4 chapter 14, problem 6.9

Problem Find Laurent series for $f(z) = \frac{1+\cos z}{(z-\pi)^2}$ around $z = \pi$

Solution

We just need to expand $\cos z$ around $z = \pi$ here. This gives

$$
\cos z = -1 + \frac{1}{2}(z-\pi)^2 - \frac{1}{2}(z-\pi)^4 + \cdots
$$

Hence $f(z)$ becomes

$$
\begin{aligned}
f(z) &= \frac{1 + \left(-1 + \frac{1}{2}(z-\pi)^2 - \frac{1}{2}(z-\pi)^4 + \cdots\right)}{(z-\pi)^2} \\
&= \frac{\frac{1}{2}(z-\pi)^2 - \frac{1}{2}(z-\pi)^4 + \cdots}{(z-\pi)^2} \\
&= \frac{1}{2} - \frac{1}{2}(z-\pi)^4 + \cdots
\end{aligned}
$$

Residue is zero.

5 chapter 14, problem 6.8

Problem Find Laurent series for $f(z) = \frac{1}{z^2-5z+6}$ around $z = 2$

Solution

$$
f(z) = \frac{1}{(z-3)(z-2)}
$$

Let $u = z - 2$ or $z = u + 2$ and the above becomes

$$
\begin{aligned}
f(z) &= \frac{1}{(u-1)u} \\
&= \frac{-1}{u}\left(\frac{1}{1-u}\right) \\
&= \frac{-1}{u}\left(1 + u + u^2 + u^3 + \cdots\right) \\
&= -\left(\frac{1}{u} + 1 + u + u^2 + \cdots\right)
\end{aligned}
$$

But $u = z - 2$ hence the above becomes

$$
\begin{aligned}
f(z) &= -\left(\frac{1}{z-2} + 1 + (z-2) + (z-2)^2 + \cdots\right) \\
&= \frac{-1}{z-2} - 1 - (z-2) - (z-2)^2 - \cdots
\end{aligned}
$$

Reside is -1.

6 chapter 14, problem 6.15

Problem Find residue at $z=\frac{1}{2}$ and $z=\frac{4}{5}$for $f(z)=\frac{1}{(1-2z)(5z-4)}$
Solution

$$
\begin{aligned}
\text{Residue}\left(\frac{1}{2}\right) &= \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)f(z)\\
&= \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)\frac{1}{(1-2z)(5z-4)}\\
&= \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)\frac{1}{\left(2\left(\frac{1}{2}-z\right)\right)(5z-4)}\\
&= \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)\frac{-1}{\left(2\left(z-\frac{1}{2}\right)\right)(5z-4)}\\
&= \lim_{z\to\frac{1}{2}}\frac{-1}{2(5z-4)}\\
&= \frac{-1}{2\left(5\left(\frac{1}{2}\right)-4\right)}\\
&= \frac{1}{3}
\end{aligned}
$$

And

$$
\begin{aligned}
\text{Residue}\left(\frac{4}{5}\right) &= \lim_{z\to\frac{4}{5}}\left(z-\frac{4}{5}\right)f(z)\\
&= \lim_{z\to\frac{4}{5}}\left(z-\frac{4}{5}\right)\frac{1}{(1-2z)(5z-4)}\\
&= \lim_{z\to\frac{4}{5}}\left(z-\frac{4}{5}\right)\frac{1}{(1-2z)\,5\left(z-\frac{4}{5}\right)}\\
&= \lim_{z\to\frac{4}{5}}\frac{1}{5(1-2z)}\\
&= \frac{1}{5\left(1-2\left(\frac{4}{5}\right)\right)}\\
&= \frac{-1}{2\left(5\left(\frac{1}{2}\right)-4\right)}\\
&= -\frac{1}{3}
\end{aligned}
$$

7 chapter 14, problem 6.19

Problem Find residue at $z=\frac{\pi}{2}$ for $f(z)=\frac{\sin^2 z}{2z-\pi}$
Solution

$$
\begin{aligned}
\text{Residue}\left(\frac{\pi}{2}\right) &= \lim_{z\to\frac{\pi}{2}}\left(z-\frac{\pi}{2}\right)\frac{\sin^2 z}{2z-\pi}\\
&= \lim_{z\to\frac{\pi}{2}}\left(z-\frac{\pi}{2}\right)\frac{\sin^2 z}{2\left(z-\frac{\pi}{2}\right)}\\
&= \lim_{z\to\frac{\pi}{2}}\frac{\sin^2 z}{2}\\
&= \frac{\sin^2\left(\frac{\pi}{2}\right)}{2}\\
&= \frac{1}{2}
\end{aligned}
$$

8 chapter 14, problem 6.23

Problem Find residue at $z=\frac{2i}{3}$ for $f(z)=\frac{e^{iz}}{9z^2+4}$
Solution

$$
\begin{aligned}
\text{Residue}\left(\frac{2i}{3}\right) &= \lim_{z\to\frac{2i}{3}}\left(z-\frac{2i}{3}\right)\frac{e^{iz}}{9z^2+4}\\
&= \lim_{z\to\frac{2i}{3}}\left(z-\frac{2i}{3}\right)\frac{e^{iz}}{9\left(z^2+\frac{4}{9}\right)}\\
&= \frac{1}{9}\lim_{z\to\frac{2i}{3}}\left(z-\frac{2i}{3}\right)\frac{e^{iz}}{\left(z-\frac{2i}{3}\right)\left(z+\frac{2i}{3}\right)}\\
&= \frac{1}{9}\lim_{z\to\frac{2i}{3}}\frac{e^{iz}}{\left(z+\frac{2i}{3}\right)}\\
&= \frac{1}{9}\frac{e^{i\frac{2i}{3}}}{\left(\frac{2i}{3}+\frac{2i}{3}\right)}\\
&= \frac{1}{9}\frac{e^{\frac{-2}{3}}}{\frac{4i}{3}}\\
&= \frac{1}{3}\frac{e^{\frac{-2}{3}}}{4i}\\
&= -i\frac{e^{-\frac{2}{3}}}{12}
\end{aligned}
$$

9 chapter 14, problem 6.31

Problem Find residue at $z=0$ for $f(z)=\frac{e^{3z}-3z-1}{z^4}$
Solution
Pole is of order $m=4$, so we use the formula.

$$
\begin{aligned}
\text{Residue}(0) &= \lim_{z\to 0}\frac{1}{(m-1)!}\left(\frac{d^{m-1}}{z^m}(z-0)^m f(z)\right)\\
&= \lim_{z\to 0}\frac{1}{3!}\left(\frac{d^3}{z^3}\left(z^4\frac{e^{3z}-3z-1}{z^4}\right)\right)\\
&= \lim_{z\to 0}\frac{1}{3!}\left(\frac{d^3}{z^3}\left(e^{3z}-3z-1\right)\right)\\
&= \lim_{z\to 0}\frac{1}{3!}\left(\frac{d^2}{z^2}\left(3e^{3z}-3\right)\right)\\
&= \lim_{z\to 0}\frac{1}{3!}\left(\frac{d}{z}9e^{3z}\right)\\
&= \lim_{z\to 0}\frac{1}{3!}\left(27e^{3z}\right)\\
&= \frac{27}{6}\\
&= \frac{9}{2}
\end{aligned}
$$

10 chapter 14, problem 7.4

Problem Evaluate $I=\int_0^{2\pi}\frac{\sin^2\theta}{5+3\cos\theta}d\theta$
Solution let $z=e^{i\theta}$ then

$$
\begin{aligned}
dz &= ie^{i\theta}d\theta\\
d\theta &= \frac{dz}{ie^{i\theta}}\\
&= -i\frac{dz}{z}
\end{aligned}
$$

Hence $d\theta = -i\frac{dz}{z}$. Now using

$$\begin{aligned}
\sin^2\theta &= \frac{1-\cos(2\theta)}{2} \\
&= \frac{1-\left(\frac{e^{i2\theta}+e^{-i2\theta}}{2}\right)}{2} \\
&= \frac{1-\left(\frac{z^2+z^{-2}}{2}\right)}{2} \\
&= \frac{\frac{2-z^2-z^{-2}}{2}}{2} \\
&= \frac{2-z^2-\frac{1}{z^2}}{4} \\
&= \frac{2z^2-z^4-1}{4z^2}
\end{aligned}$$

And

$$\cos\theta = \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right) = \left(\frac{z+z^{-1}}{2}\right)$$

Then the integral becomes

$$\begin{aligned}
I &= \int_0^{2\pi} \frac{\sin^2\theta}{5+3\cos\theta} d\theta \\
&= \oint \frac{\frac{2z^2-z^4-1}{4z^2}}{5+3\left(\frac{z+z^{-1}}{2}\right)}\left(-i\frac{dz}{z}\right) \\
&= -i\oint \frac{\frac{2z^2-z^4-1}{4z^2}}{z\left(5+3\left(\frac{z+z^{-1}}{2}\right)\right)} dz \\
&= -i\oint \frac{2\left(2z^2-z^4-1\right)}{4z^3\left(10+3\left(z+z^{-1}\right)\right)} \\
&= \frac{-i}{2}\oint \frac{2z^2-z^4-1}{z^3\left(\frac{10z+3(z^2+1)}{z}\right)} dz \\
&= \frac{-i}{2}\oint \frac{2z^2-z^4-1}{z^2\left(10z+3z^2+3\right)} dz
\end{aligned}$$

Using residue theorem,

$$\begin{aligned}
\oint \frac{2z^2-z^4-1}{z^2\left(10z+3z^2+3\right)} dz &= \oint f(z)\ dz \\
&= 2\pi i \sum \text{residues of } f(z) \text{ inside}
\end{aligned}$$

Hence

$$\begin{aligned}
I &= \frac{-i}{2}\oint \frac{2z^2-z^4-1}{z^2\left(10z+3z^2+3\right)} dz \\
&= \frac{-i}{2}\left[2\pi i \sum R(f(z))\right] \\
&= -i^2\pi \sum R(f(z)) \\
&= \pi \sum R(f(z))
\end{aligned}$$

Now we need to find residues of $f(z)$

$$\frac{2z^2-z^4-1}{z^2\left(10z+3z^2+3\right)} = f(z) = \frac{g(z)}{h(z)}$$

factoring $h(z) = z^2(3+z)(1+3z)$ gives

$$f(z) = \frac{g(z)}{h(z)} = \frac{2z^2-z^4-1}{z^2(3+z)(1+3z)}$$

Need to find residue inside a unit circle. $3z=-1$ or $z_0 = -1/3$ is inside the unit circle, also $z_1 = 0$ is inside the circle

$$\begin{aligned}
\lim_{z\to z_0}(z-z_0)\frac{g(z)}{h(z)} &= \lim_{z\to -1/3}\left(z+\frac{1}{3}\right)\frac{2z^2-z^4-1}{z^2(3+z)(1+3z)}\\
&= \lim_{z\to -1/3}\frac{2z^2-z^4-1}{3z^2(3+z)}\\
&= \frac{2\left(-\frac{1}{3}\right)^2-\left(-\frac{1}{3}\right)^4-1}{3\left(-\frac{1}{3}\right)^2\left(3-\frac{1}{3}\right)}\\
&= \frac{\frac{2}{9}-\frac{1}{81}-1}{3\left(\frac{1}{9}\right)\left(3-\frac{1}{3}\right)}\\
&= \frac{\frac{18-1-81}{81}}{\left(\frac{1}{3}\right)\left(\frac{9-1}{3}\right)}\\
&= \frac{\frac{-64}{81}}{\left(\frac{8}{9}\right)}\\
&= \frac{-8}{9}
\end{aligned}$$

To find residue at zero, since it is order $m=2$

$$\begin{aligned}
\lim_{z\to z_1}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}(z-z_1)^2\frac{g(z)}{h(z)} &= \lim_{z\to 0}\frac{1}{(2-1)!}\frac{d}{dz}(z)^2\frac{2z^2-z^4-1}{z^2(3+z)\,3\left(\frac{1}{3}+z\right)}\\
&= \lim_{z\to 0}\frac{d}{dz}\frac{2z^2-z^4-1}{(3+z)\,3\left(\frac{1}{3}+z\right)}\\
&= \lim_{z\to 0}\frac{d}{dz}\frac{2z^2-z^4-1}{3+10z+3z^2}\\
&= \lim_{z\to 0}\frac{1}{3+10z+3z^2}\left(4z-4z^3\right)+\left[\left(2z^2-z^4-1\right)\left(\frac{-1}{\left(3+10z+3z^2\right)^2}\right)(10+6z)\right]\\
&= \frac{1}{3}(0)+(-1)\left(\frac{-1}{(3)^2}\right)(10)\\
&= \frac{10}{9}
\end{aligned}$$

Hence

$$\begin{aligned}
I &= \frac{-i}{2}\oint\frac{2z^2-z^4-1}{z^2\left(10z+3z^2+3\right)}dz\\
&= \pi\sum R(f(z))\\
&= \pi\left(\frac{10}{9}-\frac{8}{9}\right)
\end{aligned}$$

Therefore

$$\begin{aligned}
I &= \int_0^{2\pi}\frac{\sin^2\theta}{5+3\cos\theta}d\theta\\
&= \frac{2}{9}\pi
\end{aligned}$$

11 chapter 14, problem 7.5

Problem Evaluate $\int_0^{\pi}\frac{d\theta}{1-2r\cos\theta+r^2}$ for $0\le r<1$
Solution
Since even function then,

$$\int_0^{\pi}\frac{d\theta}{1-2r\cos\theta+r^2}=\frac{1}{2}\int_0^{2\pi}\frac{d\theta}{1-2r\cos\theta+r^2}$$

Using $\cos\theta = \frac{z+z^{-1}}{2}$ and $z = e^{i\theta}$ then $dz = ie^{i\theta}d\theta = izd\theta$. The integral becomes

$$
\begin{aligned}
I &= \frac{1}{2}\oint_C \frac{1}{1-2r\left(\frac{z+z^{-1}}{2}\right)+r^2}\frac{dz}{iz} \\
&= -\frac{1}{2}i\oint_C \frac{1}{1-r\left(z+\frac{1}{z}\right)+r^2}\frac{dz}{z} \\
&= -\frac{1}{2}i\oint_C \frac{1}{z-r\left(z^2+1\right)+zr^2}dz \\
&= -\frac{1}{2}i\oint_C \frac{1}{z-rz^2-r+zr^2}dz \\
&= -\frac{1}{2}i\oint_C \frac{1}{z^2\left(-r\right)+z\left(1+r^2\right)-r}dz \\
&= \frac{1}{2}\frac{i}{r}\oint_C \frac{1}{z^2+z\left(\frac{-1}{r}-r\right)+1}dz \\
&= \frac{1}{2}\frac{i}{r}\oint_C \frac{1}{\left(z-r\right)\left(z-\frac{1}{r}\right)}dz
\end{aligned}
$$

Since $|r|<1$ then only pole inside the unit circle is $z=r$. We do not need to find residue for the pole at $z=\frac{1}{r}$ since it is outside. Hence

$$
\begin{aligned}
I &= \frac{i}{2r}2\pi i\,\text{Residue}\left(r\right) \\
&= \frac{-\pi}{r}\,\text{Residue}\left(r\right)
\end{aligned}
$$

Where $f\left(z\right) = \frac{1}{(z-r)(z-\frac{1}{r})}$ for purpose of finding residue.

$$
\begin{aligned}
\text{Residue}\left(r\right) &= \lim_{z\rightarrow r}\left(z-r\right)\frac{1}{\left(z-r\right)\left(z-\frac{1}{r}\right)} \\
&= \lim_{z\rightarrow r}\frac{1}{\left(z-\frac{1}{r}\right)} \\
&= \frac{1}{\left(r-\frac{1}{r}\right)} \\
&= \frac{r}{r^2-1}
\end{aligned}
$$

Hence

$$
\begin{aligned}
I &= \frac{-\pi}{r}\left(\frac{r}{r^2-1}\right) \\
&= \frac{-\pi}{r^2-1} \\
&= \frac{\pi}{1-r^2}
\end{aligned}
$$

12 chapter 14, problem 7.12

Problem Evaluate $\int_0^\infty \frac{x^2}{x^4+16}dx$

Solution

The integrand is even. Hence $I = \int_0^\infty \frac{x^2}{x^4+16}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{x^2}{x^4+16}dx$. First we find location of poles. These are roots of

$$
\begin{aligned}
x^4+16 &= 0 \\
x^4 &= -16 \\
x &= \left(-16\right)^{\frac{1}{4}} \\
&= \left(16\right)^{\frac{1}{4}}\left(-1\right)^{\frac{1}{4}} \\
&= 2\left(-1\right)^{\frac{1}{4}}
\end{aligned}
$$

To find $\left(-1\right)^{\frac{1}{4}}$, we write it as

$$
\begin{aligned}
\left(-1\right)^{\frac{1}{4}} &= \left(e^{i\pi}\right)^{\frac{1}{4}} \\
&= e^{i\frac{(\pi+2n\pi)}{4}} \qquad n=0,1,2,3
\end{aligned}
$$

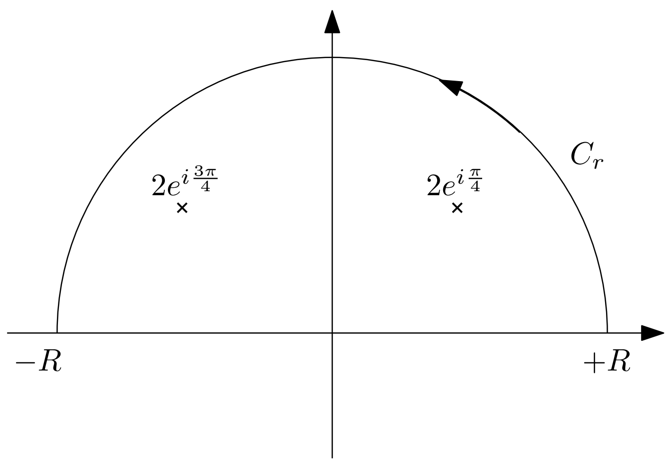
Therefore the four roots are

$$2e^{i\frac{\pi}{4}}, 2e^{i\frac{3}{4}\pi}, 2e^{i\frac{5}{4}\pi}, 2e^{i\frac{7}{4}\pi}$$

Or

$$2e^{\pm i\frac{\pi}{4}}, 2e^{\pm i\frac{3}{4}\pi}$$

Now we use the following contour to do the integration, such that the upper half circle include inside it the first two roots above, since these are the only ones in the upper half plane.



$$\oint_C f(z)\,dz = 2\pi i \sum \text{sum of residues inside}$$

Where C above is the contour from $-R$ to $+R$ and around C_r as shown. So all what we need to do now if found the residues. There are two poles inside C. These are $z_1 = 2e^{i\frac{\pi}{4}}$ and $z_2 = 2e^{i\frac{3}{4}\pi}$. Therefore using

$$\begin{aligned}
\text{Residue}(z_1) &= \lim_{z\to z_1}(z-z_1) f(z) \\
\text{Residue}\left(2e^{i\frac{\pi}{4}}\right) &= \lim_{z\to 2e^{i\frac{\pi}{4}}}\left(z-2e^{i\frac{\pi}{4}}\right)\frac{z^2}{z^4+16} \\
&= \lim_{z\to 2e^{i\frac{\pi}{4}}}\frac{z^3-2z^2e^{\frac{i\pi}{4}}}{z^4+16}
\end{aligned}$$

Applying L'Hopitals rules gives

$$\begin{aligned}
\text{Residue}(z_1) &= \lim_{z\to 2e^{i\frac{\pi}{4}}}\frac{3z^2-4ze^{\frac{i\pi}{4}}}{4z^3} \\
&= \frac{3\left(2e^{i\frac{\pi}{4}}\right)^2-4\left(2e^{i\frac{\pi}{4}}\right)e^{\frac{i\pi}{4}}}{4\left(2e^{i\frac{\pi}{4}}\right)^3} \\
&= \frac{12e^{i\frac{\pi}{2}}-8e^{i\frac{\pi}{2}}}{32e^{i\frac{3\pi}{4}}} \\
&= \frac{4e^{i\frac{\pi}{2}}}{32e^{i\frac{3\pi}{4}}} \\
&= \frac{1}{8}e^{-i\frac{\pi}{4}}
\end{aligned} \tag{1}$$

Similarly

$$\begin{aligned}
\text{Residue}(z_2) &= \lim_{z\to z_2}(z-z_2) f(z) \\
\text{Residue}\left(2e^{i\frac{\pi}{4}}\right) &= \lim_{z\to 2e^{i\frac{3}{4}\pi}}\left(z-2e^{i\frac{3}{4}\pi}\right)\frac{z^2}{z^4+16} \\
&= \lim_{z\to 2e^{i\frac{3}{4}\pi}}\frac{z^3-2z^2e^{i\frac{3}{4}\pi}}{z^4+16}
\end{aligned}$$

Applying L'Hopitals rules gives

$$
\begin{aligned}
\text{Residue}\left(z_2\right) &= \lim_{z\to 2e^{i\frac{3}{4}\pi}} \frac{3z^2 - 4ze^{i\frac{3}{4}\pi}}{4z^3} \\
&= \frac{3\left(2e^{i\frac{3}{4}\pi}\right)^2 - 4\left(2e^{i\frac{3}{4}\pi}\right)e^{i\frac{3}{4}\pi}}{4\left(2e^{i\frac{3}{4}\pi}\right)^3} \\
&= \frac{12e^{i\frac{3}{2}\pi} - 8e^{i\frac{3}{2}\pi}}{32e^{i\frac{9}{4}\pi}} \\
&= \frac{4e^{i\frac{3}{2}\pi}}{32e^{i\frac{9}{4}\pi}} \\
&= \frac{1}{8}e^{i\left(\frac{3\pi}{2}-\frac{9}{4}\pi\right)} \\
&= \frac{1}{8}e^{-i\frac{3}{4}\pi}
\end{aligned} \tag{2}
$$

From (1) and (2) then

$$
\begin{aligned}
\oint_C f(z)\,dz &= 2\pi i\left(\frac{1}{8}e^{-i\frac{\pi}{4}} + \frac{1}{8}e^{-i\frac{3}{4}\pi}\right) \\
&= 2\pi i\left(-\frac{1}{8}i\sqrt{2}\right) \\
&= \frac{1}{4}\sqrt{2}\pi
\end{aligned}
$$

Therefore, since

$$
\oint_C f(z)\,dz = \lim_{R\to\infty}\int_{-R}^{R}\frac{x^2}{x^4+16}dx + \lim_{R\to\infty}\int_{CR}\frac{z^2}{z^4+16}dz
$$

Then

$$
\frac{1}{4}\sqrt{2}\pi = \lim_{R\to\infty}\int_{-R}^{R}\frac{x^2}{x^4+16}dx + \lim_{R\to\infty}\int_{CR}\frac{z^2}{z^4+16}dz \tag{3}
$$

The only thing left is to show that $\lim_{R\to\infty}\int_{CR}\frac{z^2}{z^4+16}dz = 0$. Let $z = R\,ie^{\theta}$, then $\frac{dz}{d\theta} = Rie^{i\theta}$, therefore the second integral above can be written as

$$
\begin{aligned}
\lim_{R\to\infty}\int_{CR}\frac{z^2}{z^4+16}dz &= \lim_{R\to\infty}\int_0^{\pi}\frac{R^2\left(ie^{i\theta}\right)^2}{\left(Rie^{i\theta}\right)^4+16}Rie^{i\theta}d\theta \\
&= \lim_{R\to\infty}\int_0^{\pi}\frac{R^3\left(ie^{i\theta}\right)^2}{R^4e^{i4\theta}+16}ie^{i\theta}d\theta
\end{aligned}
$$

As $R\to\infty$ the integrand goes to zero, since the numerator has R^3 and the denominator has R^4. In other words, $\frac{R^3}{R^4+16}$ or $\frac{\frac{1}{R^4}}{1+\frac{16}{R^4}}$ which goes to zero as $R\to\infty$. Hence (3) simplifies to

$$
\begin{aligned}
\frac{1}{4}\sqrt{2}\pi &= \lim_{R\to\infty}\int_{-R}^{R}\frac{x^2}{x^4+16}dx \\
\int_{-\infty}^{\infty}\frac{x^2}{x^4+16}dx &= \frac{1}{4}\sqrt{2}\pi
\end{aligned}
$$

Therefore

$$
\int_0^{\infty}\frac{x^2}{x^4+16}dx = \frac{1}{8}\sqrt{2}\pi
$$

13 chapter 14, problem 7.18

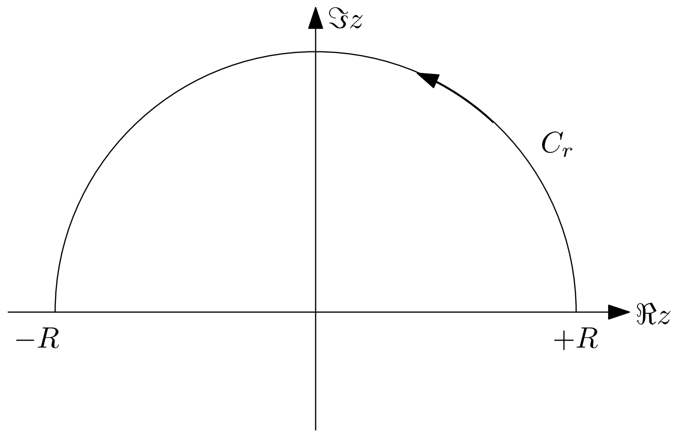
Problem Evaluate

$$
I = \int_0^{\infty}\frac{\cos\pi x}{1+x^2+x^4}dx
$$

Solution
Since this is an even function then

$$
\begin{aligned}
\int_0^{\infty}\frac{\cos\pi x}{1+x^2+x^4}dx &= \frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos\pi x}{1+x^2+x^4}dx \\
&= \frac{1}{2}\operatorname{Re}\int_{-\infty}^{\infty}\frac{e^{i\pi x}}{1+x^2+x^4}dx
\end{aligned} \tag{1}
$$

Now consider the contour shown



Then considering the integral

$$I = \oint f(z)\,dz$$

$$= \lim_{R\to\infty}\left(\int_{-R}^{R}\frac{e^{i\pi x}}{1+x^2+x^4}dx + \int_C \frac{e^{i\pi z}}{1+z^2+z^4}dz\right) \tag{2}$$

But by Cauchy theorem,

$$2\pi i \sum \text{residues of } f(z) \text{ inside contour} = \oint f(z)\,dz$$

The second integral to the right in (1) can be shown to go to zero as $R \to \infty$ (See below). Hence the above simplifies to

$$2\pi i \sum \text{residues of } f(z) \text{ inside contour} = \lim_{R\to\infty}\int_{-R}^{R}\frac{e^{i\pi x}}{1+x^2+x^4}dx \tag{3}$$

Now we need to find residues of

$$f(z) = \frac{e^{i\pi z}}{1+z^2+z^4}$$

Looking at $1+z^2+z^4$, let $z^2=\beta$, then find root of $1+\beta+\beta^2=0$, the roots are

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-1\pm\sqrt{1-4}}{2}$$

$$= \frac{-1\pm i\sqrt{3}}{2}$$

Hence

$$\left(\beta - \frac{-1+i\sqrt{3}}{2}\right)\left(\beta - \frac{-1-i\sqrt{3}}{2}\right) = 0$$

Replacing $z^2=\beta$ gives

$$\left(z^2 - \frac{-1+i\sqrt{3}}{2}\right)\left(z^2 - \frac{-1-i\sqrt{3}}{2}\right) = 0$$

looking at each term. $z^2 - \frac{-1+i\sqrt{3}}{2} = 0$ results in

$$z = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$= \frac{0\pm\sqrt{0-4\left(-\frac{-1+i\sqrt{3}}{2}\right)}}{2}$$

$$= \pm\sqrt{\frac{-1+i\sqrt{3}}{2}}$$

$$= \pm\sqrt{\frac{-1}{2}+\frac{i\sqrt{3}}{2}}$$

Hence the first 2 roots are

$$\left(\frac{-1}{2} - \frac{i\sqrt{3}}{2}\right),\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)$$

Now to find the second 2 roots, looking at second $\left(z^2 - \frac{-1-i\sqrt{3}}{2}\right) = 0$ results in

$$\begin{aligned} z &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{0 \pm \sqrt{-4\left(-\frac{-1-i\sqrt{3}}{2}\right)}}{2} \\ &= \frac{\pm\sqrt{-2 - 2i\sqrt{3}}}{2} \\ &= \pm\sqrt{-\frac{1}{2} - \frac{i}{2}\sqrt{3}} \end{aligned}$$

Hence

$$= \pm\sqrt{\left(\frac{1}{2} - \frac{i}{2}\sqrt{3}\right)\left(\frac{1}{2} - \frac{i}{2}\sqrt{3}\right)} = \pm\sqrt{\left(\frac{1}{2} - \frac{i}{2}\sqrt{3}\right)^2} = \pm\left(\frac{1}{2} - \frac{i}{2}\sqrt{3}\right)$$

And the second two roots are

$$\left(\frac{1}{2} - \frac{i}{2}\sqrt{3}\right), \left(-\frac{1}{2} + \frac{i}{2}\sqrt{3}\right)$$

So the poles of $f(z)$ are

$$\begin{aligned} z_1 &= \left(\frac{-1}{2} - \frac{i\sqrt{3}}{2}\right) \\ z_2 &= \left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) \\ z_3 &= \left(\frac{1}{2} - \frac{i}{2}\sqrt{3}\right) \\ z_4 &= \left(-\frac{1}{2} + \frac{i}{2}\sqrt{3}\right) \end{aligned}$$

Now need to find which ones are inside the contour. Need the poles with a positive imaginary parts. Hence looking at above, these are z_2 and z_4. To find the residue of $f(z)$ at z_2

$$\begin{aligned} \lim_{z\to z_2}(z - z_2) f(z) &= \lim_{z\to z_2}\left(z - \left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\right)\frac{e^{i\pi z}}{1 + z^2 + z^4} \\ &= \lim_{z\to z_2}\frac{ze^{i\pi z} - \left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)e^{i\pi z}}{1 + z^2 + z^4} \end{aligned}$$

Applying L'Hopitals

$$\begin{aligned} \text{Residue}(z_2) &= \lim_{z\to z_2}\frac{e^{i\pi z} + i\pi z e^{i\pi z} - i\pi\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)e^{i\pi z}}{2z + 4z^3} \\ &= \frac{e^{i\pi\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)} + i\pi\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)e^{i\pi\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)} - i\pi\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)e^{i\pi\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)}}{2\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) + 4\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^3} \\ &= \frac{e^{\left(\frac{i\pi}{2}-\frac{\sqrt{3}\pi}{2}\right)} + i\pi\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)e^{i\pi\left(\frac{i\pi}{2}-\frac{\sqrt{3}\pi}{2}\right)} - i\pi\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)e^{i\pi\left(\frac{i\pi}{2}-\frac{\sqrt{3}}{2}\pi\right)}}{i\sqrt{3} - 3} \\ &= \frac{ie^{-\frac{1}{2}\sqrt{3}\pi}}{i\sqrt{3} - 3} \end{aligned}$$

Now to find the second residue

$$\begin{aligned} \lim_{z\to z_2}(z - z_4) f(z) &= \lim_{z\to z_4}\left(z - \left(-\frac{1}{2} + \frac{i}{2}\sqrt{3}\right)\right)\frac{e^{i\pi z}}{1 + z^2 + z^4} \\ &= \lim_{z\to z_4}\frac{ze^{i\pi z} - \left(-\frac{1}{2} + \frac{i}{2}\sqrt{3}\right)e^{i\pi z}}{1 + z^2 + z^4} \end{aligned}$$

Applying L'Hopitals

$$
\begin{aligned}
\text{Residue}\left(z_2\right) &= \lim_{z\to z_2} \frac{e^{i\pi z} + i\pi z e^{i\pi z} - i\pi\left(-\frac{1}{2}+\frac{i}{2}\sqrt{3}\right)e^{i\pi z}}{2z+4z^3}\\
&= \frac{e^{i\pi\left(-\frac{1}{2}+\frac{i}{2}\sqrt{3}\right)} + i\pi\left(-\frac{1}{2}+\frac{i}{2}\sqrt{3}\right)e^{i\pi\left(-\frac{1}{2}+\frac{i}{2}\sqrt{3}\right)} - i\pi\left(-\frac{1}{2}+\frac{i}{2}\sqrt{3}\right)e^{i\pi\left(-\frac{1}{2}+\frac{i}{2}\sqrt{3}\right)}}{2\left(-\frac{1}{2}+\frac{i}{2}\sqrt{3}\right)+4\left(-\frac{1}{2}+\frac{i}{2}\sqrt{3}\right)^3}\\
&= \frac{e^{-\frac{1}{2}\pi\left(\sqrt{3}+i\right)}}{i\sqrt{3}+3}
\end{aligned}
$$

The above is the second residue. Hence Sum of residues is

$$
\begin{aligned}
\frac{i\exp\left(-\frac{\pi\sqrt{3}}{2}\right)}{i\sqrt{3}-3} - \frac{i\ \exp\left(-\frac{\pi\sqrt{3}}{2}\right)}{i\sqrt{3}+\ 3} &= i\exp\left(-\frac{\pi\sqrt{3}}{2}\right)\left(\frac{1}{i\sqrt{3}-3}-\frac{1}{i\sqrt{3}+\ 3}\right)\\
&= i\exp\left(-\frac{\pi\sqrt{3}}{2}\right)\left(\frac{\left(i\sqrt{3}+\ 3\right)-\left(i\sqrt{3}-3\right)}{\left(i\sqrt{3}-3\right)\left(i\sqrt{3}+\ 3\right)}\right)\\
&= i\exp\left(-\frac{\pi\sqrt{3}}{2}\right)\left(\frac{6}{-3-9}\right)\\
&= i\exp\left(-\frac{\pi\sqrt{3}}{2}\right)\left(\frac{-1}{2}\right)\\
&= -\frac{i}{2}\exp\left(-\frac{\pi\sqrt{3}}{2}\right)
\end{aligned}
$$

Now, from (3) we obtain

$$
\begin{aligned}
\lim_{R\to\infty}\int_{-R}^{+R}\frac{e^{i\pi x}}{1+x^2+x^4}dx &= 2\pi i\left(-\frac{i}{2}\exp\left(-\frac{\pi\sqrt{3}}{2}\right)\right)\\
&= \pi\ \exp\left(-\frac{\pi\sqrt{3}}{2}\right)
\end{aligned}
$$

Taking the real part of both sides above gives

$$
\int_{-\infty}^{+\infty}\frac{\cos\pi x}{1+x^2+x^4}dx = \pi\ \exp\left(-\frac{\pi\sqrt{3}}{2}\right)
$$

Using the above in (1) gives

$$
\begin{aligned}
\int_0^{\infty}\frac{\cos\pi x}{1+x^2+x^4}dx &= \frac{1}{2}\int_{-\infty}^{+\infty}\frac{\cos\pi x}{1+x^2+x^4}dx\\
&= \frac{\pi}{2}\exp\left(-\frac{\pi\sqrt{3}}{2}\right)
\end{aligned}
$$

What is left is to show that $\lim_{R\to\infty}\int_C e^{imz}f(z)\,dz = 0$. To do this, we use Jordan Lemma, which says that integral $\lim_{R\to\infty}\int_C e^{imz}f(z)\,dz \to 0$ if $|f(z)|_{\max}\to 0$ as $|z|\to\infty$. But

$$
\begin{aligned}
\lim_{R\to\infty}|f(z)|_{\max} &= \lim_{R\to\infty}\frac{1}{|1+z^2+z^4|_{\min}}\\
&= \lim_{R\to\infty}\frac{1}{\left|1+\left(R\,e^{i\theta}\right)^2+\left(R\,e^{i\theta}\right)^4\right|_{\min}}\\
&= \lim_{R\to\infty}\frac{1}{1+R^2+R^4}\\
&= \lim_{R\to\infty}\frac{\frac{1}{R^4}}{\frac{1}{R^4}+\frac{1}{R^2}+1}\\
&= \frac{0}{1}\\
&= 0
\end{aligned}
$$

Therefore $\lim_{R\to\infty}\int_C e^{imz}f(z)\,dz = 0$, which is all what we needed to show to complete the solution.

14 chapter 14, problem 7.20

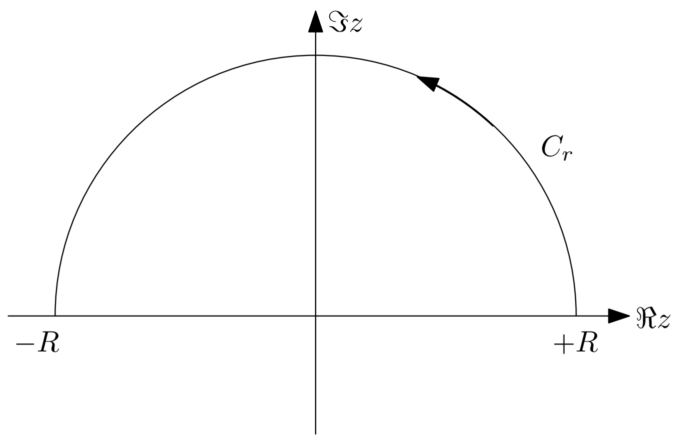
Problem Evaluate

$$I=\int_0^{\infty}\frac{\cos x}{\left(1+9x^2\right)^2}dx$$

Solution consider

$$\oint\frac{e^{iz}}{\left(1+9z^2\right)^2}dz$$

over the contour shown.



By Cauchy theorem,

$$\oint\frac{e^{iz}}{\left(1+9z^2\right)^2}dz=2\pi i\sum\text{residues of } f(z) \text{ inside contour}$$

But

$$\oint\frac{e^{iz}}{\left(1+9z^2\right)^2}dz=\int_{-R}^{+R}\frac{e^{ix}}{\left(1+9z^2\right)^2}dx+\int_C\frac{e^{iz}}{\left(1+9z^2\right)^2}dz$$

second integral to the right above can be shown to go to zero as $R\longrightarrow\infty$ as was done in earlier problem. Hence

$$\oint\frac{e^{iz}}{\left(1+9z^2\right)^2}dz=\int_{-\infty}^{+\infty}\frac{e^{ix}}{\left(1+9x^2\right)^2}dx=2\pi i\sum\text{residues } f(z)$$

Or

$$\int_{-\infty}^{+\infty}\frac{e^{ix}}{\left(1+9x^2\right)^2}dx=2\pi i\sum\text{residues } f(z)$$

Now we need to find residues of

$$f(z)=\frac{e^{iz}}{\left(1+9z^2\right)^2}$$

Looking at

$$\begin{aligned}1+9z^2&=0\\ z^2&=-\frac{1}{9}\\ z&=\pm\frac{i}{3}\end{aligned}$$

Hence the poles of $f(z)$ are

$$\begin{aligned}z_1&=-\frac{i}{3}\\ z_2&=+\frac{i}{3}\end{aligned}$$

Each is of order $m=2$. Hence, $\left(1+9z^2\right)^2$ can be written as $\left(9\left(z-\frac{i}{3}\right)\left(z+\frac{i}{3}\right)\right)^2$ or $81\left(z-\frac{i}{3}\right)^2\left(z+\frac{i}{3}\right)^2$. Therefore

$$f(z)=\frac{1}{81}\frac{e^{iz}}{\left(z-\frac{i}{3}\right)^2\left(z+\frac{i}{3}\right)^2}$$

Now need to find which ones are inside the contour. Need the poles with a positive imaginary parts. Looking at above, z_2 is the pole we need to find residue for. Hence, to find the residue R of $f(z)$ at z_2, using $m = 2$

$$
\begin{aligned}
R &= \lim_{z\to z_2} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}(z-z_2)^m f(z) \\
&= \lim_{z\to z_2} \frac{d}{dz}(z-z_2)^2 \ \frac{1}{81}\frac{e^{iz}}{(z-z_1)^2(z-z_2)^2} \\
&= \frac{1}{81}\lim_{z\to z_2}\frac{d}{dz}\frac{e^{iz}}{(z-z_1)^2} \\
&= \frac{1}{81}\lim_{z\to z_2}\frac{1}{(z-z_1)^2}(ie^{iz}) + e^{iz}\left(-2\frac{1}{(z-z_1)^3}\right) \\
&= \frac{1}{81}\left(\frac{1}{(z_2-z_1)^2}(ie^{iz_2}) + e^{iz_2}\left[-2\frac{1}{(z_2-z_1)^3}\right]\right) \\
&= \frac{1}{81}e^{iz_2}\left(\frac{i}{(z_2-z_1)^2} - \frac{2}{(z_2-z_1)^3}\right) \\
&= \frac{1}{81}e^{i\frac{i}{3}}\left(\frac{i}{\left(\frac{i}{3}-\left(-\frac{i}{3}\right)\right)^2} - \frac{2}{\left(\frac{i}{3}-\left(-\frac{i}{3}\right)\right)^3}\right) \\
&= \frac{1}{81}e^{\frac{-1}{3}}\left(\frac{i}{\left(\frac{2i}{3}\right)^2} - \frac{2}{\left(\frac{2i}{3}\right)^3}\right) \\
&= \frac{1}{81}e^{\frac{-1}{3}}\left(\frac{-9i}{4} + \frac{54}{8i}\right) \\
&= \frac{1}{81}e^{\frac{-1}{3}}\left(\frac{-18i-54i}{8}\right) \\
&= \frac{1}{81}e^{\frac{-1}{3}}(-9i)
\end{aligned}
$$

Hence

$$
R = -\frac{1}{9}ie^{\frac{-1}{3}}
$$

Since

$$
\int_{-\infty}^{+\infty}\frac{e^{ix}}{(1+9x^2)^2}dx = 2\pi i\sum \text{residues} \ \ f(z)
$$

Then

$$
\begin{aligned}
\int_{-\infty}^{+\infty}\frac{e^{ix}}{(1+9x^2)^2}dx &= 2\pi i\left(-\frac{1}{9}ie^{\frac{-1}{3}}\right) \\
&= \left(\frac{1}{9}\right)2\pi\left(e^{\frac{-1}{3}}\right) \\
&= \frac{2}{9}\pi\left(e^{\frac{-1}{3}}\right)
\end{aligned}
$$

Taking the real part of both sides above gives

$$
\int_{-\infty}^{+\infty}\frac{\cos x}{(1+9x^2)^2}dx = \frac{2}{9}\pi\left(e^{\frac{-1}{3}}\right)
$$

Since $\frac{\cos x}{(1+9x^2)^2}$ is an even function, then

$$
\begin{aligned}
\int_{-\infty}^{+\infty}\frac{\cos x}{(1+9x^2)^2}dx &= 2\int_{0}^{+\infty}\frac{\cos x}{(1+9x^2)^2}dx \\
&= \frac{\pi}{9}\left(e^{\frac{-1}{3}}\right)
\end{aligned}
$$

15 chapter 14, problem 7.29

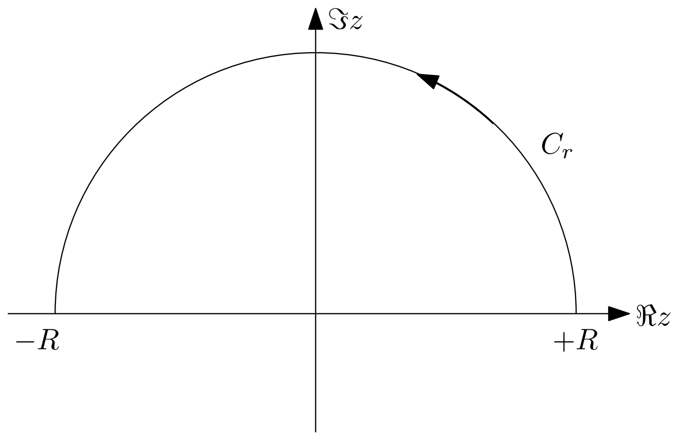
Evaluate the following integral. Find principal value if necessary

$$
I = \int_0^\infty \frac{\sin \ ax}{x}dx
$$

consider

$$
\oint \frac{e^{iaz}}{z}dz
$$

over the contour shown.



The contour avoids the singularity at $z = 0$, hence by Cauchy theorem,

$$\oint \frac{e^{iaz}}{z} dz = 0$$

since $f(z)$ is analytic on and everywhere inside the contour. Then

$$\oint \frac{e^{iaz}}{z} dz = 0 = \int_{-R}^{-r} \frac{e^{iax}}{x} dx + \int_{C'} \frac{e^{iaz}}{z} dz + \int_{+r}^{+R} \frac{e^{iax}}{x} dx + \int_{C} \frac{e^{iaz}}{z} dz$$

Looking at $\int_{C'} \frac{e^{iaz}}{z} dz$ let $z = re^{i\theta}, dz = ire^{i\theta} d\theta, \frac{dz}{z} = id\theta$, then

$$\int_{C'} \frac{e^{iaz}}{z} dz = \int_{\pi}^{0} e^{i\ a\ re^{i\theta}}\ id\theta$$

As $r \longrightarrow 0$, $e^{i\ a\ re^{i\theta}} \longrightarrow 1$, hence

$$\int_{C'} \frac{e^{iaz}}{z} dz \longrightarrow \int_{\pi}^{0} id\theta = [i\theta]_{\pi}^{0} = -i\pi$$

Hence, as $r \longrightarrow 0$ and $R \longrightarrow \infty$

$$0 = \int_{-\infty}^{-0} \frac{e^{iax}}{x} dx - i\pi + \int_{+0}^{+\infty} \frac{e^{iax}}{x} dx + \int_{C} \frac{e^{iaz}}{z} dz$$

But $\int_{C} \frac{e^{iaz}}{z} dz \longrightarrow 0$ as $R \longrightarrow \infty$ as from before and as shown in book page 603. Hence

$$\begin{aligned} 0 &= \int_{-\infty}^{-0} \frac{e^{iax}}{x} dx - i\pi + \int_{+0}^{+\infty} \frac{e^{iax}}{x} dx \\ i\pi &= \int_{-\infty}^{-0} \frac{e^{iax}}{x} dx + \int_{+0}^{+\infty} \frac{e^{iax}}{x} dx \\ i\pi &= \int_{-\infty}^{+\infty} \frac{e^{iax}}{x} dx \end{aligned}$$

Equating imaginary and real parts part of the above equation

$$i\pi = \int_{-\infty}^{+\infty} \frac{\cos ax + i \sin ax}{x} dx$$

Hence

$$\begin{aligned} \pi &= \int_{-\infty}^{+\infty} \frac{\sin ax}{x} dx \\ 0 &= \int_{-\infty}^{+\infty} \frac{\cos ax}{x} dx \end{aligned}$$

Now, $\frac{\sin ax}{x}$ is an even function, since $f(-x) = \frac{\sin(-ax)}{-x} = \frac{-\sin ax}{-x} = \frac{\sin ax}{x} = f(x)$. Therefore

$$\pi = \int_{-\infty}^{+\infty} \frac{\sin ax}{x} dx = 2 \int_{0}^{+\infty} \frac{\sin ax}{x} dx$$

And

$$\int_{0}^{+\infty} \frac{\sin ax}{x} dx = \frac{\pi}{2}$$

16 Chapter 14, problem 7.30 part (a)

Evaluate the following integral by the method of example 2

$$I = \int_0^\infty \frac{1}{1+x^4} dx$$

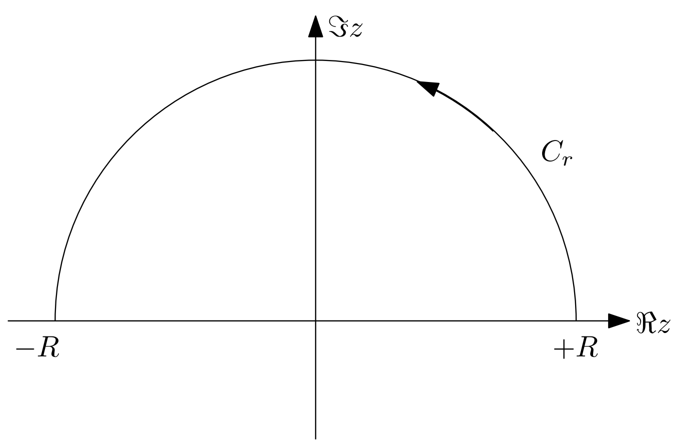
First, let me find what function we have.

$$\begin{aligned} f(-x) &= \frac{1}{1+(-x)^4} \\ &= \frac{1}{1+(x)^4} \\ &= f(x) \end{aligned}$$

Hence an even function. Consider

$$\oint \frac{1}{1+z^4} dz$$

over the contour shown.



Hence

$$\begin{aligned} \oint \frac{1}{1+z^4} dz &= \int_{-R}^{+R} \frac{1}{1+x^4} dx + \int_C \frac{1}{1+z^4} dz \\ &= 2\pi i \sum \text{residues } f(z) \text{ inside} \end{aligned}$$

As $R \to \infty$, $\int_C \frac{1}{1+z^4} dz \to 0$ as shown before and as shown in book page 603. Hence

$$\begin{aligned} \oint \frac{1}{1+z^4} dz = \int_{-\infty}^{+\infty} \frac{1}{1+x^4} dx &= 2\pi i \sum \text{residues } f(z) \text{ inside} \\ \int_{-\infty}^{+\infty} \frac{1}{1+x^4} dx = 2\pi i \sum \text{residues } f(z) \text{ inside} \end{aligned}$$

Now to find residues of

$$f(z) = \frac{1}{1+z^4}$$

At poles inside C. Finding roots of polynomial $1 + z^4 = 0$

$$\begin{aligned} z^4 &= -1 \\ z &= -1^{\frac{1}{4}} = \left(e^{i\pi}\right)^{\frac{1}{4}} = e^{i\frac{(\pi+2\pi n)}{4}} \qquad n = 0, 1, 2, 3 \end{aligned}$$

Hence the roots are

$$\left(e^{i(\pi)}\right)^{\frac{1}{4}}, \left(e^{-i(\pi+2\pi)}\right)^{\frac{1}{4}}, \left(e^{-i(\pi+4\pi)}\right)^{\frac{1}{4}}, \left(e^{-i(\pi+6\pi)}\right)^{\frac{1}{4}}$$
$$e^{i\left(\frac{\pi}{4}\right)}, e^{i\left(\frac{3}{4}\pi\right)}, e^{i\left(\frac{5}{4}\pi\right)}, e^{i\left(\frac{7}{4}\pi\right)}$$

The poles are

$$\begin{aligned} z_1 &= e^{i\left(\frac{\pi}{4}\right)} \\ z_2 &= e^{i\left(\frac{3}{4}\pi\right)} \\ z_3 &= e^{-i\left(\frac{\pi}{4}\right)} \\ z_4 &= e^{-i\left(\frac{3}{4}\pi\right)} \end{aligned}$$

Out of these zeros, we want the ones with positive imaginary parts since those are the ones inside the contour. From above, those are z_1 and z_2. Therefore to find residue at z_1

$$\begin{aligned}\lim_{z\to z_1}(z-z_1)f(z) &= \lim_{z\to z_1}\left(z-e^{i\frac{\pi}{4}}\right)\frac{1}{1+z^4}\\ &= \lim_{z\to z_1}\frac{z-e^{i\frac{\pi}{4}}}{1+z^4}\end{aligned}$$

Applying L'Hopitals

$$\begin{aligned}\text{Residue}\,(z_1) &= \lim_{z\to z_1}\frac{1}{4z^3}\\ &= \frac{1}{4\left(e^{i\frac{\pi}{4}}\right)^3}\\ &= \frac{1}{4}e^{-i\frac{3\pi}{4}}\end{aligned}$$

Now we find residue at the other pole, at z_2

$$\begin{aligned}\lim_{z\to z_2}(z-z_2)f(z) &= \lim_{z\to z_2}\left(z-e^{i\frac{3}{4}\pi}\right)\frac{1}{1+z^4}\\ &= \lim_{z\to z_1}\frac{z-e^{i\frac{3}{4}\pi}}{1+z^4}\end{aligned}$$

Applying L'Hopitals

$$\begin{aligned}\text{Residue}\,(z_1) &= \lim_{z\to z_1}\frac{1}{4z^3}\\ &= \frac{1}{4\left(e^{i\frac{3}{4}\pi}\right)^3}\\ &= \frac{1}{4}e^{-i\frac{9\pi}{4}}\\ &= \frac{1}{4}e^{-i\frac{\pi}{4}}\end{aligned}$$

Hence sum of residues is

$$\begin{aligned}\sum\text{residues} &= \frac{1}{4}e^{-i\frac{3\pi}{4}}+\frac{1}{4}e^{-i\frac{\pi}{4}}\\ &= \frac{-\sqrt{2}}{4}i\end{aligned}$$

Therefore

$$\begin{aligned}\int_{-\infty}^{+\infty}\frac{1}{1+x^4}dx &= 2\pi i\sum\text{residues } f(z)\text{ inside}\\ &= 2\pi i\left(\frac{-\sqrt{2}}{4}i\right)\\ &= \pi\frac{\sqrt{2}}{2}\end{aligned}$$

But $\frac{1}{1+x^4}$ is an even function, hence

$$\int_0^{+\infty}\frac{1}{1+x^4}dx = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{1}{1+x^4}dx = \frac{1}{2}\pi\frac{\sqrt{2}}{2} = \pi\frac{\sqrt{2}}{4}$$

Therefore

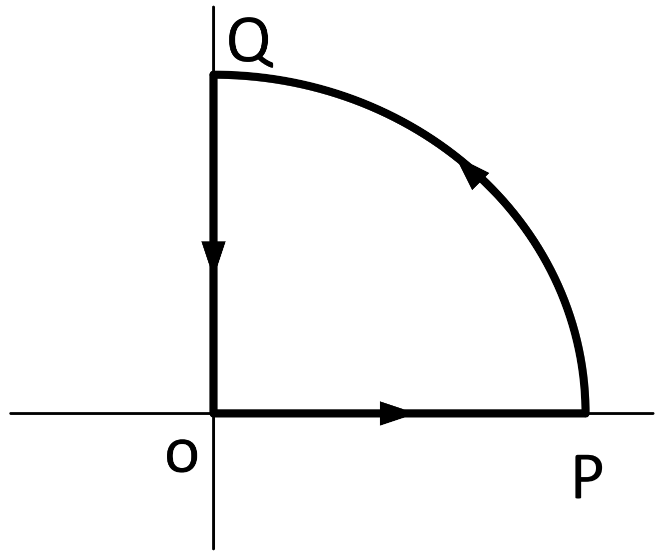
$$\int_0^{+\infty}\frac{1}{1+x^4}dx = \pi\frac{\sqrt{2}}{4}$$

17 chapter 14, problem 7.45

Problem determine in which quadrants the roots of the following equation lie

$$f(z) = z^3+z^2+z+4=0$$

Look at $f(z)$ in the first quadrant, put a contour as shown in figure,



And using theorem 7.8, which says

$$N - P = \frac{1}{2\pi}\left(\text{change of angle of } f(z) \text{ around contour}\right)$$

Where N is number of zeros of $f(z)$ INSIDE contour, and P is number of poles of $f(z)$ INSIDE contour. The main idea is that we want to see how much does the argument changes as the complex number is mapped by $f(z)$ from $z \longrightarrow z^3 + z^2 + z + 4$, then using 7.8 theorem this will tell us the number of zeros in the first quadrant. From this was can find where the other 2 roots are.
First, along path OP, $f(z)$ is real and equals $x^3 + x^2 + x + 4$ which is always > 0. On path QO, i.e. on imaginary y axis,

$$\begin{aligned} f(z) &= (iy)^3 + (iy)^2 + iy + 4 = -iy^3 - y^2 + iy + 4 \\ &= i\left(-y^3 + y\right) + \left(4 - y^2\right) \end{aligned}$$

If this to be zero, then $\left(4 - y^2\right) = 0$, or $y^2 = 4$, or $y = \pm 2$.Then looking at the imaginary part of $f(z)$ which is $\left(-y^3 + y\right)$ and substitute these y values, we get , when $y = +2$, $\left(-2^3 + 2\right) = -6 \neq 0$, and when $y = -2$, we get $\left(-(-2)^3 - 2\right) = 6 \neq 0$, hence $f(z)$ is not zero on OP and not zero on QO.
Now on the arc PQ, we can make the radius as large as we want to contains all zeros of $f(z)$ inside, so to make no zeros on PQ or outside. So, now we need to find the angle change of $f(z)$. on OP, z is real x, so any point on x is mapped to a point on x , hence no argument is changed, i.e. angle change is zero by the function. On arc PQ write $z = re^{i\theta}$, so mapping of $f(z)$ results in

$$\begin{aligned} f(z) &= \left(re^{i\theta}\right)^3 + \left(re^{i\theta}\right)^2 + re^{i\theta} + 4 \\ &= \left(re^{i\theta}\right)^3 + \left(re^{i\theta}\right)^2 + re^{i\theta} + 4 \\ &= r^3 e^{3i\theta} + r^2 e^{2i\theta} + re^{i\theta} + 4 \end{aligned}$$

For very large radius r, r^3term dominates, and so $f(z) \approx r^3 e^{3i\theta}$, so as z moves from 0 to $\frac{\pi}{2}$, the change in angle cause by $f(z)$ will be 3θ or $3 \times \frac{\pi}{2} = \frac{3\pi}{2}$. On QO, i.e. the imaginary axis, $z = iy$ then

$$f(z) = (iy)^3 + (iy)^2 + iy + 4 = i\left(-y^3 + y\right) + \left(4 - y^2\right)$$

Hence in the w domain, i.e. looking at the output of $w = f(z)$, the angle that the complex number w makes is $\tan\Theta = \frac{(y-y^3)}{(4-y^2)}$.
For very large y, $\tan\Theta \approxeq -y = -\infty$ and from above we see that the angle start at $\frac{3\pi}{2}$ when y is very large. now we decrease y as we move down to the origin and see how the angle changes. Looking at $\tan\Theta = \frac{(y-y^3)}{(4-y^2)}$, as y get smaller $y - y^3$ becomes smaller but remains negative, this is until $y - y^3 = 0$, or $y = 0$ at which time the angle is 2π (note the final angle is 2π, for $\arctan 0$ and not angle of zero, since the tangent was negative when we started and continued to be negative but smaller and smaller, hence the final angle is 2π).
Since we got to $y = 0$, this completes the contour. So angle change is 2π. From

$$N - P = \frac{1}{2\pi}\left(\text{change of angle of } f(z) \text{ around contour}\right)$$

And since poles of $f(z)$ do not exist (it has no denominator), then

$$N = \frac{1}{2\pi}(2\pi) = 1$$

Hence $f(z)$ has ONE zero in the first quadrant. Now using the same argument as on page 611 of text book, we know that a polynomial of real coefficients, when it has a complex root, then they come in conjugate pairs, hence the second complex root will be in the 4th quadrant (since when take a conjugate of a complex number in the first quadrant, we get a complex number in the fourth quadrant).
Now since a 3rd order polynomial must have number of zeros as its order, then the 3rd zero must be real (it can't be complex since complex roots come in pairs). In addition the third root(the real root) must be on the negative x-axis to make x^3+x^2+x+4 a zero quantity.

18 chapter 14, problem 8.15

Evaluate the following integral by computing residue at ∞ check answer by computing residues at all finite poles.

$$\oint \frac{z^2\,dz}{(2z+1)(z^2+9)}$$

around $|z|=5$, where in the above, the circle is going in the positive direction, i.e. anticlockwise. We know that

$$\oint_{\substack{\text{clockwise}\\ \text{around zero}}} f(z) = -\oint_{\substack{\text{anticlockwise}\\ \text{around } \infty}} f(z)$$

but $\oint_{\text{clockwise}} f(z) = 2\pi i \sum$residues of $f(z)$ inside circle, therefore $\oint_{\text{clockwise}} f(z) = -2\pi i\,(\text{residues}\ \ f(z)\ \ at\ \ \infty)$
Hence we need to find residue of $f(z)$ at ∞. We know that residue of $f(z)$ at ∞ is residue of $-\frac{1}{z^2}f\left(\frac{1}{z}\right)$ at zero. Therefore

$$\begin{aligned}
-\frac{1}{z^2}f\left(\frac{1}{z}\right) &= -\frac{1}{z^2}\frac{(1/z)^2}{(2(1/z)+1)\left((1/z)^2+9\right)}\\
&= -\frac{1}{z^4}\frac{1}{\left(\frac{2}{z}+1\right)\left(\frac{1}{z^2}+9\right)}\\
&= -\frac{1}{z^4}\frac{1}{\left(\frac{2+z}{z}\right)\left(\frac{1+9z^2}{z^2}\right)}\\
&= -\frac{1}{z^4}\frac{z^3}{(2+z)(1+9z^2)}\\
&= \frac{-1}{z(2+z)(1+9z^2)}
\end{aligned}$$

So we have a simple pole at $z=0$, hence residue of the above function at zero is

$$\lim_{z\to 0} z\frac{-1}{z(2+z)(1+9z^2)} = \frac{-1}{2}$$

Hence

$$\oint \frac{z^2\,dz}{(2z+1)(z^2+9)} = -2\pi i\left(\frac{-1}{2}\right) = \pi i$$

To find the same integral by standard method, we write

$$\oint \frac{z^2\,dz}{(2z+1)(z^2+9)} = 2\pi i \sum \text{residues}\ \ f(z)$$

Poles are at $\frac{-1}{2}$ and at $z=\pm 3i$. Notice that all poles are inside $|z|=5$ so must add residue of $f(z)$ for each pole. Residue at $\frac{-1}{2}$is

$$\begin{aligned}
&\lim_{z\to-\frac{1}{2}}\left(z+\frac{1}{2}\right)\frac{z^2}{(2z+1)(z^2+9)}\\
&= \lim_{z\to-\frac{1}{2}}\frac{z^2}{2(z^2+9)}\\
&= \frac{\left(-\frac{1}{2}\right)^2}{2\left(\left(-\frac{1}{2}\right)^2+9\right)}\\
&= \frac{\frac{1}{4}}{2\left(\frac{1}{4}+9\right)}\\
&= \frac{\frac{1}{4}}{2\left(\frac{1}{4}+9\right)}\\
&= \frac{1}{2(37)}\\
&= \frac{1}{74}
\end{aligned}$$

Residue at $3i$ is

$$\begin{aligned}
\lim_{z\to 3i}(z-3i)\frac{z^2}{(2z+1)(z^2+9)} &= \lim_{z\to 3i}(z-3i)\frac{z^2}{(2z+1)(z-3i)(z+3i)}\\
&= \lim_{z\to 3i}\frac{z^2}{(2z+1)\ (z+3i)}\\
&= \frac{(3i)^2}{(2(3i)+1)\ (3i+3i)}\\
&= \frac{-9}{(6i+1)(6i)}\\
&= \frac{-9}{-36+6i}\\
&= \frac{-3}{-12+2i}\\
&= \frac{-3}{-12+2i}\left(\frac{-12-2i}{-12-2i}\right)\\
&= \frac{36+6i}{144+4}\\
&= \frac{36+6i}{148}\\
&= \frac{18+3i}{74}
\end{aligned}$$

Hence the residue at the last pole will be $\frac{18-3i}{74}$ symmetric. Hence sum of residues is

$$\begin{aligned}
\frac{18-3i}{74}+\frac{18+3i}{74}+\frac{1}{74} &= \frac{18+18+1}{74}\\
&= \frac{37}{74}
\end{aligned}$$

Hence

$$\begin{aligned}
\oint\frac{z^2\,dz}{(2z+1)(z^2+9)} &= 2\pi i\frac{37}{74}\\
&= \pi i
\end{aligned}$$

which agrees with my answer earlier using the residue around ∞ approach. I think the approach using residue around ∞ required less effort to do.

19 chapter 14, problem 8.4

Problem Determine if ∞ is a regular point, essential singularity or a pole (and of what order) and find the residue

$$f(z)=\frac{2z+3}{(z+2)^2}$$

Solution We start by doing the transformation $w=\frac{1}{z}$ and examine the function $f(w)$ at zero.

$$\begin{aligned}
f(w) &= f\left(\frac{1}{z}\right)\\
&= \frac{2\frac{1}{z}+3}{\left(\frac{1}{z}+2\right)^2}\\
&= \frac{\frac{2+3z}{z}}{\left(\frac{1+2z}{z}\right)^2}\\
&= \frac{\frac{2+3z}{z}}{\frac{(1+2z)^2}{z^2}}\\
&= \frac{z(2+3z)}{(1+2z)^2}
\end{aligned}$$

At $z=0$, $f\left(\frac{1}{z}\right)=0$, hence $f(z)$ is a *regular function* at ∞. To find the residue at ∞, we want to find the residue of

$$\left(-\frac{1}{(z-z_0)^2}\right)f\left(\frac{1}{z}\right)$$

at $z_0 = 0$. The term $-\frac{1}{(z-z_0)^2}$ comes from doing $Z = 1/z$, $dZ = -\frac{1}{z^2}dz$. Hence, we want to the residue of

$$\begin{aligned}\left(-\frac{1}{(z-z_0)^2}\right) f\left(\frac{1}{z}\right) &= \left(-\frac{1}{(z-z_0)^2}\right)\frac{z(2+3z)}{(1+2z)^2}\\ &= \left(-\frac{1}{z^2}\right)\frac{z(2+3z)}{(1+2z)^2}\\ &= \frac{-(2+3z)}{z(1+2z)^2}\end{aligned}$$

To find the residue of $\frac{-(2+3z)}{z(1+2z)^2}$ at 0, we see the function has a pole of order 2 at $-1/2$ and a simple pole of at $z = 0$, therefore

$$\begin{aligned}\text{residue} &= \lim_{z\to 0}(z)\frac{-(2+3z)}{z(1+2z)^2}\\ &= \frac{-(2+3(0))}{(1+2(0))^2}\\ &= \frac{-2}{(1)^2}\\ &= -2\end{aligned}$$

Or

$$\text{residue of} \frac{2z+3}{(z+2)^2} \text{at } \infty \text{ is} -2$$

20 chapter 14, problem 8.5

Problem Determine if ∞ is a regular point, essential singularity or a pole (and of what order) and find the residue

$$f(z) = \sin\frac{1}{z}$$

We start by doing the transformation $w = \frac{1}{z}$ and examine the function $f(w)$ at zero.

$$f(w) = f\left(\frac{1}{z}\right) = \sin z$$

at $z = 0$, $f\left(\frac{1}{z}\right) = 0$, hence $f(z) = \sin\frac{1}{z}$ is a *regular function* at ∞. To find the residue at ∞, we want to find the residue of

$$\left(-\frac{1}{(z-z_0)^2}\right) f\left(\frac{1}{z}\right)$$

at $z_0 = 0$. The $-\frac{1}{(z-z_0)^2}$ term above comes from doing $Z = 1/z$, $dZ = -\frac{1}{z^2}dz$. Hence, we want to the residue of

$$\begin{aligned}\left(-\frac{1}{(z-z_0)^2}\right) f\left(\frac{1}{z}\right) &= \left(-\frac{1}{(z-z_0)^2}\right)\sin z\\ &= -\frac{1}{z^2}\sin z\end{aligned}$$

To find the residue of $-\frac{1}{z^2}\sin z$ at 0, we see the function has a pole of order $m = 2$ at $z = 0$, hence

$$\begin{aligned}\text{residue} &= \frac{1}{m-1}\lim_{z\to z_0}\frac{d^{m-1}}{dz^{m-1}}(z-z_0)^m f(z)\\ &= \lim_{z\to z_0}\frac{d}{dz}(z-z_0)^2\left(-\frac{1}{z^2}\right)\sin z\\ &= \lim_{z\to z_0}\frac{d}{dz} - \sin z = \lim_{z\to z_0} -\cos z\\ &= -\cos z_0\\ &= -\cos 0\\ &= -1\end{aligned}$$

Hence

$$\text{residue of } \sin\frac{1}{z} \text{at } \infty \text{ is } -1$$