HW 13, Math 121 A
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Problem
Write and solve the Euler equation to make the following integral stationary
$\int_{x_1}^{x_2} \sqrt{x}\sqrt{1+y'^2}dx$
Solution
Let $F=(x,y,y')=\sqrt{x}\sqrt{1+y'^2}$
The Euler equation is

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)-\frac{\partial F}{\partial y}=0$$
$$\frac{\partial F}{\partial y}=\frac{\partial}{\partial y}\left(\sqrt{x}\sqrt{1+y'^2}\right)=0$$

Hence the Euler equation becomes
$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0$$

This means that $\frac{\partial F}{\partial y'}=C$ for some constant C.

$$\frac{\partial F}{\partial y'}=\sqrt{x}\frac{y'}{\sqrt{1+y'^2}}$$

Hence

$$\sqrt{x}\frac{y'}{\sqrt{1+y'^2}}=C$$
$$y'^2=\frac{C^2\left(1+y'^2\right)}{x}$$
$$x=\frac{C^2+C^2y'^2}{y'^2}$$
$$x=\frac{C^2}{y'^2}+k$$
$$y'^2=\frac{C^2}{x-C^2}$$
$$y'=\frac{C}{\sqrt{x-C^2}}$$
$$y\left(x\right)=\frac{2C}{\sqrt{x-C^2}}+C_1$$
$$\frac{y\left(x\right)}{2C}-\frac{C_1}{C}=\frac{1}{\sqrt{x-C^2}}$$

Let $\frac{C_1}{C}=-b$ (some constant), and Let $\frac{1}{2C}=a$ (constant), Hence above becomes

$$a\,y+b=\frac{1}{\sqrt{x-\frac{1}{4a^2}}}$$
$$a\,y+b=\frac{2a}{\sqrt{4a^2\,x-1}}$$

This is equation of a parabola.

2 chapter 9, problem 2.3

Problem
Write and solve the Euler equation to make the following integral stationary $\int_{x_1}^{x_2} x\sqrt{1-y'^2}dx$
Solution
Let $F=(x,y,y')=x\sqrt{1-y'^2}$. The Euler equation is

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)-\frac{\partial F}{\partial y}=0$$
$$\frac{\partial F}{\partial y}=\frac{\partial}{\partial y}\left(x\sqrt{1-y'^2}\right)=0$$

Hence Euler equation becomes

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0$$

This means that $\frac{\partial F}{\partial y'}=C$ for some constant C.

$$\frac{\partial F}{\partial y'}=\frac{-x\,y'}{\sqrt{1-y'^2}}$$

Hence

$$\begin{aligned}\frac{-x\,y'}{\sqrt{1-y'^2}}&=C\\ y'^2&=\frac{C^2\left(1-y'^2\right)}{x^2}\\ x^2&=\frac{C^2-C^2y'^2}{y'^2}\\ x^2&=\frac{C^2}{y'^2}-C^2\\ y'^2&=\frac{C^2}{x^2+C^2}\\ y'&=\frac{C}{\sqrt{x^2+C^2}}\\ y\left(x\right)&=C\operatorname{arcsinh}\left(\frac{x}{C}\right)+C_1\\ \frac{y-C_1}{C}&=\operatorname{arcsinh}\left(\frac{x}{C}\right)\\ \frac{x}{C}&=\sinh\left(\frac{y}{C}-\frac{C_1}{C}\right)\end{aligned}$$

Let $\frac{C_1}{C}=-b$ (some constant). Let $\frac{1}{C}=a$ (some constant). Hence the above becomes

$$a\,x=\sinh\left(a\,y+b\right)$$

3 chapter 9, problem 2.6

Problem
Write and solve the Euler equation to make the following integral stationary $\int_{x_1}^{x_2}\left(y'^2+\sqrt{y}\right)dx$
Solution

Let $F(x,y,y') = y'^2 + \sqrt{y}$. Since F does not depend on x, we change the integration variable to y. Let $y' = \frac{1}{x'}$, then $dx = \frac{dx}{dy}dy$. Hence the integral becomes

$$\int_{y_1}^{y_2} \left(\frac{1}{x'^2} + \sqrt{y} \right) x' \, dy = \int_{y_1}^{y_2} \left(\frac{1}{x'} + x'\sqrt{y} \right) \, dy$$

Now $F(y,x') = \left(\frac{1}{x'} + x'\sqrt{y} \right)$. The Euler equation changes from $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$ to $\frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) - \frac{\partial F}{\partial x} = 0$. Now, $\frac{\partial F}{\partial x} = 0$ since F does not depend on x,Hence the Euler equation reduces to

$$\begin{aligned} \frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) &= 0 \\ \frac{d}{dy}\left(-\frac{1}{x'^2} + \sqrt{y}\right) &= 0 \end{aligned}$$

Hence $-\frac{1}{x'^2} + \sqrt{y} = C$ where C is some constant

$$\begin{aligned} -\frac{1}{x'^2} &= C - \sqrt{y} \\ -\frac{1}{C - \sqrt{y}} &= x'^2 \\ \frac{1}{b + \sqrt{y}} &= x'^2 \qquad \text{where } b \text{ is a new constant} = -C \\ \frac{1}{\sqrt{k + \sqrt{y}}} &= \frac{dx}{dy} \\ \int \frac{dy}{\sqrt{b + \sqrt{y}}} &= \int dx \\ \frac{4}{3}\left(-2b + \sqrt{y}\right)\left(\sqrt{b + \sqrt{y}}\right) &= x + a \qquad \text{Where } a \text{ is constant of integration} \end{aligned}$$

Hence the solution is

$$\frac{4}{3}\left(\sqrt{y} - 2b\right)\left(\sqrt{b + \sqrt{y}}\right) = x + a$$

4 chapter 9, problem 3.2

Problem

Write and solve the Euler equation to make the following integral stationary $\int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{y^2} dx$

Solution

Let $F(x,y,y') = \frac{\sqrt{1+y'^2}}{y^2}$. Since F does not depend on x, we change the integration variable to y. Let $y' = \frac{1}{x'}$, hence $dx = \frac{dx}{dy}dy$. The integral becomes

$$\int_{y_1}^{y_2} \left(\frac{\sqrt{1 + \frac{1}{x'^2}}}{y^2} \right) x' \, dy = \int_{y_1}^{y_2} \frac{\sqrt{x'^2 + 1}}{y^2} \, dy$$

Now $F(y,x') = \frac{\sqrt{x'^2+1}}{y^2}$. The Euler equation changes from $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$ to $\frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) - \frac{\partial F}{\partial x} = 0$.

But $\frac{\partial F}{\partial x} = 0$ since F does not depend on x,Hence the Euler equation reduces to

$$\frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) = 0$$

$$\begin{aligned}\frac{\partial F}{\partial x'} &= \frac{\partial}{\partial x'}\left(\frac{\sqrt{x'^2+1}}{y^2}\right) \\ &= \frac{x'}{y^2\sqrt{x'^2+1}}\end{aligned}$$

Hence

$$\frac{d}{dy}\left(\frac{x'}{y^2\sqrt{x'^2+1}}\right) = 0$$

Hence $\frac{x'}{y^2\sqrt{x'^2+1}} = C$ where C is some constant

$$\begin{aligned}\frac{x'^2}{x'^2+1} &= C\,y^4 \\ \frac{x'^2+1}{x'^2} &= \frac{1}{C\,y^4} \\ 1+\frac{1}{x'^2} &= \frac{1}{C\,y^4} \\ \frac{1}{x'^2} &= \frac{1-Cy^4}{C\,y^4} \\ \frac{\sqrt{C}\,y^2}{\sqrt{1-Cy^4}} &= x' \\ \frac{\sqrt{C}\,y^2}{\sqrt{1-Cy^4}} &= \frac{dx}{dy} \\ \int\frac{\sqrt{C}\,y^2}{\sqrt{1-Cy^4}} &= \int dx\end{aligned}$$

The solution is

$$\frac{\sqrt{C}\,y^3}{3\sqrt{1-Cy^4}} = x + C_1$$

Where C_1 is constant of integration. Let $C_1 = a$, $C = b$ hence solution can be written as

$$\frac{\sqrt{b}\,y^3}{3\sqrt{1-by^4}} = x + a$$

5 chapter 9, problem 3.4

Problem

Write and solve the Euler equation to make the following integral stationary $\int_{x_1}^{x_2} y\sqrt{y'^2+y^2}\,dx$

Solution

Let $F(x,y,y') = y\sqrt{y'^2+y^2}$. Since F does not depend on x, we change the integration variable to y. Let $y' = \frac{1}{x'}$ and $dx = \frac{dx}{dy}dy$. Hence the integral becomes

$$\int_{y_1}^{y_2}\left(y\sqrt{\frac{1}{x'^2}+y^2}\right)x'\,dy = \int_{y_1}^{y_2} y\sqrt{1+x'^2\,y^2}\,dy$$

Now $F(y,x') = y\sqrt{1+x'^2\,y^2}$. The Euler equation changes from $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$ to $\frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) - \frac{\partial F}{\partial x} = 0$. But $\frac{\partial F}{\partial x} = 0$ since F does not depend on x,Hence the Euler equation reduces to

$$\frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) = 0$$

$$\begin{aligned}\frac{\partial F}{\partial x'} &= \frac{\partial}{\partial x'}\left(y\sqrt{1+x'^2\,y^2}\right)\\ &= y\left(\frac{x'\,y^2}{\sqrt{1+x'^2\,y^2}}\right)\\ &= \frac{x'\,y^3}{\sqrt{1+x'^2\,y^2}}\end{aligned}$$

Hence

$$\frac{d}{dy}\left(\frac{x'\,y^3}{\sqrt{1+x'^2\,y^2}}\right) = 0$$

Hence $\frac{x'\,y^3}{\sqrt{1+x'^2\,y^2}} = C$ where C is some constant

$$\begin{aligned}x'\,y^3 &= C\sqrt{1+x'^2\,y^2}\\ x'^2\,y^6 &= C^2\left(1+x'^2\,y^2\right)\\ x'^2\,y^6 &= C^2 + C^2\,x'^2\,y^2\\ x'^2\left(y^6 - C^2\,y^2\right) &= C^2\\ x'^2 &= \frac{C^2}{\left(y^6 - C^2\,y^2\right)}\\ x' &= \frac{C}{y\sqrt{y^4-C^2}}\\ \int dx &= C\int \frac{1}{y\sqrt{y^4-C^2}}dy\end{aligned}$$

The solution is (using Mathematica)

$$a\,x = -\frac{1}{2}i\log\left(\frac{-2iC + 2\sqrt{-C^2+y^4}}{y^2}\right)$$

6 chapter 9, problem 3.6

Problem
Write and solve the Euler equation to make the following integral stationary $\int_{x_1}^{x_2}\frac{y\,y'^2}{1+y\,y'}\,dx$
Solution
Let $F(x,y,y') = \frac{y\,y'^2}{1+y\,y'}$. Since F does not depend on x, we change the integration variable to y. Let

$y' = \frac{1}{x'}$, $dx = \frac{dx}{dy}dy$. Hence the integral becomes

$$\begin{aligned}\int_{y_1}^{y_2}\left(\frac{y\ \frac{1}{x'^2}}{1+y\ \frac{1}{x'}}\right)x'\ dy &= \int_{y_1}^{y_2}\left(\frac{y\ \frac{1}{x'^2}}{\frac{x'+\ y}{x'}}\right)x'\ dy\\ &= \int_{y_1}^{y_2}\left(\frac{y\ \frac{1}{x'}}{x'\ +\ y}\right)x'\ dy\\ &= \int_{y_1}^{y_2}\left(\frac{y}{x'+y}\right)\ dy\end{aligned}$$

Now $F(y,x') = \left(\frac{y}{x'+y}\right)$. The Euler equation changes from $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$ to $\frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) - \frac{\partial F}{\partial x} = 0$. $\frac{\partial F}{\partial x} = 0$ since F does not depend on x,Hence the Euler equation reduces to

$$\frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) = 0$$

$$\begin{aligned}\frac{\partial F}{\partial x'} &= \frac{\partial}{\partial x'}\left(\frac{y}{x'+y}\right)\\ &= y\left(-\frac{1}{(x'+y)^2}\right)\\ &= \frac{-y}{(x'+y)^2}\end{aligned}$$

Hence

$$\frac{d}{dy}\left(\frac{-y}{(x'+y)^2}\right) = 0$$

Hence $\frac{-y}{(x'+y)^2} = C$ where C is some constant

$$-y = C(x'+y)^2$$

Let $C = -k$

$$\begin{aligned}y &= k\ \ (x'+y)^2\\ \sqrt{\frac{y}{k}} &= x' + y\\ \sqrt{\frac{y}{k}} - y &= x'\\ \sqrt{\frac{y}{k}} - y &= \frac{dx}{dy}\\ \int\sqrt{\frac{y}{k}} - y\ dy &= \int dx\\ -\frac{y^2}{2} + \frac{2}{3}y\sqrt{\frac{y}{k}} &= x + \beta\end{aligned}$$

Where β is the integration constant. Let $\frac{1}{\sqrt{k}} = \alpha$ a new constant

$$x = -\frac{1}{2}y^2 + \frac{2}{3}\alpha\ y^{\frac{3}{2}} - \beta$$

Let $\frac{2}{3}\alpha = a$ a new integration constant, let $-\beta = b$ a new constant, we get

$$x = a\ y^{\frac{3}{2}} - \frac{1}{2}y^2 + b$$

7 chapter 9, problem 3.9

Problem

Write and solve the Euler equation to make the following integral stationary $\int_{\phi_1}^{\phi_2} \sqrt{\theta'^2 + \sin^2\theta}\, d\phi$, $\theta' = \frac{d\theta}{d\phi}$

Solution

Here $F(x, y(x), y'(x))$ becomes $F(\phi, \theta(\phi), \theta'(\phi))$. So now $x \to \phi, y \to \theta, y' \to \theta'$. Since $F(\theta', \theta)$ does not depend on ϕ, we change the integration variable to θ, so we want to change from $\theta' = \frac{d\theta}{d\phi}$ to $\phi' = \frac{d\phi}{d\theta}$. Let $\theta' = \frac{1}{\phi'}, d\phi = \frac{d\phi}{d\theta} d\theta$. Hence the integral becomes

$$\int_{\theta_1}^{\theta_2} \left(\sqrt{\frac{1}{\phi'^2} + \sin^2\theta} \right) \phi'\, d\theta = \int_{\theta_1}^{\theta_2} \sqrt{1 + \phi'^2 \sin^2\theta}\, d\theta$$

So now

$$F(\phi', \theta) = \sqrt{1 + \phi'^2 \sin^2\theta}$$

The Euler equation changes from $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$ to $\frac{d}{d\theta}\left(\frac{\partial F}{\partial \phi'}\right) - \frac{\partial F}{\partial \phi} = 0$. $\frac{\partial F}{\partial \phi} = 0$ since F does not depend on ϕ,Hence the Euler equation reduces to

$$\frac{d}{d\theta}\left(\frac{\partial F}{\partial \phi'}\right) = 0$$

$$\begin{aligned} \frac{\partial F}{\partial \phi'} &= \frac{\partial}{\partial \phi'}\left(\sqrt{1 + \phi'^2 \sin^2\theta}\right) \\ &= \frac{\phi' \sin^2\theta}{\sqrt{1 + \phi'^2 \sin^2\theta}} \end{aligned}$$

Hence

$$\frac{d}{d\theta}\left(\frac{\phi' \sin^2\theta}{\sqrt{1 + \phi'^2 \sin^2\theta}}\right) = 0$$

Hence $\frac{\phi' \sin^2\theta}{\sqrt{1 + \phi'^2 \sin^2\theta}} = C$ where C is some constant

$$\begin{aligned} \phi' \sin^2\theta &= C\sqrt{1 + \phi'^2 \sin^2\theta} \\ \phi'^2 \sin^4\theta &= C^2 \left(1 + \phi'^2 \sin^2\theta\right) \\ \phi'^2 \sin^4\theta &= C^2 + C^2 \phi'^2 \sin^2\theta \\ \phi'^2 &= \frac{C^2}{\sin^4\theta - C^2 \sin^2\theta} \\ \phi' &= \frac{C}{\sin\theta\sqrt{\sin^2\theta - C^2}} \\ \int d\phi &= \int \frac{C}{\sin\theta\sqrt{\sin^2\theta - C^2}} d\theta \\ \phi + \alpha &= -\frac{C \tanh^{-1}\left(\frac{\sqrt{2}\sqrt{C^2}\cos(\theta)}{\sqrt{1 - 2C^2 - \cos(2\theta)}}\right)}{\sqrt{-C^2}} \end{aligned}$$

The last integral value was found using mathematica. Hence

$$\frac{\sqrt{-C^2}\,(\phi+\alpha)}{-C} = \operatorname{arctanh}\left(\frac{\sqrt{2}\sqrt{C^2}\cos(\theta)}{\sqrt{1-2C^2-\cos(2\theta)}}\right)$$

Let $\frac{\sqrt{-C^2}}{-C} = A$, let $\sqrt{2}\sqrt{C^2} = B$, $1-2C^2 = D$, then

$$A\ (\phi+\alpha) = \operatorname{arctanh}\left(\frac{B\cos(\theta)}{\sqrt{D-\cos(2\theta)}}\right)$$
$$\tanh(A\ (\phi+\alpha)) = \frac{B\cos(\theta)}{\sqrt{D-\cos(2\theta)}}$$

8 chapter 9, problem 5.2

Problem
Set up Lagrange equations in cylindrical coordinates for a particle of mass m in a potential field $V(r,\theta,z)$
Solution
$L = T - V$ where T is the K.E. and V the potential energy. $T = \frac{1}{2}mv^2$, But

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2$$

As shown on page 219 equation 4.4 , now differentiate both sides w.r.t. time

$$2\ ds\ \frac{ds}{dt} = 2dr\ \dot{r} + \left(r^2\ 2\ d\theta\ \dot{\theta} + 2r\ \dot{r}\ d\theta^2\right) + 2dz\ \dot{z}$$
$$\frac{ds}{dt} = \frac{dr\ \dot{r} + r^2\ \ d\theta\ \dot{\theta} + r\ \dot{r}\ d\theta^2 + dz\ \dot{z}}{\sqrt{dr^2 + r^2 d\theta^2 + dz^2}}$$

Hence

$$v^2 = \frac{\left(dr\ \dot{r} + r^2 d\theta\ \dot{\theta} + r\ \dot{r}\ d\theta^2 + dz\ \dot{z}\right)^2}{dr^2 + r^2 d\theta^2 + dz^2}$$

I used Mathematica to simplify this getting

$$v^2 = \ \dot{r}^2 + r^2\ \dot{\theta}^2 +\ \dot{z}^2$$

Hence,

$$L = \overbrace{\frac{1}{2}m\ \left(\dot{r}^2 + r^2\ \dot{\theta}^2 +\ \dot{z}^2\right)}^{\text{K.E.}} - \overbrace{V(r,\theta,z)}^{\text{P.E.}}$$

The Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0$$

Hence, we get

$$\begin{aligned} \frac{d}{dt}(m\,\dot{r}) - \left(mr\dot{\theta}^2 - \frac{\partial V}{\partial r}\right) &= 0 \\ \frac{d}{dt}\left(mr^2\dot{\theta}\right) + \frac{\partial V}{\partial \theta} &= 0 \\ \frac{d}{dt}(m\dot{z}) + \frac{\partial V}{\partial z} &= 0 \end{aligned}$$

Now differentiating w.r.t. time, and remembering that $r(t)$ also changes with time.

$$\begin{aligned} m\,\ddot{r} - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} &= 0 \\ m\left(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}\right) + \frac{\partial V}{\partial \theta} &= 0 \\ m\ddot{z} + \frac{\partial V}{\partial z} &= 0 \end{aligned}$$

Hence finally we get

$$\begin{aligned} m\left(\ddot{r} - r\dot{\theta}^2\right) &= -\frac{\partial V}{\partial r} \\ m\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right) &= -\frac{1}{r}\frac{\partial V}{\partial \theta} \\ m\ddot{z} &= -\frac{\partial V}{\partial z} \end{aligned}$$

9 chapter 9, problem 5.6

Problem
A particle moves on the surface of a sphere of radius a under the action of the earth gravitational field. Find the θ, ϕ equations of motion. (this is called the spherical pendulum).
Solution
$L = T - V$ where T is the K.E. and V the potential energy. Using spherical coordinates.

$$x = a\sin\theta\cos\phi,\; y = a\sin\theta\sin\phi,\; z = a\cos\theta$$

Hence a position vector

$$\mathbf{r} = \mathbf{i}\; a\sin\theta\cos\phi + \mathbf{j}\; a\sin\theta\sin\phi + \mathbf{k}\; a\cos\theta$$

So velocity is

$$\begin{aligned} \mathbf{r} &= \mathbf{i}\,\frac{d}{dt}(a\sin\theta\cos\phi) + \mathbf{j}\,\frac{d}{dt}(a\sin\theta\sin\phi) + \mathbf{k}\,\frac{d}{dt}(a\cos\theta) \\ &= \mathbf{i}\;\left(-a\sin\theta\sin\phi\;\dot{\phi} + a\cos\theta\;\dot{\theta}\;\cos\phi\right) + \mathbf{j}\;\left(a\sin\theta\cos\phi\dot{\phi} + a\cos\theta\;\dot{\theta}\;\sin\phi\right) + \mathbf{k}\;\left(-a\sin\theta\;\dot{\theta}\right) \end{aligned}$$

Hence

$$\dot{r} = \|\mathbf{r}\| = \sqrt{\left(-a\sin\theta\sin\phi\;\dot{\phi} + a\cos\theta\;\dot{\theta}\cos\phi\right)^2 + \left(a\sin\theta\cos\phi\;\dot{\phi} + a\cos\theta\;\dot{\theta}\;\sin\phi\right)^2 + \left(-a\sin\theta\;\dot{\theta}\right)^2}$$

Then

$$v^2 = \dot{r}^2 = \left(-a\sin\theta\sin\phi\,\dot{\phi} + a\cos\theta\,\dot{\theta}\cos\phi\right)^2 + \left(a\sin\theta\cos\phi\dot{\phi} + a\cos\theta\,\dot{\theta}\,\sin\phi\right)^2 + \left(-a\sin\theta\,\dot{\theta}\right)^2$$

$$= \left(a^2\sin^2\theta\sin^2\phi\,\dot{\phi}^2 + a^2\cos^2\theta\,\dot{\theta}^2\cos^2\phi - \overbrace{2a^2\sin\theta\sin\phi\,\dot{\phi}\cos\theta\,\dot{\theta}\cos\phi}\right)$$
$$+ \left(a^2\sin^2\theta\cos^2\phi\,\dot{\phi}^2 + a^2\cos^2\theta\,\dot{\theta}^2\,\sin^2\phi\overbrace{+2a^2\sin\theta\cos\phi\dot{\phi}\,\cos\theta\,\dot{\theta}\,\sin\phi}\right) + \left(a^2\sin^2\theta\,\dot{\theta}^2\right)$$

$$= \overbrace{a^2\sin^2\theta\sin^2\phi\,\dot{\phi}^2} + a^2\widehat{\cos^2\theta\,\dot{\theta}^2}\cos^2\phi + \overbrace{a^2\sin^2\theta\cos^2\phi\,\dot{\phi}^2} + a^2\widehat{\cos^2\theta\,\dot{\theta}^2}\,\sin^2\phi + a^2\sin^2\theta\,\dot{\theta}^2$$

$$= a^2\dot{\phi}^2\sin^2\theta\,\overbrace{\left(\sin^2\phi + \cos^2\phi\right)}^{=1} + a^2\dot{\theta}^2\cos^2\theta\,\overbrace{\left(\cos^2\phi + \sin^2\phi\right)}^{=1} + a^2\sin^2\theta\,\dot{\theta}^2$$

$$= a^2\dot{\phi}^2\sin^2\theta + a^2\dot{\theta}^2\,\overbrace{\left(\cos^2\theta + \sin^2\theta\right)}^{=1}$$

$$= a^2\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right)$$

Hence $T = \frac{1}{2}mv^2$. For a particle, taking mass as one unit. Hence

$$T = \frac{1}{2}a^2\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right)$$

The P.E. is $mga\,\cos\theta$. Hence the Lagrangian is

$$L = T - V$$
$$L = \frac{1}{2}a^2\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) - ga\,\cos\theta$$

We have 2 independent variables, hence we need 2 Lagrangian equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) - \frac{\partial L}{\partial\phi} = 0$$

$$\frac{\partial L}{\partial\dot{\theta}} = a^2\dot{\theta}$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = a^2\ddot{\theta}$$
$$\frac{\partial L}{\partial\theta} = a^2\left(\dot{\phi}^2\sin\theta\,\cos\theta\right) + ga\,\sin\theta$$

Hence the first equation becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$
$$a^2\ddot{\theta} - a^2\left(\dot{\phi}^2 \sin\theta\ \cos\theta\right) - ga\ \sin\theta = 0$$
$$a\ \ddot{\theta} - a\ \left(\dot{\phi}^2 \sin\theta\ \cos\theta\right) - g\ \sin\theta = 0$$

To find the second equation

$$\frac{\partial L}{\partial \dot{\phi}} = a^2\left(2\dot{\phi}\sin^2\theta\right)$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{d}{dt}\left(a^2\left(2\dot{\phi}\sin^2\theta\right)\right)$$
$$\frac{\partial L}{\partial \phi} = 0$$

Hence the second equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0$$
$$\frac{d}{dt}\left(a^2\left(2\dot{\phi}\sin^2\theta\right)\right) = 0$$
$$\frac{d}{dt}\left(\ 2\dot{\phi}\sin^2\theta\right) = 0$$
$$\frac{d}{dt}\left(\ 2\dot{\phi}\sin^2\theta\right) = 0$$

10 chapter 9, problem 6.1

Problem
Find surface of revolution formed by rotating the curve around the x-axis that has a minimum area subject to a curve of give length l joining 2 points.
Solution
Area is

$$I = \int_{x_1}^{x_2} 2\pi y\sqrt{1 + y'^2}dx \tag{1}$$

Since integrand does not depend on x we change the independent variable to y. $dx = \frac{dx}{dy}dy, y' = \frac{1}{x'}$. Hence (1) becomes

$$I = \int_{y_1}^{y_2} 2\pi y\sqrt{1 + \frac{1}{x'^2}}\ x'dy \tag{1}$$
$$= \int_{y_1}^{y_2} 2\pi y\sqrt{x'^2 + 1}\ dy$$

Hence $F(y, x', x) = 2\pi y\sqrt{x'^2 + 1}$. Now finding the constraint

$$g = \int ds = l$$
$$= \int_{x_1}^{x_2} \sqrt{1 + y'^2}dx$$

Since integrand does not depend on x we change the independent variable to y. $dx = \frac{dx}{dy}dy, y' = \frac{1}{x'}$. Hence

$$\begin{aligned} g &= \int_{y_1}^{y_2} \sqrt{1 + \frac{1}{x'^2}}\, x' dy \\ &= \int_{y_1}^{y_2} \sqrt{x'^2 + 1}\; dy \end{aligned}$$

So $G = \sqrt{x'^2 + 1}$. Hence we get

$$F + \lambda G = \left(2\pi y \sqrt{x'^2 + 1}\right) + \lambda \sqrt{x'^2 + 1}$$

As the new Euler equation (with constrains). Solving

$$\begin{aligned} \frac{d}{dy}\left(\frac{\partial}{\partial x'}(F + \lambda G)\right) - \overbrace{\frac{\partial}{\partial x}(F + \lambda G)}^{0 \text{ since does not depend on } x} &= 0 \\ \frac{d}{dy}\left(\frac{\partial}{\partial x'}\left(2\pi y \sqrt{x'^2 + 1} + \lambda \sqrt{x'^2 + 1}\right)\right) &= 0 \\ \frac{d}{dy}\left(\frac{2\pi y x'}{\sqrt{x'^2 + 1}} + \frac{\lambda x'}{\sqrt{x'^2 + 1}}\right) &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
\frac{2\pi y x'}{\sqrt{x'^2+1}} + \frac{\lambda x'}{\sqrt{x'^2+1}} &= c \\
\frac{2\pi y x' + \lambda x'}{\sqrt{x'^2+1}} &= c \\
x'\left(2\pi y + \lambda\right) &= c\sqrt{x'^2+1} \\
x'^2\left(2\pi y + \lambda\right)^2 &= c^2\left(x'^2 + 1\right) \\
\frac{x'^2}{\left(x'^2+1\right)} &= \frac{c^2}{\left(2\pi y + \lambda\right)^2} \\
\frac{\left(x'^2+1\right)}{x'^2} &= \frac{\left(2\pi y + \lambda\right)^2}{c^2} \\
1 + \frac{1}{x'^2} &= \frac{\left(2\pi y + \lambda\right)^2}{c^2} \\
\frac{1}{x'^2} &= \frac{\left(2\pi y + \lambda\right)^2 - c^2}{c^2} \\
\frac{c^2}{\left(2\pi y + \lambda\right)^2 - c^2} &= x'^2 \\
\frac{c}{\sqrt{\left(2\pi y + \lambda\right)^2 - c^2}} &= x' \\
\frac{dx}{dy} &= \frac{c}{\sqrt{\left(2\pi y + \lambda\right)^2 - c^2}} \\
\int dx &= \int \frac{c}{\sqrt{\left(2\pi y + \lambda\right)^2 - c^2}} dy \\
\int dx &= \int \frac{1}{\sqrt{\left(\frac{2\pi y + \lambda}{c}\right)^2 - 1}} dy \\
x &= \frac{c}{2\pi} \operatorname{arccosh}\left(\frac{2\pi y + \lambda}{c}\right) + c_1
\end{aligned}
$$

To express this as y a function of x we get

$$
\begin{aligned}
\frac{2\pi}{c}\left(x - c_1\right) &= \operatorname{arccosh}\left(\frac{2\pi y + \lambda}{c}\right) \\
\cosh\left(\frac{2\pi}{c}\left(x - c_1\right)\right) &= \frac{2\pi y + \lambda}{c} \\
\frac{c\cosh\left(\frac{2\pi}{c}\left(x - c_1\right)\right) - \lambda}{2\pi} &= y
\end{aligned}
$$

We have 3 unknowns, c, c_1, λ that we can use boundary conditions, and length l to determine.

11 chapter 9, problem 6.2

Problem

Find the equation of the curve subject to a curve of give length l joining 2 points so that the plane area between the curve and straight line joining the points is a maximum.

Solution
Area is $\int y\,dx$. Hence area is $I=\int_{x_1}^{x_2} y\,dx$ subject to constraint that $\int\ ds=l$ or $g=\int_{x_1}^{x_2}\sqrt{1+y'^2}dx=l$. Hence the Euler equation with constrains now becomes

$$F+\lambda G=y\ +\lambda\sqrt{y'^2+1}$$

Therefore

$$\begin{aligned}\frac{d}{dx}\left(\frac{\partial}{\partial y'}(F+\lambda G)\right)-\frac{d}{dy}(F+\lambda G)&=0\\ \frac{d}{dy}\left(\frac{\lambda\,y'}{\sqrt{y'^2+1}}\right)-1&=0\\ \frac{\lambda\,y'}{\sqrt{y'^2+1}}&=x+c\end{aligned}$$

This simplifies to

$$\begin{aligned}\int dy&=\int\frac{(x+c)}{\sqrt{\lambda^2-(x+c)^2}}dx\\ y+c_1&=-\sqrt{\lambda^2-(x+c)^2}\\ (y+c_1)^2&=\lambda^2-(x+c)^2\\ (y+c_1)^2+(x+c)^2&=\lambda^2\end{aligned}$$

This is the equation of a circle.

12 chapter 9, problem 6.5

Problem
Given surface area of solid of revolution, finds its shape to make its volume a maximum.
Solution
Volume is $\int\pi y^2 ds$ where ds is a small segment of the curve length. Hence

$$I=\int_{x_1}^{x_2}\pi y^2\sqrt{1+y'^2}dx \tag{1}$$

Constraint is that area is given, say A. Hence

$$g=\int_{x_1}^{x_2}2\pi y\sqrt{1+y'^2}dx=A \tag{2}$$

Since both integrands do not depend on x we change the independent variable to y. $dx=\frac{dx}{dy}dy, y'=\frac{1}{x'}$. Hence (1) becomes

$$\begin{aligned}I&=\int_{x_1}^{x_2}\pi y^2\sqrt{1+\frac{1}{x'^2}}x'dy\\ &=\int_{x_1}^{x_2}\pi y^2\sqrt{x'^2+1}dy\end{aligned}$$

And (2) becomes

$$\begin{aligned} g &= \int_{y_1}^{y_2} 2\pi y \sqrt{1 + \frac{1}{x'^2}}\, x' dy \\ &= \int_{y_1}^{y_2} 2\pi y \sqrt{x'^2 + 1}\, dy \end{aligned}$$

Hence we get

$$F + \lambda G = \left(\pi y^2 \sqrt{x'^2 + 1}\right) + 2\lambda \pi y \sqrt{x'^2 + 1}$$

as the new Euler equation (with constrains) to solve.

$$\begin{aligned} \frac{d}{dy}\left(\frac{\partial}{\partial x'}(F + \lambda G)\right) - \overbrace{\frac{\partial}{\partial x}(F + \lambda G)}^{0 \text{ since does not depend on } x} &= 0 \\ \frac{d}{dy}\left(\frac{\partial}{\partial x'}\left(\pi y^2 \sqrt{x'^2 + 1} + 2\lambda \pi y \sqrt{x'^2 + 1}\right)\right) &= 0 \\ \frac{d}{dy}\left(\frac{\pi y^2 x'}{\sqrt{x'^2 + 1}} + \frac{2\lambda \pi y x'}{\sqrt{x'^2 + 1}}\right) &= 0 \end{aligned}$$

Hence

$$\frac{\pi y^2 x'}{\sqrt{x'^2+1}} + \frac{2\lambda\pi y x'}{\sqrt{x'^2+1}} = c$$
$$\frac{\pi y^2 x' + 2\lambda\pi y x'}{\sqrt{x'^2+1}} = c$$
$$\pi y^2 x' + 2\lambda\pi y x' = c\sqrt{x'^2 + 1}$$
$$x'^2 \left(\pi y^2 + 2\lambda\pi y\right)^2 = c^2\left(x'^2 + 1\right)$$
$$\frac{x'^2}{(x'^2+1)} = \frac{c^2}{(\pi y^2 + 2\lambda\pi y)^2}$$
$$\frac{\left(x'^2+1\right)}{x'^2} = \frac{\left(\pi y^2 + 2\lambda\pi y\right)^2}{c^2}$$
$$1 + \frac{1}{x'^2} = \frac{\left(\pi y^2 + 2\lambda\pi y\right)^2}{c^2}$$
$$\frac{1}{x'^2} = \frac{\left(\pi y^2 + 2\lambda\pi y\right)^2 - c^2}{c^2}$$
$$\frac{c^2}{(\pi y^2 + 2\lambda\pi y)^2 - c^2} = x'^2$$
$$\frac{c}{\sqrt{(\pi y^2 + 2\lambda\pi y)^2 - c^2}} = x'$$
$$\frac{dx}{dy} = \frac{c}{\sqrt{(\pi y^2 + 2\lambda\pi y)^2 - c^2}}$$
$$\int dx = \int \frac{c}{\sqrt{(\pi y^2 + 2\lambda\pi y)^2 - c^2}} dy$$
$$x = \int \frac{c}{\sqrt{(\pi y^2 + 2\lambda\pi y)^2 - c^2}} dy$$
$$x = \int \frac{1}{\sqrt{\left(\frac{\pi y^2 + 2\lambda\pi y}{c}\right)^2 - 1}} dy$$

Hence

$$x = \left(\frac{c}{2y\pi + 2\lambda\pi}\right)\cosh^{-1}\left(\frac{\pi y^2 + 2\lambda\pi y}{c}\right)$$

13 chapter 15, problem 8.12

Problem
Solve $y'' + y = f(x)$ with $y(0) = y\left(\frac{\pi}{2}\right) = 0$ using 8.17:

$$y(x) = -\cos x \int_0^x \sin(x') f(x')\, dx' - \sin x \int_x^{\frac{\pi}{2}} \cos(x') f(x')\, dx'$$

when $f(x) = \sec x$
Solution

$$y(x) = -\cos x \int_0^x \sin(x') \sec x' \, dx' - \sin x \int_x^{\frac{\pi}{2}} \cos(x') \sec x' \, dx'$$

Since $\sec x' = \frac{1}{\cos x'}$ we get

$$y(x) = -\cos x \int_0^x \tan x' \, dx' - \sin x \int_x^{\frac{\pi}{2}} dx'$$

But $\int_0^x \tan x' \, dx' = -\log(\cos(x))$, Hence

$$\begin{aligned} y(x) &= \cos(x) \log(\cos(x)) - \sin x \left(\frac{1}{2}\pi - x\right) \\ &= \cos(x) \log(\cos(x)) - \frac{1}{2}\pi \sin x + x \sin x \end{aligned}$$

14 chapter 15, problem 8.15

Problem
Use Green function method and the given solutions of the homogeneous equation to find a particular solution to $y'' - y = \sec h(x)$, where $y_1(x) = \sinh(x), \quad y_2(x) = \cosh(x)$
Solution

$$y_p = y_2 \int \frac{y_1 f}{W} dx - y_1 \int \frac{y_2 f}{W} dx \tag{1}$$

Where $f = \sec h(x)$

$$\begin{aligned} W &= \begin{vmatrix} y_1' & y_2' \\ y_1 & y_2 \end{vmatrix} \\ &= \begin{vmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{vmatrix} \\ &= \cosh^2 x - \sinh^2 x \\ &= 1 \end{aligned}$$

So from (1) we get

$$y_p = \cosh(x) \int \sinh(x) \ \sec h(x) \ dx - \sinh(x) \int \cosh(x) \sec h(x) \ dx$$

But $\sec h(x) = \frac{1}{\cosh x}$, Hence

$$\begin{aligned} y_p &= \cosh(x) \int \sinh(x) \ \frac{1}{\cosh x} \ dx - \sinh(x) \int \cosh(x) \frac{1}{\cosh x} \ dx \\ &= \cosh(x) \int \tan sh(x) \ dx - \sinh(x) \int dx \end{aligned}$$

But $\int \tan sh(x) \ dx = \log(\cosh(x))$, Hence

$$y_p = \cosh(x) \log(\cosh(x)) - x \ \sinh(x)$$

15 chapter 15, problem 8.17

Problem

Use Green function method and the given solutions of the homogeneous equation to find a particular solution to $y'' - 2\left(\csc^2(x)\right) y = \sin^2(x)$, where $y_1(x) = \cot x,\ \ y_2(x) = 1 - x\ \cot(x)$

Solution

Note $\cot(x) = \frac{1}{\tan(x)} = \frac{\cos(x)}{\sin(x)}$, $\csc(x) = \frac{1}{\sin(x)}$

$$y_p = y_2 \int \frac{y_1\, f}{W} dx - y_1 \int \frac{y_2\, f}{W} dx \tag{1}$$

Where $f = \sin^2(x)$.

$$\begin{aligned} y_1' &= \frac{d}{dx}(\cot(x)) = -\cot^2 x - 1 \\ &= -\frac{1}{\sin^2(x)} \end{aligned}$$

And

$$\begin{aligned} y_2' &= \frac{d}{dx}(1 - x\ \cot(x)) \\ &= -\frac{\cos(x)}{\sin(x)} + \frac{x}{\sin^2(x)} \end{aligned}$$

Therefore

$$\begin{aligned} W &= \begin{vmatrix} y_1' & y_2' \\ y_1 & y_2 \end{vmatrix} \\ &= \begin{vmatrix} -\frac{1}{\sin^2(x)} & -\frac{\cos(x)}{\sin(x)} + \frac{x}{\sin^2(x)} \\ \frac{\cos(x)}{\sin(x)} & 1 - \frac{x\ \cos(x)}{\sin(x)} \end{vmatrix} \\ &= \left(-\frac{1}{\sin^2(x)}\right)\left(1 - \frac{x\ \cos(x)}{\sin(x)}\right) - \left(-\frac{\cos(x)}{\sin(x)} + \frac{x}{\sin^2(x)}\right)\frac{\cos(x)}{\sin(x)} \\ &= -\frac{1}{\sin^2(x)} + \overbrace{\frac{x\ \cos(x)}{\sin^3(x)}} + \frac{\cos^2(x)}{\sin^2(x)} - \overbrace{\frac{x\cos(x)}{\sin^3(x)}} \\ &= -\frac{1}{\sin^2(x)} + \frac{\cos^2(x)}{\sin^2(x)} \end{aligned}$$

So from (1) we get

$$\begin{aligned} y_p &= \left(1 - \frac{x\cos x}{\sin x}\right)\int \frac{\frac{\cos x}{\sin x}\ \sin^2(x)}{-\frac{1}{\sin^2(x)} + \frac{\cos^2(x)}{\sin^2(x)}} dx - \frac{\cos x}{\sin x}\int \frac{\left(1 - \frac{x\cos x}{\sin(x)}\right)\ \sin^2(x)}{-\frac{1}{\sin^2(x)} + \frac{\cos^2(x)}{\sin^2(x)}} dx \\ &= \left(1 - \frac{x\cos x}{\sin x}\right)\int \frac{\cos x\ \sin x}{\frac{-1+\cos^2 x}{\sin^2(x)}} dx - \frac{\cos x}{\sin x}\int \frac{\sin^2 x - x\ \cos x\ \sin x}{\frac{-1+\cos^2 x}{\sin^2(x)}} dx \\ &= \left(1 - \frac{x\cos x}{\sin x}\right)\int \frac{\cos x\ \sin^3 x}{-1+\cos^2 x} dx - \frac{\cos x}{\sin x}\int \frac{\sin^4 x - x\ \cos x\ \sin^3 x}{-1+\cos^2 x} dx \end{aligned}$$

but $I=\int \frac{\cos x\ \sin^3 x}{\cos^2 x-1}=\int \frac{\cos x\ \sin^3 x}{-\sin^2 x}=\int -\cos x\ \sin x=\frac{1}{2}\cos^2 x$ And

$$\begin{aligned}
I &= \int \frac{\sin^4 x - x\ \cos x\ \sin^3 x}{-1+\cos^2 x}\\
&= \int \frac{\sin^4 x - x\ \cos x\ \sin^3 x}{-\sin^2 x}\\
&= \int -\sin^2 x + x\ \cos x\ \sin x\\
&= -\int \sin^2(x)\ dx + \int x\ \cos(x)\ \sin(x)\ dx
\end{aligned}$$

But $\int \sin^2(x)\ dx = \frac{x}{2}-\frac{1}{4}\sin(2x)$ and $\int x\ \cos(x)\ \sin(x)\ dx = -\frac{1}{4}x\cos(2x)+\frac{1}{8}\sin(2x)$, therefore

$$\begin{aligned}
-\int \sin^2(x)\ dx + \int x\ \cos(x)\ \sin(x)\ dx &= \left(-\frac{x}{2}+\frac{1}{4}\sin(2x)\right)+\left(-\frac{1}{4}x\cos(2x)+\frac{1}{8}\sin(2x)\right)\\
&= -\frac{x}{2}+\frac{1}{4}\sin(2x)-\frac{1}{4}x\cos(2x)+\frac{1}{8}\sin(2x)\\
&= \frac{3}{8}\sin 2x-\frac{1}{2}x-\frac{1}{4}x\cos 2x
\end{aligned}$$

Hence (2) becomes

$$\begin{aligned}
y_p(x) &= \left(1-\frac{x\cos x}{\sin x}\right)\left(\frac{1}{2}\cos^2 x\right)-\frac{\cos x}{\sin x}\left(\frac{3}{8}\sin 2x-\frac{1}{2}x-\frac{1}{4}x\cos 2x\right)\\
&= \left(\frac{1}{2}\cos^2 x-\frac{1}{2}\frac{x\cos^3 x}{\sin x}\right)-\left(\frac{3}{8}\sin 2x\frac{\cos x}{\sin x}-\frac{1}{2}x\frac{\cos x}{\sin x}-\frac{1}{4}x\cos 2x\frac{\cos x}{\sin x}\right)\\
&= \frac{1}{2}\cos^2 x-\frac{1}{2}\frac{x\cos^3 x}{\sin x}-\frac{3}{8}\sin 2x\frac{\cos x}{\sin x}+\frac{1}{2}x\frac{\cos x}{\sin x}+\frac{1}{4}x\cos 2x\frac{\cos x}{\sin x}\\
&= \frac{1}{4}\cot x\ (x-\cos x\ \sin x)
\end{aligned}$$

16 chapter 15, problem 8.2

Problem
Solve $y''+\omega^2 y=f(t)$ using $y(t)=\int_0^t \frac{1}{\omega}\sin\omega(t-t')\ f(t')\ dt'$ when $f(t)=\sin\omega t$
Solution

$$\begin{aligned}
y(t) &= \int_0^t \frac{1}{\omega}\sin\omega(t-t')f(t')\ dt'\\
&= \int_0^t \frac{1}{\omega}\sin\omega(t-t')\sin\omega t'\ dt' \qquad (1)
\end{aligned}$$

But $\sin\alpha\sin\beta=\frac{1}{2}\cos(\alpha-\beta)-\frac{1}{2}\cos(\alpha+\beta)$, hence

$$\begin{aligned}
\sin\omega(t-t')\sin\omega t' &= \frac{1}{2}\cos(\omega(t-t')-\omega t')-\frac{1}{2}\cos(\omega(t-t')+\omega t')\\
&= \frac{1}{2}\cos(t\omega-2\omega t')-\frac{1}{2}\cos\omega t
\end{aligned}$$

Hence (1) becomes

$$
\begin{aligned}
y(t) &= \int_0^t \frac{1}{\omega}\frac{1}{2}\cos(\omega t - 2\omega t') - \frac{1}{2}\cos\omega t \; dt' \\
&= \frac{1}{2\omega}\int_0^t \cos(\omega t - 2\omega t') \; dt' - \frac{1}{2}\cos\omega t \int_0^t dt' \\
&= \frac{1}{2\omega}\left[\frac{\sin(\omega t - 2\omega t')}{-2\omega}\right]_0^t - \frac{1}{2} t\cos t\omega \\
&= \frac{-1}{4\omega^2}\left(\sin(\omega t - 2\omega t) - \sin(\omega t)\right) - \frac{1}{2} t\cos t\omega \\
&= \frac{1}{2\omega^2}\sin t\omega - \frac{1}{2} t\cos t\omega \\
&= \frac{1}{2\omega^2}\left(\sin t\omega - \omega t\cos t\omega\right)
\end{aligned}
$$

$$
y(t) = \frac{1}{2\omega^2}\left(\sin t\omega - \omega t\cos t\omega\right)
$$

17 chapter 15, problem 8.3

Problem
Solve $y'' + \omega^2 y = f(t)$ using $y(t) = \int_0^t \frac{1}{\omega}\sin\omega(t - t') \; f(t') \; dt'$ when $f(t) = e^{-t}$
Solution

$$
\begin{aligned}
y(t) &= \int_0^t \frac{1}{\omega}\sin\omega(t - t') \; f(t') \; dt' \\
&= \frac{1}{\omega}\int_0^t \sin\omega(t - t') \; e^{-t'} \; dt' \qquad (1)
\end{aligned}
$$

$$
\text{Let } I = \int_0^t \sin\omega(t - t') \; e^{-t'} \; dt'
$$

Integrate by part, let $u = \sin(\omega t - \omega t')$, $v = -e^{-t'}$

$$
\begin{aligned}
I &= [\sin\omega(t - t') \; \left(-e^{-t'}\right)]_0^t - \omega\int_0^t \cos(\omega t - \omega t') \; e^{-t'} \; dt' \\
&= \sin\omega t \; - \omega\int_0^t \cos(\omega t - \omega t') \; e^{-t'} \; dt'
\end{aligned}
$$

Integrate by parts again. $u = \cos(\omega t - \omega t')$, $v = -e^{-t'}$

$$
\begin{aligned}
I &= \sin\omega t - \omega\left([\cos(\omega t - \omega t')\left(-e^{-t'}\right)]_0^t + \omega\int_0^t \sin\omega(t - t')\, e^{-t'} \; dt'\right) \\
I &= \sin\omega t - \omega\left([-e^{-t} + \cos(\omega t)] + \omega I\right) \\
I &= \sin\omega t + \omega e^{-t} - \omega\cos(\omega t) - \omega^2 I \\
I + \omega^2 I &= \sin\omega t + \omega e^{-t} - \omega\cos(\omega t) \\
I &= \frac{\sin\omega t + \omega e^{-t} - \omega\cos(\omega t)}{1 + \omega^2}
\end{aligned}
$$

Hence from (1)

$$
y(t) = \frac{1}{\omega}\frac{\omega e^{-t} - \omega\cos(\omega t) + \sin(\omega t)}{1 + \omega^2}
$$

18 chapter 9, problem 3.1

Problem
Change the independent variable to simplify the Euler equation and then find the first integral of it.
$\int_{x_2}^{x_1} y^{\frac{3}{2}} ds$
Solution

$$ds = \sqrt{(dx)^2 + (dy)^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = dx\sqrt{1 + y'^2}$$

Hence

$$I = \int_{x_2}^{x_1} y^{\frac{3}{2}} ds = \int_{x_2}^{x_1} y^{\frac{3}{2}} \sqrt{1 + y'^2} dx$$

Since integrand does not depend on x, changing the independent variable to y in order to simplify solution. Using $dx = \frac{dx}{dy} dy \to y' = \frac{1}{x'}$. The integral now becomes

$$\begin{aligned} I &= \int_{x_2}^{x_1} y^{\frac{3}{2}} \sqrt{1 + \frac{1}{x'^2}} x' \, dy \\ &= \int_{x_2}^{x_1} y^{\frac{3}{2}} \sqrt{x'^2 + 1} \, dy \end{aligned}$$

$$F(y, x', x) = y^{\frac{3}{2}} \sqrt{x'^2 + 1}$$

The Euler equation is

$$\begin{aligned} \frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) - \overbrace{\frac{\partial F}{\partial x}}^{0} &= 0 \\ \frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) &= 0 \\ \frac{\partial F}{\partial x'} &= c \\ y^{\frac{3}{2}} \frac{x'}{\sqrt{x'^2 + 1}} &= c \end{aligned}$$

Simplifying gives

$$\begin{aligned} x' &= \frac{c}{\sqrt{y^3 - c^2}} \\ \frac{dx}{dy} &= \frac{c}{\sqrt{y^3 - c^2}} \\ x &= \int \frac{1}{\sqrt{\frac{y^3}{c^2} - 1}} dy \end{aligned}$$

We can stop here as the problem did not ask to fully solve the integral.