HW 13, Math 121 A
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1 chapter 9, problem 2.1

Problem

Write and solve the Euler equation to make the following integral stationary

$\int_{x_1}^{x_2}\sqrt{x}\sqrt{1+y'^2}dx$

Solution

Let $F=(x,y,y')=\sqrt{x}\sqrt{1+y'^2}$

The Euler equation is

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)-\frac{\partial F}{\partial y}=0$$

$$\frac{\partial F}{\partial y}=\frac{\partial}{\partial y}\left(\sqrt{x}\sqrt{1+y'^2}\right)=0$$

Hence the Euler equation becomes

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0$$

This means that $\frac{\partial F}{\partial y'} = C$ for some constant C.

$$\frac{\partial F}{\partial y'} = \sqrt{x}\frac{y'}{\sqrt{1+y'^2}}$$

Hence

$$\begin{aligned}
\sqrt{x}\frac{y'}{\sqrt{1+y'^2}} &= C \\
y'^2 &= \frac{C^2\left(1+y'^2\right)}{x} \\
x &= \frac{C^2+C^2y'^2}{y'^2} \\
x &= \frac{C^2}{y'^2}+k \\
y'^2 &= \frac{C^2}{x-C^2} \\
y' &= \frac{C}{\sqrt{x-C^2}} \\
y(x) &= \frac{2C}{\sqrt{x-C^2}}+C_1 \\
\frac{y(x)}{2C}-\frac{C_1}{C} &= \frac{1}{\sqrt{x-C^2}}
\end{aligned}$$

Let $\frac{C_1}{C} = -b$ (some constant), and Let $\frac{1}{2C} = a$ (constant), Hence above becomes

$$\begin{aligned}
a\,y+b &= \frac{1}{\sqrt{x-\frac{1}{4a^2}}} \\
a\,y+b &= \frac{2a}{\sqrt{4a^2\,x-1}}
\end{aligned}$$

This is equation of a parabola.

2 chapter 9, problem 2.3

Problem
Write and solve the Euler equation to make the following integral stationary $\int_{x_1}^{x_2} x\sqrt{1-y'^2}dx$
Solution
Let $F=(x,y,y') = x\sqrt{1-y'^2}$. The Euler equation is

$$\begin{aligned}
\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)-\frac{\partial F}{\partial y} &= 0 \\
\frac{\partial F}{\partial y} &= \frac{\partial}{\partial y}\left(x\sqrt{1-y'^2}\right) = 0
\end{aligned}$$

Hence Euler equation becomes

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

This means that $\frac{\partial F}{\partial y'} = C$ for some constant C.

$$\frac{\partial F}{\partial y'} = \frac{-x\,y'}{\sqrt{1-y'^2}}$$

Hence

$$
\begin{aligned}
\frac{-x\, y'}{\sqrt{1-y'^2}} &= C \\
y'^2 &= \frac{C^2\left(1-y'^2\right)}{x^2} \\
x^2 &= \frac{C^2 - C^2 y'^2}{y'^2} \\
x^2 &= \frac{C^2}{y'^2} - C^2 \\
y'^2 &= \frac{C^2}{x^2+C^2} \\
y' &= \frac{C}{\sqrt{x^2+C^2}} \\
y(x) &= C \operatorname{arcsinh}\left(\frac{x}{C}\right) + C_1 \\
\frac{y-C_1}{C} &= \operatorname{arcsinh}\left(\frac{x}{C}\right) \\
\frac{x}{C} &= \sinh\left(\frac{y}{C} - \frac{C_1}{C}\right)
\end{aligned}
$$

Let $\frac{C_1}{C} = -b$ (some constant). Let $\frac{1}{C} = a$ (some constant). Hence the above becomes

$$a\, x = \sinh(a\, y + b)$$

3 chapter 9, problem 2.6

Problem

Write and solve the Euler equation to make the following integral stationary $\int_{x_1}^{x_2}\left(y'^2+\sqrt{y}\right)dx$

Solution

Let $F(x,y,y') = y'^2 + \sqrt{y}$. Since F does not depend on x, we change the integration variable to y. Let $y' = \frac{1}{x'}$, then $dx = \frac{dx}{dy}dy$. Hence the integral becomes

$$\int_{y_1}^{y_2}\left(\frac{1}{x'^2}+\sqrt{y}\right)x'\, dy = \int_{y_1}^{y_2}\left(\frac{1}{x'}+x'\sqrt{y}\right)\, dy$$

Now $F(y,x') = \left(\frac{1}{x'}+x'\sqrt{y}\right)$. The Euler equation changes from $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$ to $\frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) - \frac{\partial F}{\partial x} = 0$. Now, $\frac{\partial F}{\partial x} = 0$ since F does not depend on x,Hence the Euler equation reduces to

$$
\begin{aligned}
\frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) &= 0 \\
\frac{d}{dy}\left(-\frac{1}{x'^2}+\sqrt{y}\right) &= 0
\end{aligned}
$$

Hence $-\frac{1}{x'^2}+\sqrt{y} = C$ where C is some constant

$$
\begin{aligned}
-\frac{1}{x'^2} &= C - \sqrt{y} \\
-\frac{1}{C-\sqrt{y}} &= x'^2 \\
\frac{1}{b+\sqrt{y}} &= x'^2 \qquad \text{where } b \text{ is a new constant} = -C \\
\frac{1}{\sqrt{k+\sqrt{y}}} &= \frac{dx}{dy} \\
\int \frac{dy}{\sqrt{b+\sqrt{y}}} &= \int dx \\
\frac{4}{3}\left(-2b+\sqrt{y}\right)\left(\sqrt{b+\sqrt{y}}\right) &= x + a \qquad \text{Where } a \text{ is constant of integration}
\end{aligned}
$$

Hence the solution is

$$\frac{4}{3}\left(\sqrt{y}-2b\right)\left(\sqrt{b+\sqrt{y}}\right) = x + a$$

4 chapter 9, problem 3.2

Problem
Write and solve the Euler equation to make the following integral stationary $\int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{y^2} dx$
Solution
Let $F(x, y, y') = \frac{\sqrt{1+y'^2}}{y^2}$. Since F does not depend on x, we change the integration variable to y. Let $y' = \frac{1}{x'}$, hence $dx = \frac{dx}{dy}dy$. The integral becomes

$$\int_{y_1}^{y_2} \left(\frac{\sqrt{1 + \frac{1}{x'^2}}}{y^2} \right) x' \, dy = \int_{y_1}^{y_2} \frac{\sqrt{x'^2 + 1}}{y^2} \, dy$$

Now $F(y, x') = \frac{\sqrt{x'^2+1}}{y^2}$. The Euler equation changes from $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$ to $\frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) - \frac{\partial F}{\partial x} = 0$. But $\frac{\partial F}{\partial x} = 0$ since F does not depend on x,Hence the Euler equation reduces to

$$\frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) = 0$$

$$\frac{\partial F}{\partial x'} = \frac{\partial}{\partial x'}\left(\frac{\sqrt{x'^2 + 1}}{y^2}\right) = \frac{x'}{y^2\sqrt{x'^2 + 1}}$$

Hence

$$\frac{d}{dy}\left(\frac{x'}{y^2\sqrt{x'^2 + 1}}\right) = 0$$

Hence $\frac{x'}{y^2\sqrt{x'^2+1}} = C$ where C is some constant

$$\begin{aligned}
\frac{x'^2}{x'^2 + 1} &= C\,y^4 \\
\frac{x'^2 + 1}{x'^2} &= \frac{1}{C\,y^4} \\
1 + \frac{1}{x'^2} &= \frac{1}{C\,y^4} \\
\frac{1}{x'^2} &= \frac{1 - Cy^4}{C\,y^4} \\
\frac{\sqrt{C}\,y^2}{\sqrt{1 - Cy^4}} &= x' \\
\frac{\sqrt{C}\,y^2}{\sqrt{1 - Cy^4}} &= \frac{dx}{dy} \\
\int \frac{\sqrt{C}\,y^2}{\sqrt{1 - Cy^4}} &= \int dx
\end{aligned}$$

The solution is

$$\frac{\sqrt{C}\,y^3}{3\sqrt{1 - Cy^4}} = x + C_1$$

Where C_1 is constant of integration. Let $C_1 = a, C = b$ hence solution can be written as

$$\frac{\sqrt{b}\,y^3}{3\sqrt{1 - by^4}} = x + a$$

5 chapter 9, problem 3.4

Problem
Write and solve the Euler equation to make the following integral stationary $\int_{x_1}^{x_2} y\sqrt{y'^2 + y^2}\, dx$
Solution
Let $F(x, y, y') = y\sqrt{y'^2 + y^2}$. Since F does not depend on x, we change the integration variable to y. Let $y' = \frac{1}{x'}$ and $dx = \frac{dx}{dy}dy$. Hence the integral becomes

$$\int_{y_1}^{y_2} \left(y\sqrt{\frac{1}{x'^2} + y^2} \right) x' \, dy = \int_{y_1}^{y_2} y\sqrt{1 + x'^2\,y^2}\, dy$$

Now $F(y,x') = y\sqrt{1+x'^2\ y^2}$. The Euler equation changes from $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$ to $\frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) - \frac{\partial F}{\partial x} = 0$. But $\frac{\partial F}{\partial x} = 0$ since F does not depend on x,Hence the Euler equation reduces to

$$\frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) = 0$$

$$\begin{aligned}
\frac{\partial F}{\partial x'} &= \frac{\partial}{\partial x'}\left(y\sqrt{1+x'^2\ y^2}\right) \\
&= y\left(\frac{x'\ y^2}{\sqrt{1+x'^2\ y^2}}\right) \\
&= \frac{x'\ y^3}{\sqrt{1+x'^2\ y^2}}
\end{aligned}$$

Hence

$$\frac{d}{dy}\left(\frac{x'\ y^3}{\sqrt{1+x'^2\ y^2}}\right) = 0$$

Hence $\frac{x'\ y^3}{\sqrt{1+x'^2\ y^2}} = C$ where C is some constant

$$\begin{aligned}
x'\ y^3 &= C\ \sqrt{1+x'^2\ y^2} \\
x'^2\ y^6 &= C^2\ \left(1+x'^2\ y^2\right) \\
x'^2\ y^6 &= C^2 + C^2\ x'^2\ \ y^2 \\
x'^2\ \left(y^6 - C^2\ y^2\right) &= C^2 \\
x'^2 &= \frac{C^2}{\left(y^6 - C^2\ y^2\right)} \\
x' &= \frac{C}{y\sqrt{y^4 - C^2}} \\
\int dx &= C\int \frac{1}{y\sqrt{y^4 - C^2}} dy
\end{aligned}$$

The solution is (using Mathematica)

$$a\ x = -\frac{1}{2} i \log\left(\frac{-2iC + 2\sqrt{-C^2 + y^4}}{y^2}\right)$$

6 chapter 9, problem 3.6

Problem

Write and solve the Euler equation to make the following integral stationary $\int_{x_1}^{x_2} \frac{y\ y'^2}{1+y\ y'}\ dx$

Solution

Let $F(x,y,y') = \frac{y\ y'^2}{1+y\ y'}$. Since F does not depend on x, we change the integration variable to y. Let $y' = \frac{1}{x'}$, $dx = \frac{dx}{dy} dy$. Hence the integral becomes

$$\begin{aligned}
\int_{y_1}^{y_2}\left(\frac{y\ \frac{1}{x'^2}}{1+y\ \frac{1}{x'}}\right) x'\ dy &= \int_{y_1}^{y_2}\left(\frac{y\ \frac{1}{x'^2}}{\frac{x'+\ y}{x'}}\right) x'\ dy \\
&= \int_{y_1}^{y_2}\left(\frac{y\ \frac{1}{x'}}{x'+\ y}\right) x'\ dy \\
&= \int_{y_1}^{y_2}\left(\frac{y}{x'+y}\right)\ dy
\end{aligned}$$

Now $F(y,x') = \left(\frac{y}{x'+y}\right)$. The Euler equation changes from $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$ to $\frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) - \frac{\partial F}{\partial x} = 0$. $\frac{\partial F}{\partial x} = 0$ since F does not depend on x,Hence the Euler equation reduces to

$$\frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) = 0$$

$$\begin{aligned}
\frac{\partial F}{\partial x'} &= \frac{\partial}{\partial x'}\left(\frac{y}{x'+y}\right) \\
&= y\left(-\frac{1}{(x'+y)^2}\right) \\
&= \frac{-y}{(x'+y)^2}
\end{aligned}$$

Hence

$$\frac{d}{dy}\left(\frac{-y}{(x'+y)^2}\right)=0$$

Hence $\frac{-y}{(x'+y)^2}=C$ where C is some constant

$$-y=C\left(x'+y\right)^2$$

Let $C=-k$

$$y=k\ \ (x'+y)^2$$

$$\sqrt{\frac{y}{k}}=x'+y$$

$$\sqrt{\frac{y}{k}}-y=x'$$

$$\sqrt{\frac{y}{k}}-y=\frac{dx}{dy}$$

$$\int\sqrt{\frac{y}{k}}-y\ dy=\int dx$$

$$-\frac{y^2}{2}+\frac{2}{3}y\sqrt{\frac{y}{k}}=x+\beta$$

Where β is the integration constant. Let $\frac{1}{\sqrt{k}}=\alpha$ a new constant

$$x=-\frac{1}{2}y^2+\frac{2}{3}\alpha\ y^{\frac{3}{2}}-\beta$$

Let $\frac{2}{3}\alpha=a$ a new integration constant, let $-\beta=b$ a new constant, we get

$$x=a\ y^{\frac{3}{2}}-\frac{1}{2}y^2+b$$

7 chapter 9, problem 3.9

Problem
Write and solve the Euler equation to make the following integral stationary $\int_{\phi_1}^{\phi_2}\sqrt{\theta'^2\ +\sin^2\theta}\ d\phi$, $\theta'=\frac{d\theta}{d\phi}$
Solution
Here $F(x,y(x),y'(x))$ becomes $F(\phi,\theta(\phi),\theta'(\phi))$. So now $x\rightarrow\phi, y\rightarrow\theta, y'\rightarrow\theta'$. Since $F(\theta',\theta)$ does not depend on ϕ, we change the integration variable to θ, so we want to change from $\theta'=\frac{d\theta}{d\phi}$ to $\phi'=\frac{d\phi}{d\theta}$. Let $\theta'=\frac{1}{\phi'}$, $d\phi=\frac{d\phi}{d\theta}d\theta$. Hence the integral becomes

$$\int_{\theta_1}^{\theta_2}\left(\sqrt{\frac{1}{\phi'^2}\ +\sin^2\theta}\right)\phi'\ d\theta=\int_{\theta_1}^{\theta_2}\sqrt{1\ +\phi'^2\sin^2\theta}\ d\theta$$

So now

$$F(\phi',\theta)=\sqrt{1\ +\phi'^2\sin^2\theta}$$

The Euler equation changes from $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)-\frac{\partial F}{\partial y}=0$ to $\frac{d}{d\theta}\left(\frac{\partial F}{\partial\phi'}\right)-\frac{\partial F}{\partial\phi}=0$. $\frac{\partial F}{\partial\phi}=0$ since F does not depend on ϕ,Hence the Euler equation reduces to

$$\frac{d}{d\theta}\left(\frac{\partial F}{\partial\phi'}\right)=0$$

$$\begin{aligned}\frac{\partial F}{\partial\phi'}&=\frac{\partial}{\partial\phi'}\left(\sqrt{1+\phi'^2\sin^2\theta}\right)\\&=\frac{\phi'\sin^2\theta}{\sqrt{1+\phi'^2\sin^2\theta}}\end{aligned}$$

Hence

$$\frac{d}{d\theta}\left(\frac{\phi'\sin^2\theta}{\sqrt{1+\phi'^2\sin^2\theta}}\right)=0$$

Hence $\frac{\phi' \sin^2\theta}{\sqrt{1+\phi'^2\sin^2\theta}} = C$ where C is some constant

$$\begin{aligned}
\phi' \sin^2\theta &= C\sqrt{1+\phi'^2\sin^2\theta} \\
\phi'^2 \sin^4\theta &= C^2\left(1+\phi'^2\sin^2\theta\right) \\
\phi'^2 \sin^4\theta &= C^2 + C^2\,\phi'^2\sin^2\theta \\
\phi'^2 &= \frac{C^2}{\sin^4\theta - C^2\sin^2\theta} \\
\phi' &= \frac{C}{\sin\theta\sqrt{\sin^2\theta - C^2}} \\
\int d\phi &= \int \frac{C}{\sin\theta\sqrt{\sin^2\theta - C^2}} d\theta \\
\phi + \alpha &= -\frac{C\ \tanh^{-1}\left(\frac{\sqrt{2}\sqrt{C^2}\cos(\theta)}{\sqrt{1-2C^2-\cos(2\theta)}}\right)}{\sqrt{-C^2}}
\end{aligned}$$

The last integral value was found using mathematica. Hence

$$\frac{\sqrt{-C^2}\,(\phi+\alpha)}{-C} = \operatorname{arctanh}\left(\frac{\sqrt{2}\sqrt{C^2}\cos(\theta)}{\sqrt{1-2C^2-\cos(2\theta)}}\right)$$

Let $\frac{\sqrt{-C^2}}{-C} = A$, let $\sqrt{2}\sqrt{C^2} = B$, $1-2C^2 = D$, then

$$\begin{aligned}
A\ (\phi+\alpha) &= \operatorname{arctanh}\left(\frac{B\cos(\theta)}{\sqrt{D-\cos(2\theta)}}\right) \\
\tanh\left(A\ (\phi+\alpha)\right) &= \frac{B\cos(\theta)}{\sqrt{D-\cos(2\theta)}}
\end{aligned}$$

8 chapter 9, problem 5.2

Problem

Set up Lagrange equations in cylindrical coordinates for a particle of mass m in a potential field $V(r,\theta,z)$

Solution

$L = T - V$ where T is the K.E. and V the potential energy. $T = \frac{1}{2}mv^2$, But

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2$$

As shown on page 219 equation 4.4 , now differentiate both sides w.r.t. time

$$\begin{aligned}
2\,ds\,\frac{ds}{dt} &= 2dr\,\dot{r} + \left(r^2\ 2\,d\theta\ \dot{\theta} + 2r\ \dot{r}\ d\theta^2\right) + 2dz\ \dot{z} \\
\frac{ds}{dt} &= \frac{dr\ \dot{r} + r^2\ \ d\theta\ \dot{\theta} + r\ \dot{r}\ d\theta^2 + dz\ \dot{z}}{\sqrt{dr^2 + r^2 d\theta^2 + dz^2}}
\end{aligned}$$

Hence

$$v^2 = \frac{\left(dr\ \dot{r} + r^2 d\theta\ \dot{\theta} + r\ \dot{r}\ d\theta^2 + dz\ \dot{z}\right)^2}{dr^2 + r^2 d\theta^2 + dz^2}$$

I used Mathematica to simplify this getting

$$v^2 = \dot{r}^2 + r^2\ \dot{\theta}^2 + \ \dot{z}^2$$

Hence,

$$L = \overbrace{\frac{1}{2}m\ \left(\dot{r}^2 + r^2\ \dot{\theta}^2 + \ \dot{z}^2\right)}^{\text{K.E.}} - \overbrace{V(r,\theta,z)}^{\text{P.E.}}$$

The Lagrange equations are

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} &= 0 \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= 0 \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} &= 0
\end{aligned}$$

Hence, we get

$$
\begin{aligned}
\frac{d}{dt}(m\,\dot{r}) - \left(mr\dot{\theta}^2 - \frac{\partial V}{\partial r}\right) &= 0\\
\frac{d}{dt}\left(mr^2\dot{\theta}\right) + \frac{\partial V}{\partial \theta} &= 0\\
\frac{d}{dt}(m\dot{z}) + \frac{\partial V}{\partial z} &= 0
\end{aligned}
$$

Now differentiating w.r.t. time, and remembering that $r(t)$ also changes with time.

$$
\begin{aligned}
m\,\ddot{r} - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} &= 0\\
m\left(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}\right) + \frac{\partial V}{\partial \theta} &= 0\\
m\ddot{z} + \frac{\partial V}{\partial z} &= 0
\end{aligned}
$$

Hence finally we get

$$
\begin{aligned}
m\left(\ddot{r} - r\dot{\theta}^2\right) &= -\frac{\partial V}{\partial r}\\
m\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right) &= -\frac{1}{r}\frac{\partial V}{\partial \theta}\\
m\ddot{z} &= -\frac{\partial V}{\partial z}
\end{aligned}
$$

9 chapter 9, problem 5.6

Problem
A particle moves on the surface of a sphere of radius a under the action of the earth gravitational field. Find the θ, ϕ equations of motion. (this is called the spherical pendulum).
Solution
$L = T - V$ where T is the K.E. and V the potential energy. Using spherical coordinates.

$$x = a\sin\theta\cos\phi,\ y = a\sin\theta\sin\phi,\ z = a\cos\theta$$

Hence a position vector

$$\mathbf{r} = \mathbf{i}\ a\sin\theta\cos\phi + \mathbf{j}\ a\sin\theta\sin\phi + \mathbf{k}\ a\cos\theta$$

So velocity is

$$
\begin{aligned}
\mathbf{r} &= \mathbf{i}\ \frac{d}{dt}(a\sin\theta\cos\phi) + \mathbf{j}\ \frac{d}{dt}(a\sin\theta\sin\phi) + \mathbf{k}\ \frac{d}{dt}(a\cos\theta)\\
&= \mathbf{i}\ \left(-a\sin\theta\sin\phi\ \dot{\phi} + a\cos\theta\ \dot{\theta}\ \cos\phi\right) + \mathbf{j}\ \left(a\sin\theta\cos\phi\dot{\phi} + a\cos\theta\ \dot{\theta}\ \sin\phi\right) + \mathbf{k}\ \left(-a\sin\theta\ \dot{\theta}\right)
\end{aligned}
$$

Hence

$$\dot{r} = \|\mathbf{r}\| = \sqrt{\left(-a\sin\theta\sin\phi\ \dot{\phi} + a\cos\theta\ \dot{\theta}\cos\phi\right)^2 + \left(a\sin\theta\cos\phi\ \dot{\phi} + a\cos\theta\ \dot{\theta}\ \sin\phi\right)^2 + \left(-a\sin\theta\ \dot{\theta}\right)^2}$$

Then

$$v^2 = \dot{r}^2 = \left(-a\sin\theta\sin\phi\,\dot{\phi} + a\cos\theta\,\dot{\theta}\cos\phi\right)^2 + \left(a\sin\theta\cos\phi\dot{\phi} + a\cos\theta\,\dot{\theta}\ \sin\phi\right)^2 + \ \left(-a\sin\theta\,\dot{\theta}\right)^2$$

$$= \left(a^2\sin^2\theta\sin^2\phi\,\dot{\phi}^2 + a^2\cos^2\theta\,\dot{\theta}^2\cos^2\phi - \overbrace{2a^2\sin\theta\sin\phi\,\dot{\phi}\cos\theta\,\dot{\theta}\cos\phi}\right)$$
$$+ \left(a^2\sin^2\theta\cos^2\phi\,\dot{\phi}^2 + a^2\cos^2\theta\,\dot{\theta}^2\ \sin^2\phi\,\overbrace{+2a^2\sin\theta\cos\phi\dot{\phi}\ \cos\theta\,\dot{\theta}\ \sin\phi}\right) + \ \left(a^2\sin^2\theta\,\dot{\theta}^2\right)$$

$$= \overbrace{a^2\sin^2\theta\sin^2\phi\,\dot{\phi}^2} + a^2\overbrace{\cos^2\theta\,\dot{\theta}^2}\cos^2\phi + \overbrace{a^2\sin^2\theta\cos^2\phi\,\dot{\phi}^2} + a^2\overbrace{\cos^2\theta\,\dot{\theta}^2}\ \sin^2\phi + \ a^2\sin^2\theta\,\dot{\theta}^2$$

$$= a^2\dot{\phi}^2\sin^2\theta\,\overbrace{\left(\sin^2\phi + \cos^2\phi\right)}^{=1} + a^2\dot{\theta}^2\cos^2\theta\,\overbrace{\left(\cos^2\phi + \sin^2\phi\right)}^{=1} + a^2\sin^2\theta\,\dot{\theta}^2$$

$$= a^2\dot{\phi}^2\sin^2\theta\ + a^2\dot{\theta}^2\,\overbrace{\left(\cos^2\theta + \sin^2\theta\right)}^{=1}$$

$$= a^2\left(\dot{\phi}^2\sin^2\theta\ + \dot{\theta}^2\right)$$

Hence $T = \frac{1}{2}mv^2$. For a particle, taking mass as one unit. Hence

$$T = \frac{1}{2}a^2\left(\dot{\phi}^2\sin^2\theta\ + \dot{\theta}^2\right)$$

The P.E. is $mga\ \cos\theta$. Hence the Lagrangian is

$$L = T - V$$
$$L = \frac{1}{2}a^2\left(\dot{\phi}^2\sin^2\theta\ + \dot{\theta}^2\right) - ga\ \cos\theta$$

We have 2 independent variables, hence we need 2 Lagrangian equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) - \frac{\partial L}{\partial\phi} = 0$$

$$\frac{\partial L}{\partial\dot{\theta}} = a^2\dot{\theta}$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = a^2\ddot{\theta}$$
$$\frac{\partial L}{\partial\theta} = a^2\left(\dot{\phi}^2\sin\theta\ \cos\theta\right) + ga\ \sin\theta$$

Hence the first equation becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0$$
$$a^2\ddot{\theta} - a^2\left(\dot{\phi}^2\sin\theta\ \cos\theta\right) - ga\ \sin\theta = 0$$
$$a\,\ddot{\theta} - a\ \left(\dot{\phi}^2\sin\theta\ \cos\theta\right) - g\ \sin\theta = 0$$

To find the second equation

$$\frac{\partial L}{\partial\dot{\phi}} = a^2\left(2\dot{\phi}\sin^2\theta\right)$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) = \frac{d}{dt}\left(a^2\left(2\dot{\phi}\sin^2\theta\right)\right)$$
$$\frac{\partial L}{\partial\phi} = 0$$

Hence the second equation is

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} &= 0\\
\frac{d}{dt}\left(a^2\left(2\dot{\phi}\sin^2\theta\right)\right) &= 0\\
\frac{d}{dt}\left(2\dot{\phi}\sin^2\theta\right) &= 0\\
\frac{d}{dt}\left(2\dot{\phi}\sin^2\theta\right) &= 0
\end{aligned}$$

10 chapter 9, problem 6.1

Problem
Find surface of revolution formed by rotating the curve around the x-axis that has a minimum area subject to a curve of give length l joining 2 points.
Solution
Area is

$$I = \int_{x_1}^{x_2} 2\pi y\sqrt{1+y'^2}dx \tag{1}$$

Since integrand does not depend on x we change the independent variable to y. $dx = \frac{dx}{dy}dy, y' = \frac{1}{x'}$. Hence (1) becomes

$$\begin{aligned}
I &= \int_{y_1}^{y_2} 2\pi y\sqrt{1+\frac{1}{x'^2}}\,x'dy \qquad (1)\\
&= \int_{y_1}^{y_2} 2\pi y\sqrt{x'^2+1}\,dy
\end{aligned}$$

Hence $F(y,x',x) = 2\pi y\sqrt{x'^2+1}$. Now finding the constraint

$$\begin{aligned}
g &= \int ds = l\\
&= \int_{x_1}^{x_2} \sqrt{1+y'^2}dx
\end{aligned}$$

Since integrand does not depend on x we change the independent variable to y. $dx = \frac{dx}{dy}dy, y' = \frac{1}{x'}$. Hence

$$\begin{aligned}
g &= \int_{y_1}^{y_2} \sqrt{1+\frac{1}{x'^2}}\,x'dy\\
&= \int_{y_1}^{y_2} \sqrt{x'^2+1}\;dy
\end{aligned}$$

So $G = \sqrt{x'^2+1}$. Hence we get

$$F + \lambda G = \left(2\pi y\sqrt{x'^2+1}\right) + \lambda\sqrt{x'^2+1}$$

As the new Euler equation (with constrains). Solving

$$\begin{aligned}
\frac{d}{dy}\left(\frac{\partial}{\partial x'}(F+\lambda G)\right) - \overbrace{\frac{\partial}{\partial x}(F+\lambda G)}^{0 \text{ since does not depend on } x} &= 0\\
\frac{d}{dy}\left(\frac{\partial}{\partial x'}\left(2\pi y\sqrt{x'^2+1}+\lambda\sqrt{x'^2+1}\right)\right) &= 0\\
\frac{d}{dy}\left(\frac{2\pi y x'}{\sqrt{x'^2+1}}+\frac{\lambda x'}{\sqrt{x'^2+1}}\right) &= 0
\end{aligned}$$

Hence

$$
\begin{aligned}
\frac{2\pi y x'}{\sqrt{x'^2+1}} + \frac{\lambda x'}{\sqrt{x'^2+1}} &= c \\
\frac{2\pi y x' + \lambda x'}{\sqrt{x'^2+1}} &= c \\
x'\left(2\pi y + \lambda\right) &= c\sqrt{x'^2+1} \\
x'^2\left(2\pi y + \lambda\right)^2 &= c^2\left(x'^2 + 1\right) \\
\frac{x'^2}{\left(x'^2+1\right)} &= \frac{c^2}{\left(2\pi y + \lambda\right)^2} \\
\frac{\left(x'^2+1\right)}{x'^2} &= \frac{\left(2\pi y + \lambda\right)^2}{c^2} \\
1 + \frac{1}{x'^2} &= \frac{\left(2\pi y + \lambda\right)^2}{c^2} \\
\frac{1}{x'^2} &= \frac{\left(2\pi y + \lambda\right)^2 - c^2}{c^2} \\
\frac{c^2}{\left(2\pi y + \lambda\right)^2 - c^2} &= x'^2 \\
\frac{c}{\sqrt{\left(2\pi y + \lambda\right)^2 - c^2}} &= x' \\
\frac{dx}{dy} &= \frac{c}{\sqrt{\left(2\pi y + \lambda\right)^2 - c^2}} \\
\int dx &= \int \frac{c}{\sqrt{\left(2\pi y + \lambda\right)^2 - c^2}} dy \\
\int dx &= \int \frac{1}{\sqrt{\left(\frac{2\pi y + \lambda}{c}\right)^2 - 1}} dy \\
x &= \frac{c}{2\pi} \operatorname{arccosh}\left(\frac{2\pi y + \lambda}{c}\right) + c_1
\end{aligned}
$$

To express this as y a function of x we get

$$
\begin{aligned}
\frac{2\pi}{c}\left(x - c_1\right) &= \operatorname{arccosh}\left(\frac{2\pi y + \lambda}{c}\right) \\
\cosh\left(\frac{2\pi}{c}\left(x - c_1\right)\right) &= \frac{2\pi y + \lambda}{c} \\
\frac{c\cosh\left(\frac{2\pi}{c}\left(x - c_1\right)\right) - \lambda}{2\pi} &= y
\end{aligned}
$$

We have 3 unknowns, c, c_1, λ that we can use boundary conditions, and length l to determine.

11 chapter 9, problem 6.2

Problem
Find the equation of the curve subject to a curve of give length l joining 2 points so that the plane area between the curve and straight line joining the points is a maximum.
Solution
Area is $\int y\, dx$. Hence area is $I = \int_{x_1}^{x_2} y\, dx$ subject to constraint that $\int\ ds = l$ or $g = \int_{x_1}^{x_2} \sqrt{1 + y'^2} dx = l$. Hence the Euler equation with constrains now becomes

$$
F + \lambda G = y + \lambda\sqrt{y'^2 + 1}
$$

Therefore

$$
\begin{aligned}
\frac{d}{dx}\left(\frac{\partial}{\partial y'}\left(F + \lambda G\right)\right) - \frac{d}{dy}\left(F + \lambda G\right) &= 0 \\
\frac{d}{dy}\left(\frac{\lambda\, y'}{\sqrt{y'^2 + 1}}\right) - 1 &= 0 \\
\frac{\lambda\, y'}{\sqrt{y'^2 + 1}} &= x + c
\end{aligned}
$$

This simplifies to

$$
\begin{aligned}
\int dy &= \int \frac{(x+c)}{\sqrt{\lambda^2-(x+c)^2}}dx \\
y+c_1 &= -\sqrt{\lambda^2-(x+c)^2} \\
(y+c_1)^2 &= \lambda^2-(x+c)^2 \\
(y+c_1)^2+(x+c)^2 &= \lambda^2
\end{aligned}
$$

This is the equation of a circle.

12 chapter 9, problem 6.5

Problem
Given surface area of solid of revolution, finds its shape to make its volume a maximum.
Solution
Volume is $\int \pi y^2 ds$ where ds is a small segment of the curve length. Hence

$$
I = \int_{x_1}^{x_2} \pi y^2 \sqrt{1+y'^2}dx \tag{1}
$$

Constraint is that area is given, say A. Hence

$$
g = \int_{x_1}^{x_2} 2\pi y \sqrt{1+y'^2}dx = A \tag{2}
$$

Since both integrands do not depend on x we change the independent variable to y. $dx = \frac{dx}{dy}dy, y' = \frac{1}{x'}$. Hence (1) becomes

$$
\begin{aligned}
I &= \int_{x_1}^{x_2} \pi y^2 \sqrt{1+\frac{1}{x'^2}}x'dy \\
&= \int_{x_1}^{x_2} \pi y^2 \sqrt{x'^2+1}dy
\end{aligned}
$$

And (2) becomes

$$
\begin{aligned}
g &= \int_{y_1}^{y_2} 2\pi y \sqrt{1+\frac{1}{x'^2}}\,x'dy \\
&= \int_{y_1}^{y_2} 2\pi y \sqrt{x'^2+1}\,dy
\end{aligned}
$$

Hence we get

$$
F+\lambda G = \left(\pi y^2\sqrt{x'^2+1}\right) + 2\lambda\pi y\sqrt{x'^2+\ 1}
$$

as the new Euler equation (with constrains) to solve.

$$
\begin{aligned}
\frac{d}{dy}\left(\frac{\partial}{\partial x'}(F+\lambda G)\right) - \overbrace{\frac{\partial}{\partial x}(F+\lambda G)}^{0 \text{ since does not depend on } x} &= 0 \\
\frac{d}{dy}\left(\frac{\partial}{\partial x'}\left(\pi y^2\sqrt{x'^2+1}+2\lambda\pi y\sqrt{x'^2+1}\right)\right) &= 0 \\
\frac{d}{dy}\left(\frac{\pi y^2 x'}{\sqrt{x'^2+1}}+\frac{2\lambda\pi y x'}{\sqrt{x'^2+1}}\right) &= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{\pi y^2 x'}{\sqrt{x'^2+1}} + \frac{2\lambda\pi y x'}{\sqrt{x'^2+1}} &= c \\
\frac{\pi y^2 x' + 2\lambda\pi y x'}{\sqrt{x'^2+1}} &= c \\
\pi y^2 x' + 2\lambda\pi y x' &= c\sqrt{x'^2 + 1} \\
x'^2 \ \left(\pi y^2 + 2\lambda\pi y\right)^2 &= c^2 \left(x'^2 + \ 1\right) \\
\frac{x'^2}{(x'^2+1)} &= \frac{c^2}{(\pi y^2 + 2\lambda\pi y)^2} \\
\frac{\left(x'^2+1\right)}{x'^2} &= \frac{\left(\pi y^2 + 2\lambda\pi y\right)^2}{c^2} \\
1 + \frac{1}{x'^2} &= \frac{\left(\pi y^2 + 2\lambda\pi y\right)^2}{c^2} \\
\frac{1}{x'^2} &= \frac{\left(\pi y^2 + 2\lambda\pi y\right)^2 - c^2}{c^2} \\
\frac{c^2}{(\pi y^2 + 2\lambda\pi y)^2 - c^2} &= x'^2 \\
\frac{c}{\sqrt{(\pi y^2 + 2\lambda\pi y)^2 - c^2}} &= x' \\
\frac{dx}{dy} &= \frac{c}{\sqrt{(\pi y^2 + 2\lambda\pi y)^2 - c^2}} \\
\int dx &= \int \frac{c}{\sqrt{(\pi y^2 + 2\lambda\pi y)^2 - c^2}} dy \\
x &= \int \frac{c}{\sqrt{(\pi y^2 + 2\lambda\pi y)^2 - c^2}} dy \\
x &= \int \frac{1}{\sqrt{\left(\frac{\pi y^2 + 2\lambda\pi y}{c}\right)^2 - 1}} dy
\end{aligned}
$$

Hence

$$
x = \left(\frac{c}{2y\pi + 2\lambda\pi}\right) \cosh^{-1}\left(\frac{\pi y^2 + 2\lambda\pi y}{c}\right)
$$

13 chapter 15, problem 8.12

Problem
Solve $y'' + y = f(x)$ with $y(0) = y\left(\frac{\pi}{2}\right) = 0$ using 8.17:

$$
y(x) = -\cos x \int_0^x \sin(x') f(x') \ dx' - \sin x \int_x^{\frac{\pi}{2}} \cos(x') f(x') \ dx'
$$

when $f(x) = \sec x$
Solution

$$
y(x) = -\cos x \int_0^x \sin(x') \sec x' \ dx' - \sin x \int_x^{\frac{\pi}{2}} \cos(x') \sec x' \ dx'
$$

Since $\sec x' = \frac{1}{\cos x'}$ we get

$$
y(x) = -\cos x \int_0^x \tan x' \ dx' - \sin x \int_x^{\frac{\pi}{2}} dx'
$$

But $\int_0^x \tan x' \ dx' = -\log(\cos(x))$, Hence

$$
\begin{aligned}
y(x) &= \cos(x)\log(\cos(x)) - \sin x \left(\frac{1}{2}\pi - x\right) \\
&= \cos(x)\log(\cos(x)) - \frac{1}{2}\pi \sin x + x \sin x
\end{aligned}
$$

14 chapter 15, problem 8.15

Problem
Use Green function method and the given solutions of the homogeneous equation to find a particular solution to $y'' - y = \sec h(x)$, where $y_1(x) = \sinh(x)\,,\quad y_2(x) = \cosh(x)$
Solution

$$y_p = y_2 \int \frac{y_1 f}{W} dx - y_1 \int \frac{y_2 f}{W} dx \tag{1}$$

Where $f = \sec h(x)$

$$\begin{aligned} W &= \begin{vmatrix} y_1' & y_2' \\ y_1 & y_2 \end{vmatrix} \\ &= \begin{vmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{vmatrix} \\ &= \cosh^2 x - \sinh^2 x \\ &= 1 \end{aligned}$$

So from (1) we get

$$y_p = \cosh(x) \int \sinh(x)\ \sec h(x)\ dx - \sinh(x) \int \cosh(x) \sec h(x)\ dx$$

But $\sec h(x) = \frac{1}{\cosh x}$, Hence

$$\begin{aligned} y_p &= \cosh(x) \int \sinh(x) \frac{1}{\cosh x}\ dx - \sinh(x) \int \cosh(x) \frac{1}{\cosh x}\ dx \\ &= \cosh(x) \int \tan sh(x)\ dx - \sinh(x) \int dx \end{aligned}$$

But $\int \tan sh(x)\ dx = \log(\cosh(x))$, Hence

$$y_p = \cosh(x) \log(\cosh(x)) - x\ \sinh(x)$$

15 chapter 15, problem 8.17

Problem
Use Green function method and the given solutions of the homogeneous equation to find a particular solution to $y'' - 2\left(\csc^2(x)\right) y = \sin^2(x)$, where $y_1(x) = \cot x,\quad y_2(x) = 1 - x\ \cot(x)$
Solution
Note $\cot(x) = \frac{1}{\tan(x)} = \frac{\cos(x)}{\sin(x)}$, $\csc(x) = \frac{1}{\sin(x)}$

$$y_p = y_2 \int \frac{y_1 f}{W} dx - y_1 \int \frac{y_2 f}{W} dx \tag{1}$$

Where $f = \sin^2(x)$.

$$\begin{aligned} y_1' &= \frac{d}{dx}(\cot(x)) = -\cot^2 x - 1 \\ &= -\frac{1}{\sin^2(x)} \end{aligned}$$

And

$$\begin{aligned} y_2' &= \frac{d}{dx}(1 - x\ \cot(x)) \\ &= -\frac{\cos(x)}{\sin(x)} + \frac{x}{\sin^2(x)} \end{aligned}$$

Therefore

$$\begin{aligned} W &= \begin{vmatrix} y_1' & y_2' \\ y_1 & y_2 \end{vmatrix} \\ &= \begin{vmatrix} -\frac{1}{\sin^2(x)} & -\frac{\cos(x)}{\sin(x)} + \frac{x}{\sin^2(x)} \\ \frac{\cos(x)}{\sin(x)} & 1 - \frac{x\ \cos(x)}{\sin(x)} \end{vmatrix} \\ &= \left(-\frac{1}{\sin^2(x)}\right)\left(1 - \frac{x\ \cos(x)}{\sin(x)}\right) - \left(-\frac{\cos(x)}{\sin(x)} + \frac{x}{\sin^2(x)}\right)\frac{\cos(x)}{\sin(x)} \\ &= -\frac{1}{\sin^2(x)} + \overbrace{\frac{x\ \cos(x)}{\sin^3(x)}} + \frac{\cos^2(x)}{\sin^2(x)} - \overbrace{\frac{x \cos(x)}{\sin^3(x)}} \\ &= -\frac{1}{\sin^2(x)} + \frac{\cos^2(x)}{\sin^2(x)} \end{aligned}$$

So from (1) we get

$$
\begin{aligned}
y_p &= \left(1-\frac{x\cos x}{\sin x}\right)\int \frac{\frac{\cos x}{\sin x}\ \sin^2(x)}{-\frac{1}{\sin^2(x)}+\frac{\cos^2(x)}{\sin^2(x)}}dx - \frac{\cos x}{\sin x}\int \frac{\left(1-\frac{x\cos x}{\sin(x)}\right)\ \sin^2(x)}{-\frac{1}{\sin^2(x)}+\frac{\cos^2(x)}{\sin^2(x)}}dx \\
&= \left(1-\frac{x\cos x}{\sin x}\right)\int \frac{\cos x\ \sin x}{\frac{-1+\cos^2 x}{\sin^2(x)}}dx - \frac{\cos x}{\sin x}\int \frac{\sin^2 x - x\ \cos x\ \sin x}{\frac{-1+\cos^2 x}{\sin^2(x)}}dx \\
&= \left(1-\frac{x\cos x}{\sin x}\right)\int \frac{\cos x\ \sin^3 x}{-1+\cos^2 x}dx - \frac{\cos x}{\sin x}\int \frac{\sin^4 x - x\ \cos x\ \sin^3 x}{-1+\cos^2 x}dx
\end{aligned}
$$

but $I=\int \frac{\cos x\ \sin^3 x}{\cos^2 x - 1} = \int \frac{\cos x\ \sin^3 x}{-\sin^2 x} = \int -\cos x\ \sin x = \frac{1}{2}\cos^2 x$ And

$$
\begin{aligned}
I &= \int \frac{\sin^4 x - x\ \cos x\ \sin^3 x}{-1+\cos^2 x} \\
&= \int \frac{\sin^4 x - x\ \cos x\ \sin^3 x}{-\sin^2 x} \\
&= \int -\sin^2 x + x\ \cos x\ \sin x \\
&= -\int \sin^2(x)\ dx + \int x\ \cos(x)\ \sin(x)\ dx
\end{aligned}
$$

But $\int \sin^2(x)\ dx = \frac{x}{2} - \frac{1}{4}\sin(2x)$ and $\int x\ \cos(x)\ \sin(x)\ dx = -\frac{1}{4}x\cos(2x) + \frac{1}{8}\sin(2x)$, therefore

$$
\begin{aligned}
-\int \sin^2(x)\ dx + \int x\ \cos(x)\ \sin(x)\ dx &= \left(-\frac{x}{2}+\frac{1}{4}\sin(2x)\right) + \left(-\frac{1}{4}x\cos(2x)+\frac{1}{8}\sin(2x)\right) \\
&= -\frac{x}{2}+\frac{1}{4}\sin(2x) - \frac{1}{4}x\cos(2x) + \frac{1}{8}\sin(2x) \\
&= \frac{3}{8}\sin 2x - \frac{1}{2}x - \frac{1}{4}x\cos 2x
\end{aligned}
$$

Hence (2) becomes

$$
\begin{aligned}
y_p(x) &= \left(1-\frac{x\cos x}{\sin x}\right)\left(\frac{1}{2}\cos^2 x\right) - \frac{\cos x}{\sin x}\left(\frac{3}{8}\sin 2x - \frac{1}{2}x - \frac{1}{4}x\cos 2x\right) \\
&= \left(\frac{1}{2}\cos^2 x - \frac{1}{2}\frac{x\cos^3 x}{\sin x}\right) - \left(\frac{3}{8}\sin 2x\frac{\cos x}{\sin x} - \frac{1}{2}x\frac{\cos x}{\sin x} - \frac{1}{4}x\cos 2x\frac{\cos x}{\sin x}\right) \\
&= \frac{1}{2}\cos^2 x - \frac{1}{2}\frac{x\cos^3 x}{\sin x} - \frac{3}{8}\sin 2x\frac{\cos x}{\sin x} + \frac{1}{2}x\frac{\cos x}{\sin x} + \frac{1}{4}x\cos 2x\frac{\cos x}{\sin x} \\
&= \frac{1}{4}\cot x\ \left(x - \cos x\ \sin x\right)
\end{aligned}
$$

16 chapter 15, problem 8.2

Problem
Solve $y'' + \omega^2 y = f(t)$ using $y(t) = \int_0^t \frac{1}{\omega}\sin\omega(t-t')\ f(t')\ dt'$ when $f(t) = \sin\omega t$
Solution

$$
\begin{aligned}
y(t) &= \int_0^t \frac{1}{\omega}\sin\omega(t-t')\,f(t')\ dt' \\
&= \int_0^t \frac{1}{\omega}\sin\omega(t-t')\sin\omega t'\ dt' \qquad (1)
\end{aligned}
$$

But $\sin\alpha\sin\beta = \frac{1}{2}\cos(\alpha-\beta) - \frac{1}{2}\cos(\alpha+\beta)$, hence

$$
\begin{aligned}
\sin\omega(t-t')\sin\omega t' &= \frac{1}{2}\cos(\omega(t-t')-\omega t') - \frac{1}{2}\cos(\omega(t-t')+\omega t') \\
&= \frac{1}{2}\cos(t\omega - 2\omega t') - \frac{1}{2}\cos\omega t
\end{aligned}
$$

Hence (1) becomes

$$\begin{aligned}
y(t) &= \int_0^t \frac{1}{\omega}\frac{1}{2}\cos(\omega t - 2\omega t') - \frac{1}{2}\cos\omega t \, dt' \\
&= \frac{1}{2\omega}\int_0^t \cos(\omega t - 2\omega t')\, dt' - \frac{1}{2}\cos\omega t \int_0^t dt' \\
&= \frac{1}{2\omega}\left[\frac{\sin(\omega t - 2\omega t')}{-2\omega}\right]_0^t - \frac{1}{2} t\cos t\omega \\
&= \frac{-1}{4\omega^2}\left(\sin(\omega t - 2\omega t) - \sin(\omega t)\right) - \frac{1}{2} t\cos t\omega \\
&= \frac{1}{2\omega^2}\sin t\omega - \frac{1}{2} t\cos t\omega \\
&= \frac{1}{2\omega^2}\left(\sin t\omega - \omega t\cos t\omega\right)
\end{aligned}$$

$$y(t) = \frac{1}{2\omega^2}\left(\sin t\omega - \omega t\cos t\omega\right)$$

17 chapter 15, problem 8.3

Problem
Solve $y'' + \omega^2 y = f(t)$ using $y(t) = \int_0^t \frac{1}{\omega}\sin\omega(t-t')\ f(t')\ dt'$ when $f(t) = e^{-t}$
Solution

$$\begin{aligned}
y(t) &= \int_0^t \frac{1}{\omega}\sin\omega(t-t')\ f(t')\ dt' \\
&= \frac{1}{\omega}\int_0^t \sin\omega(t-t')\ e^{-t'}\ dt' \qquad (1)
\end{aligned}$$

$$\text{Let } I = \int_0^t \sin\omega(t-t')\ e^{-t'}\ dt'$$

Integrate by part, let $u = \sin(\omega t - \omega t')$, $v = -e^{-t'}$

$$\begin{aligned}
I &= [\sin\omega(t-t')\ \left(-e^{-t'}\right)]_0^t - \omega\int_0^t \cos(\omega t - \omega t')\ e^{-t'}\ dt' \\
&= \sin\omega t\ - \omega\int_0^t \cos(\omega t - \omega t')\ e^{-t'}\ dt'
\end{aligned}$$

Integrate by parts again. $u = \cos(\omega t - \omega t')$, $v = -e^{-t'}$

$$\begin{aligned}
I &= \sin\omega t - \omega\left([\cos(\omega t - \omega t')\left(-e^{-t'}\right)]_0^t + \omega\int_0^t \sin\omega(t-t')\, e^{-t'}\ dt'\right) \\
I &= \sin\omega t - \omega\left([-e^{-t} + \cos(\omega t)] + \omega I\right) \\
I &= \sin\omega t + \omega e^{-t} - \omega\cos(\omega t) - \omega^2 I \\
I + \omega^2 I &= \sin\omega t + \omega e^{-t} - \omega\cos(\omega t) \\
I &= \frac{\sin\omega t + \omega e^{-t} - \omega\cos(\omega t)}{1+\omega^2}
\end{aligned}$$

Hence from (1)

$$y(t) = \frac{1}{\omega}\frac{\omega e^{-t} - \omega\cos(\omega t) + \sin(\omega t)}{1+\omega^2}$$

18 chapter 9, problem 3.1

Problem
Change the independent variable to simplify the Euler equation and then find the first integral of it. $\int_{x_2}^{x_1} y^{\frac{3}{2}} ds$
Solution

$$ds = \sqrt{(dx)^2 + (dy)^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = dx\sqrt{1+y'^2}$$

Hence

$$I = \int_{x_2}^{x_1} y^{\frac{3}{2}} ds = \int_{x_2}^{x_1} y^{\frac{3}{2}}\sqrt{1+y'^2}dx$$

Since integrand does not depend on x, changing the independent variable to y in order to simplify solution. Using $dx = \frac{dx}{dy}dy \rightarrow y' = \frac{1}{x'}$. The integral now becomes

$$\begin{aligned} I &= \int_{x_2}^{x_1} y^{\frac{3}{2}} \sqrt{1 + \frac{1}{x'^2}} x' \, dy \\ &= \int_{x_2}^{x_1} y^{\frac{3}{2}} \sqrt{x'^2 + 1} \, dy \end{aligned}$$

$$F(y, x', x) = y^{\frac{3}{2}} \sqrt{x'^2 + 1}$$

The Euler equation is

$$\begin{aligned} \frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) - \overbrace{\frac{\partial F}{\partial x}}^{0} &= 0 \\ \frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) &= 0 \\ \frac{\partial F}{\partial x'} &= c \\ y^{\frac{3}{2}} \frac{x'}{\sqrt{x'^2 + 1}} &= c \end{aligned}$$

Simplifying gives

$$\begin{aligned} x' &= \frac{c}{\sqrt{y^3 - c^2}} \\ \frac{dx}{dy} &= \frac{c}{\sqrt{y^3 - c^2}} \\ x &= \int \frac{1}{\sqrt{\frac{y^3}{c^2} - 1}} dy \end{aligned}$$

We can stop here as the problem did not ask to fully solve the integral.