

HW 12, Math 121 A
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1 chapter 15, problem 4.12

Problem Find the exponential Fourier transform of the given $f(x)$ and write $f(x)$ as a fourier integral.

$$f(x) = \begin{cases} \sin x & |x| < \frac{\pi}{2} \\ 0, & |x| > \frac{\pi}{2} \end{cases}$$

Solution

Let $F(\alpha)$ be the Fourier transform of $f(x)$ defined as $F(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx$

$$F(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx$$

$$2\pi F(\alpha) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x e^{-i\alpha x} dx$$

Integration by parts, $u = \sin x, du = \cos x, v = \frac{e^{-i\alpha x}}{-i\alpha}$, hence $\int u dv = uv - \int du v$

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x e^{-i\alpha x} dx$$

$$= \left[\sin x \frac{e^{-i\alpha x}}{-i\alpha} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \frac{e^{-i\alpha x}}{-i\alpha} dx$$

$$= \left[\sin x \frac{e^{-i\alpha x}}{-i\alpha} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{i\alpha} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x e^{-i\alpha x} dx$$

Integration by parts the second integral again. $u = \cos x, du = -\sin x$

$$I = \left[\sin x \frac{e^{-i\alpha x}}{-i\alpha} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{i\alpha} \left\{ \left[\cos x \frac{e^{-i\alpha x}}{-i\alpha} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (-\sin x) \frac{e^{-i\alpha x}}{-i\alpha} dx \right\}$$

$$I = \left[\sin x \frac{e^{-i\alpha x}}{-i\alpha} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{i\alpha} \left\{ \left[\cos x \frac{e^{-i\alpha x}}{-i\alpha} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \left(\frac{1}{i\alpha} \right) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x e^{-i\alpha x} dx \right\}$$

$$I = \left[\sin x \frac{e^{-i\alpha x}}{-i\alpha} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{i\alpha} \left[\cos x \frac{e^{-i\alpha x}}{-i\alpha} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \frac{1}{i^2 \alpha^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x e^{-i\alpha x} dx$$

$$I = \left[\sin x \frac{e^{-i\alpha x}}{-i\alpha} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{i\alpha} \left[\cos x \frac{e^{-i\alpha x}}{-i\alpha} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{\alpha^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x e^{-i\alpha x} dx$$

But the last integral on the right above is the same as the integral we start with, so

$$\begin{aligned}
I &= \left[\sin x \frac{e^{-iax}}{-ia} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{ia} \left[\cos x \frac{e^{-iax}}{-ia} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{\alpha^2} I \\
I - \frac{1}{\alpha^2} I &= \left[\sin x \frac{e^{-iax}}{-ia} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \frac{i}{\alpha} \left[\cos x \frac{e^{-iax}}{-ia} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
I \left(1 - \frac{1}{\alpha^2} \right) &= \left[\sin\left(\frac{\pi}{2}\right) \frac{e^{-ia\frac{\pi}{2}}}{-ia} - \sin\left(-\frac{\pi}{2}\right) \frac{e^{-ia\left(-\frac{\pi}{2}\right)}}{-ia} \right] - \frac{i}{\alpha} \left[\cos\left(\frac{\pi}{2}\right) \frac{e^{-ia\frac{\pi}{2}}}{-ia} - \cos\left(-\frac{\pi}{2}\right) \frac{e^{-ia\left(-\frac{\pi}{2}\right)}}{-ia} \right] \\
I \left(\frac{\alpha^2 - 1}{\alpha^2} \right) &= \left[\frac{e^{-ia\frac{\pi}{2}}}{-ia} + \frac{e^{ia\left(\frac{\pi}{2}\right)}}{-ia} \right] - \frac{i}{\alpha} (0) \\
I &= \left(\frac{\alpha^2}{\alpha^2 - 1} \right) \left[\frac{-e^{-ia\frac{\pi}{2}}}{ia} - \frac{e^{ia\frac{\pi}{2}}}{ia} \right] \\
I &= \left(\frac{\alpha}{\alpha^2 - 1} \right) \frac{-1}{i} \left[e^{ia\frac{\pi}{2}} + e^{-ia\frac{\pi}{2}} \right] \\
I &= \left(\frac{\alpha i}{\alpha^2 - 1} \right) \left[e^{ia\frac{\pi}{2}} + e^{-ia\frac{\pi}{2}} \right] \\
I &= \left(\frac{\alpha i}{\alpha^2 - 1} \right) 2 \cos\left(\alpha \frac{\pi}{2}\right)
\end{aligned}$$

Hence the Fourier transform of $f(x)$ is

$$\begin{aligned}
2\pi F(\alpha) &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x e^{-iax} dx \\
&= \left(\frac{\alpha i}{\alpha^2 - 1} \right) 2 \cos\left(\alpha \frac{\pi}{2}\right)
\end{aligned}$$

Therefore

$$\begin{aligned}
g(\alpha) &= \frac{1}{\pi} \left(\frac{\alpha i}{\alpha^2 - 1} \right) \cos\left(\alpha \frac{\pi}{2}\right) \\
&= \frac{\alpha i \cos\left(\alpha \frac{\pi}{2}\right)}{(\alpha^2 - 1)\pi}
\end{aligned}$$

To obtain $f(x)$ given its fourier transform $F(\alpha)$, then we apply the inverse fourier transform

$$\begin{aligned}
f(x) &= \int_{-\infty}^{\infty} F(\alpha) e^{iax} d\alpha \\
&= \int_{-\infty}^{\infty} \frac{\alpha i \cos\left(\alpha \frac{\pi}{2}\right)}{(\alpha^2 - 1)\pi} e^{iax} d\alpha \\
&= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\alpha i \cos\left(\alpha \frac{\pi}{2}\right)}{\alpha^2 - 1} e^{iax} d\alpha
\end{aligned}$$

2 chapter 15, problem 4.18

Problem Find the fourier sin transform of the given $f(x)$ and write $f(x)$ as a fourier integral. Verify the answer is the same as the exponential fourier transform.

$$f(x) = \begin{cases} x & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

Solution

Let $\mathcal{F}_s f(x)$ be the Fourier sin transform of $f(x)$ defined as $g_s(\alpha) = \mathcal{F}_s f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin(ax) dx$.

Hence, for the function above we get

$$g_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^1 x \sin(ax) dx$$

Notice, we integrate from zero, not from -1, since the sin transform is defined only for positive x . Integrating by parts, $u = x, v = \frac{-\cos(ax)}{\alpha}$, hence $\int u dv = uv - \int du v$

$$\begin{aligned} g_s(\alpha) &= \sqrt{\frac{2}{\pi}} \left\{ \left[x \left(\frac{-\cos(ax)}{\alpha} \right) \right]_0^1 - \int_0^1 \frac{-\cos(ax)}{\alpha} dx \right\} \\ &= \sqrt{\frac{2}{\pi}} \left\{ \frac{-1}{\alpha} [x \cos(ax)]_0^1 + \frac{1}{\alpha} \int_0^1 \cos(ax) dx \right\} \\ &= \sqrt{\frac{2}{\pi}} \left\{ \frac{-1}{\alpha} [x \cos(ax)]_0^1 + \frac{1}{\alpha} \left[\frac{\sin(ax)}{\alpha} \right]_0^1 \right\} \\ &= \sqrt{\frac{2}{\pi}} \frac{1}{\alpha} \left\{ -[\cos(\alpha) - 0] + \frac{1}{\alpha} [\sin(ax)]_0^1 \right\} \\ &= \sqrt{\frac{2}{\pi}} \frac{1}{\alpha} \left\{ -\cos(\alpha) + \frac{1}{\alpha} [\sin(\alpha) - 0] \right\} \\ &= \sqrt{\frac{2}{\pi}} \frac{1}{\alpha} \left(-\cos(\alpha) + \frac{1}{\alpha} \sin(\alpha) \right) \\ &= \sqrt{\frac{2}{\pi}} \frac{1}{\alpha} \left(\frac{1}{\alpha} \sin(\alpha) - \cos(\alpha) \right) \end{aligned}$$

Hence the Sin Fourier transform of $f(x)$ is

$$g_s(\alpha) = \sqrt{\frac{2}{\pi}} \frac{1}{\alpha} \left(\frac{1}{\alpha} \sin(\alpha) - \cos(\alpha) \right)$$

To obtain $f(x)$ given its sin fourier transform $g_s(\alpha)$, then we apply the inverse sin fourier transform

$$\begin{aligned} f_s(x) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} g_s(\alpha) \sin ax d\alpha \\ &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sqrt{\frac{2}{\pi}} \frac{1}{\alpha} \left(\frac{1}{\alpha} \sin \alpha - \cos \alpha \right) \sin ax d\alpha \\ &= \frac{2}{\pi} \int_0^{\infty} \left(\frac{1}{\alpha^2} \sin \alpha - \frac{1}{\alpha} \cos \alpha \right) \sin ax d\alpha \\ &= \frac{2}{\pi} \int_0^{\infty} \left(\frac{\sin \alpha - \alpha \cos \alpha}{\alpha^2} \right) \sin ax d\alpha \end{aligned} \tag{A0}$$

Now we need to show that the above is the same as the inverse fourier transform found for problem 6. From back of the book, the IFT for problem 6 is given as

$$f(x) = \int_{-\infty}^{\infty} \frac{\sin \alpha - \alpha \cos \alpha}{i\pi\alpha^2} e^{i\alpha x} d\alpha$$

Need to convert the above to $f_s(x)$. Since $e^{i\alpha x} = \cos \alpha x + i \sin \alpha x$

$$\begin{aligned}
f(x) &= \int_{-\infty}^{\infty} \frac{\sin \alpha - \alpha \cos \alpha}{i\pi\alpha^2} (\cos \alpha x + i \sin \alpha x) d\alpha \\
&= \int_{-\infty}^{\infty} \frac{(\sin \alpha - \alpha \cos \alpha) \cos \alpha x}{i\pi\alpha^2} + \frac{(\sin \alpha - \alpha \cos \alpha) i \sin \alpha x}{i\pi\alpha^2} d\alpha \\
&= \int_{-\infty}^{\infty} \frac{(\sin \alpha - \alpha \cos \alpha) \cos \alpha x}{i\pi\alpha^2} d\alpha + \int_{-\infty}^{\infty} \frac{(\sin \alpha - \alpha \cos \alpha) i \sin \alpha x}{i\pi\alpha^2} d\alpha
\end{aligned} \tag{1}$$

Looking at the first integral,

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{\left(\overbrace{\sin \alpha}^{\text{odd}} - \overbrace{\alpha}^{\text{odd}} \overbrace{\cos \alpha}^{\text{even}} \right) \overbrace{\cos \alpha x}^{\text{even}}}{\overbrace{i\pi \alpha^2}^{\text{even}}} d\alpha \\
&= \int_{-\infty}^{\infty} \frac{(\text{odd} - \text{odd} \times \text{even}) \times \text{even}}{\text{even}} d\alpha \\
&= \int_{-\infty}^{\infty} \frac{(\text{odd} - \text{odd}) \times \text{even}}{\text{even}} d\alpha \\
&= \int_{-\infty}^{\infty} \frac{\text{odd} \times \text{even}}{\text{even}} d\alpha \\
&= \int_{-\infty}^{\infty} \frac{\text{odd}}{\text{even}} d\alpha \\
&= \int_{-\infty}^{\infty} \text{odd} d\alpha
\end{aligned}$$

Hence the integral vanishes. Hence (1) becomes

$$f(x) = \int_{-\infty}^{\infty} \frac{(\sin \alpha - \alpha \cos \alpha) i \sin \alpha x}{i\pi\alpha^2} d\alpha \tag{2}$$

Looking at the above

$$\begin{aligned}
\frac{\left(\overbrace{\sin \alpha}^{\text{odd}} - \overbrace{\alpha}^{\text{odd}} \overbrace{\cos \alpha}^{\text{even}} \right) \overbrace{i \sin \alpha x}^{\text{odd}}}{\overbrace{i\pi \alpha^2}^{\text{even}}} &= \frac{(\text{odd} - \text{odd} \times \text{even}) \times \text{odd}}{\text{even}} \\
&= \frac{\text{odd} \times \text{odd}}{\text{even}} \\
&= \frac{\text{even}}{\text{even}} \\
&= \text{even}
\end{aligned}$$

Since the integrand is even, then $\int_{-\infty}^{\infty} = 2 \int_0^{\infty}$ Hence (2) becomes

$$\begin{aligned}
f(x) &= 2 \int_0^{\infty} \frac{(\sin \alpha - \alpha \cos \alpha) i \sin \alpha x}{i\pi\alpha^2} d\alpha \\
&= \frac{2}{\pi} \int_0^{\infty} \frac{(\sin \alpha - \alpha \cos \alpha)}{\alpha^2} \sin \alpha x d\alpha
\end{aligned} \tag{3}$$

comparing this to equation (A1) above, we see that

$$f_s(x) = f(x)$$

Which is what we are asked to show.

3 chapter 15, problem 4.20

Problem Find the fourier sin transform of the given $f(x)$ and write $f(x)$ as a fourier integral. Verify the answer is the same as the exponential fourier transform.

$$f(x) = \begin{cases} \sin x & |x| < \frac{\pi}{2} \\ 0, & |x| > \frac{\pi}{2} \end{cases}$$

Solution

Let $\mathcal{F}_s f(x)$ be the Fourier sin transform of $f(x)$ defined as $g_s(\alpha) = \mathcal{F}_s f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin(\alpha x) dx$.

Hence, for the function above we get

$$g_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^1 \sin x \sin(\alpha x) dx$$

Notice, we integrate from zero, not from -1, since the sin transform is defined only for positive x . Since $\sin \beta \sin \gamma = \frac{1}{2} \cos(\beta - \gamma) - \frac{1}{2} \cos(\beta + \gamma)$ Then $\sin x \sin(\alpha x) = \frac{1}{2} \cos(x - \alpha x) - \frac{1}{2} \cos(x + \alpha x)$. Hence

$$\begin{aligned} g_s(\alpha) &= \sqrt{\frac{2}{\pi}} \int_0^1 \frac{1}{2} \cos(x - \alpha x) - \frac{1}{2} \cos(x + \alpha x) dx \\ &= \frac{1}{2} \sqrt{\frac{2}{\pi}} \left\{ \left[\frac{\sin(x - \alpha x)}{1 - \alpha} \right]_0^1 - \left[\frac{\sin(x + \alpha x)}{1 + \alpha} \right]_0^1 \right\} \\ &= \frac{1}{2} \sqrt{\frac{2}{\pi}} \left\{ \left[\frac{\sin(1 - \alpha)}{1 - \alpha} - 0 \right] - \left[\frac{\sin(1 + \alpha)}{1 + \alpha} - 0 \right] \right\} \\ &= \frac{1}{2} \sqrt{\frac{2}{\pi}} \left(\frac{\sin(1 - \alpha)}{1 - \alpha} - \frac{\sin(1 + \alpha)}{1 + \alpha} \right) \end{aligned}$$

Hence the Sin Fourier transform of $f(x)$ is

$$g_s(\alpha) = \frac{1}{2} \sqrt{\frac{2}{\pi}} \left(\frac{\sin(1 - \alpha)}{1 - \alpha} - \frac{\sin(1 + \alpha)}{1 + \alpha} \right)$$

Therefore, to obtain $f(x)$ given its sin fourier transform $g_s(\alpha)$, we apply the inverse sin fourier transform

$$\begin{aligned} f_s(x) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} g_s(\alpha) \sin \alpha x d\alpha \\ &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{1}{2} \sqrt{\frac{2}{\pi}} \left(\frac{\sin(1 - \alpha)}{1 - \alpha} - \frac{\sin(1 + \alpha)}{1 + \alpha} \right) \sin \alpha x d\alpha \\ &= \frac{1}{\pi} \int_0^{\infty} \left(\frac{\sin(1 - \alpha)}{1 - \alpha} - \frac{\sin(1 + \alpha)}{1 + \alpha} \right) \sin \alpha x d\alpha \end{aligned} \tag{1}$$

Now we need to show that the above is the same as the exponential inverse fourier transform found for problem 12. The exponential IFT for problem 12 is

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\alpha i \cos\left(\alpha \frac{\pi}{2}\right)}{\alpha^2 - 1} e^{i\alpha x} d\alpha \tag{2}$$

So Need to show that (1) and (2) are the same. Need to convert the above (1) to $f_s(x)$ in (2). Since $e^{iax} = \cos ax + i \sin ax$, (2) can be written as

$$\begin{aligned} f(x) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\alpha i \cos\left(\alpha \frac{\pi}{2}\right)}{\alpha^2 - 1} (\cos ax + i \sin ax) d\alpha \\ &= \frac{1}{\pi} \left[\int_{-\infty}^{\infty} \frac{\alpha i \cos\left(\alpha \frac{\pi}{2}\right)}{\alpha^2 - 1} \cos ax \, d\alpha + \int_{-\infty}^{\infty} \frac{\alpha i \cos\left(\alpha \frac{\pi}{2}\right)}{\alpha^2 - 1} i \sin ax \, d\alpha \right] \end{aligned} \quad (3)$$

Looking at the first integral,

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{\overbrace{\alpha}^{\text{odd}} \overbrace{i \cos\left(\alpha \frac{\pi}{2}\right)}^{\text{even}}}{\underbrace{\alpha^2 - 1}_{\text{even}}} \overbrace{\cos ax}^{\text{even}} \, d\alpha \\ &= \int_{-\infty}^{\infty} \frac{(\text{odd} \times \text{even}) \times \text{even}}{\text{even}} \, d\alpha \\ &= \int_{-\infty}^{\infty} \frac{\text{odd} \times \text{even}}{\text{even}} \, d\alpha \\ &= \int_{-\infty}^{\infty} \frac{\text{odd}}{\text{even}} \, d\alpha \\ &= \int_{-\infty}^{\infty} \text{odd} \, d\alpha \end{aligned}$$

Hence the integral vanishes. So (3) becomes

$$\begin{aligned} f(x) &= 0 + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\alpha i \cos\left(\alpha \frac{\pi}{2}\right)}{\alpha^2 - 1} i \sin ax \, d\alpha \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\alpha i \cos\left(\alpha \frac{\pi}{2}\right)}{\alpha^2 - 1} i \sin ax \, d\alpha \end{aligned}$$

Looking at the above integrand,

$$\begin{aligned} \frac{\overbrace{\alpha}^{\text{odd}} \overbrace{i \cos\left(\alpha \frac{\pi}{2}\right)}^{\text{even}}}{\underbrace{\alpha^2 - 1}_{\text{even}}} \overbrace{i \sin ax}^{\text{odd}} &= \frac{(\text{odd} \times \text{even}) \times \text{odd}}{\text{even}} \\ &= \frac{\text{odd} \times \text{odd}}{\text{even}} \\ &= \frac{\text{even}}{\text{even}} \\ &= \text{even} \end{aligned}$$

Since the integrand is even, then $\int_{-\infty}^{\infty} = 2 \int_0^{\infty}$. Hence (2) becomes

$$\begin{aligned} f(x) &= \frac{2}{\pi} \int_0^{\infty} \frac{\alpha i \cos\left(\alpha \frac{\pi}{2}\right)}{\alpha^2 - 1} i \sin ax \, d\alpha \\ &= \frac{-2}{\pi} \int_0^{\infty} \frac{\alpha \cos\left(\alpha \frac{\pi}{2}\right)}{\alpha^2 - 1} \sin ax \, d\alpha \\ &= \frac{2}{\pi} \int_0^{\infty} \frac{\alpha \cos\left(\alpha \frac{\pi}{2}\right)}{1 - \alpha^2} \sin ax \, d\alpha \end{aligned}$$

But $1 - \alpha^2 = (1 + \alpha)(1 - \alpha)$, therefore

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{\alpha \cos\left(\alpha \frac{\pi}{2}\right)}{(1+\alpha)(1-\alpha)} \sin \alpha x \, d\alpha$$

But $\cos\left(\alpha \frac{\pi}{2}\right) \sin \alpha x = \frac{1}{2}(-\sin(\alpha \frac{\pi}{2} - \alpha x) + \sin(\alpha \frac{\pi}{2} + \alpha x))$, hence the above becomes

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{\alpha \frac{1}{2}(-\sin(\alpha \frac{\pi}{2} - \alpha x) + \sin(\alpha \frac{\pi}{2} + \alpha x))}{(1+\alpha)(1-\alpha)} \, d\alpha$$

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \frac{\alpha (-\sin(\alpha \frac{\pi}{2} - \alpha x) + \sin(\alpha \frac{\pi}{2} + \alpha x))}{(1+\alpha)(1-\alpha)} \, d\alpha$$

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \frac{\alpha \sin(\alpha \frac{\pi}{2} + \alpha x)}{(1+\alpha)(1-\alpha)} - \frac{\alpha \sin(\alpha \frac{\pi}{2} - \alpha x)}{(1+\alpha)(1-\alpha)} \, d\alpha$$

4 chapter 15, problem 4.21

Problem Find the fourier transform of the given $f(x) = e^{\frac{-x^2}{2\sigma^2}}$

Solution

Let $F(\alpha)$ be the Fourier sin transform of $f(x)$ defined as

$$F(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} \, dx$$

So, for the function above we get

$$\begin{aligned} F(\alpha) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2\sigma^2}} e^{-i\alpha x} \, dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2\sigma^2} - i\alpha x} \, dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\frac{-x^2 - i\alpha x(2\sigma^2)}{2\sigma^2}} \, dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\frac{-x^2 - 2i\alpha\sigma^2 x}{2\sigma^2}} \, dx \end{aligned} \tag{1}$$

looking at the exponent $\frac{-x^2 - 2i\alpha\sigma^2 x}{2\sigma^2}$. completing the square in x gives

$$x^2 + 2i\alpha\sigma^2 x = (x + Z)^2 - Y$$

Solving for Z, Y gives

$$x^2 + 2i\alpha\sigma^2 x = x^2 + 2xZ + Z^2 - Y$$

Therefore $Z = i\alpha\sigma^2, Z^2 - Y = 0$, and $Y = -\alpha^2\sigma^4$. Hence Exponent can be written as

$$\begin{aligned} \frac{x^2 + 2i\alpha\sigma^2 x}{-2\sigma^2} &= \frac{(x + i\alpha\sigma^2)^2 - (-\alpha^2\sigma^4)}{-2\sigma^2} \\ &= \frac{(x + i\alpha\sigma^2)^2 + \alpha^2\sigma^4}{-2\sigma^2} \\ &= \frac{(x + i\alpha\sigma^2)^2}{-2\sigma^2} - \frac{\alpha^2\sigma^4}{2\sigma^2} \\ &= \frac{(x + i\alpha\sigma^2)^2}{-2\sigma^2} - \frac{\alpha^2\sigma^2}{2} \end{aligned}$$

The integral (1) becomes

$$\begin{aligned} F(\alpha) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\frac{(x+i\alpha\sigma^2)^2}{-2\sigma^2} - \frac{\alpha^2\sigma^2}{2}} dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\frac{(x+i\alpha\sigma^2)^2}{-2\sigma^2}} e^{-\frac{\alpha^2\sigma^2}{2}} dx \end{aligned}$$

Moving $e^{-\frac{\alpha^2\sigma^2}{2}}$ outside the integral because it does not depend on x gives

$$F(\alpha) = \frac{e^{-\frac{\alpha^2\sigma^2}{2}}}{2\pi} \int_{-\infty}^{\infty} e^{\frac{(x+i\alpha\sigma^2)^2}{-2\sigma^2}} dx$$

Let $y = x + i\alpha\sigma^2$, $dy = dx$ and the limits do not change. Hence we get

$$F(\alpha) = \frac{e^{-\frac{\alpha^2\sigma^2}{2}}}{2\pi} \int_{-\infty}^{\infty} e^{\frac{y^2}{-2\sigma^2}} dy$$

Since the exponential function is raised to a square power, then we can write $\int_{-\infty}^{\infty} e^{y^2} = 2 \int_0^{\infty} e^{y^2}$ (since even function). Hence above integral becomes

$$F(\alpha) = \frac{e^{-\frac{\alpha^2\sigma^2}{2}}}{\pi} \int_0^{\infty} e^{\frac{y^2}{-2\sigma^2}} dy$$

Let $\zeta = \frac{y}{\sqrt{2}\sigma}$, then $y = \sqrt{2}\sigma\zeta$, and $y^2 = 2\sigma^2\zeta^2$. Hence $d\zeta = \frac{1}{\sqrt{2}\sigma} dy$ and the above integral becomes

$$\begin{aligned} F(\alpha) &= \frac{e^{-\frac{\alpha^2\sigma^2}{2}}}{\pi} \int_0^{\infty} e^{-\zeta^2} \sqrt{2}\sigma d\zeta \\ F(\alpha) &= \sqrt{2}\sigma \frac{e^{-\frac{\alpha^2\sigma^2}{2}}}{\pi} \int_0^{\infty} e^{-\zeta^2} d\zeta \end{aligned} \quad (2)$$

Now from equation 9.5 on page 468

$$\begin{aligned} \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-\zeta^2} d\zeta &= 1 \\ \int_0^{\infty} e^{-\zeta^2} d\zeta &= \frac{\sqrt{\pi}}{2} \end{aligned}$$

Hence (2) becomes

$$\begin{aligned} F(\alpha) &= \sqrt{2}\sigma \frac{e^{-\frac{\alpha^2\sigma^2}{2}}}{\pi} \left(\frac{\sqrt{\pi}}{2} \right) \\ F(\alpha) &= \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{\alpha^2\sigma^2}{2}} \end{aligned}$$

Which is what we are asked to show.

5 chapter 15, problem 4.23

Problem Show that

$$\int_0^{\infty} \frac{1-\cos \pi \alpha}{\alpha} \sin \alpha \, d\alpha = \frac{\pi}{2} \text{ and } \int_0^{\infty} \frac{1-\cos \pi \alpha}{\alpha} \sin \pi \alpha \, d\alpha = \frac{\pi}{4}$$

Solution

From problem 17, the Fourier sin transform for $f(x)$ shown in problem 3 is

$$g_s(\alpha) = \frac{\sqrt{2}(1 - \cos \pi \alpha)}{\sqrt{\pi} \alpha}$$

From equation 4.14 page 651, $f(x)$ can be obtained from inverse sin transform is

$$\begin{aligned} f_s(x) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} g_s(\alpha) \sin \alpha x \, d\alpha \\ &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \left(\frac{\sqrt{2}(1 - \cos \pi \alpha)}{\sqrt{\pi} \alpha} \right) \sin \alpha x \, d\alpha \\ &= \frac{2}{\pi} \int_0^{\infty} \frac{(1 - \cos \pi \alpha)}{\alpha} \sin \alpha x \, d\alpha \end{aligned} \quad (1)$$

Now, from the definition of $f(x)$, which is

$$f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \\ 0, & |x| > \pi \end{cases}$$

We see that for $x = 1$, $f(x) = 1$, hence substitute in (1) we get

$$\begin{aligned} 1 &= \frac{2}{\pi} \int_0^{\infty} \frac{(1 - \cos \pi \alpha)}{\alpha} \sin \alpha \, d\alpha \\ \frac{\pi}{2} &= \int_0^{\infty} \frac{(1 - \cos \pi \alpha)}{\alpha} \sin \alpha \, d\alpha \end{aligned}$$

Which is the first result we required to show. For the second result, let $x = \pi$ hence $f(\pi) =$ average value of $f(x)$ at $x = \pi$. Which is given by $\frac{f(\pi_-) + f(\pi_+)}{2} = \frac{1+0}{2} = \frac{1}{2}$. Hence substitute in (1) we get

$$\begin{aligned} f_s(x = \pi) &= \frac{1}{2} = \frac{2}{\pi} \int_0^{\infty} \frac{(1 - \cos \pi \alpha)}{\alpha} \sin \alpha \pi \, d\alpha \\ \frac{\pi}{4} &= \int_0^{\infty} \frac{(1 - \cos \pi \alpha)}{\alpha} \sin \alpha \pi \, d\alpha \end{aligned}$$

Which is the second result we are asked to show.

6 chapter 15, problem 4.25

Problem Show that

(a) represent as an exponential fourier transform the function

$$f(x) = \begin{cases} \sin x & 0 < x < \pi \\ 0, & \text{otherwise} \end{cases}$$

(b) Show that the result can be written as

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \frac{\cos \alpha x + \cos \alpha(x - \pi)}{1 - \alpha^2} d\alpha$$

Solution

The exponential Fourier transform is defined as

$$g(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx$$

Applying the function $f(x)$ gives

$$g(\alpha) = \frac{1}{2\pi} \int_0^{\pi} \sin x e^{-i\alpha x} dx$$

But

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

Hence the transform can be written as

$$\begin{aligned} g(\alpha) &= \frac{1}{2\pi} \int_0^{\pi} \frac{e^{ix} - e^{-ix}}{2i} e^{-i\alpha x} dx \\ &= \frac{1}{4i\pi} \int_0^{\pi} e^{ix-i\alpha x} - e^{-ix-i\alpha x} dx \\ &= \frac{1}{4i\pi} \int_0^{\pi} e^{x(i-\alpha)} - e^{x(-i-\alpha)} dx \\ &= \frac{1}{4i\pi} \left(\left[\frac{e^{x(i-\alpha)}}{i-\alpha} \right]_0^{\pi} - \left[\frac{e^{x(-i-\alpha)}}{-i-\alpha} \right]_0^{\pi} \right) \\ &= \frac{1}{4i\pi} \left(\left[\frac{e^{\pi(i-\alpha)}}{i-\alpha} - \frac{1}{i-\alpha} \right] - \left[\frac{e^{\pi(-i-\alpha)}}{-i-\alpha} - \frac{1}{-i-\alpha} \right] \right) \\ &= \frac{1}{4i\pi} \left(\frac{1}{i-\alpha} [e^{\pi(i-\alpha)} - 1] - \frac{1}{-i-\alpha} [e^{\pi(-i-\alpha)} - 1] \right) \\ &= \frac{1}{4i\pi} \left(\frac{1}{i-\alpha} [e^{\pi(i-\alpha)} - 1] + \frac{1}{i+\alpha} [e^{\pi(-i-\alpha)} - 1] \right) \\ &= \frac{1}{i} \frac{1}{4i\pi} \left(\frac{1}{1-\alpha} [e^{\pi(i-\alpha)} - 1] + \frac{1}{1+\alpha} [e^{\pi(-i-\alpha)} - 1] \right) \\ &= \frac{-1}{4\pi} \left(\frac{e^{\pi(i-\alpha)}}{1-\alpha} - \frac{1}{1-\alpha} + \frac{e^{\pi(-i-\alpha)}}{1+\alpha} - \frac{1}{1+\alpha} \right) \\ &= \frac{-1}{4\pi} \left(\frac{(1+\alpha)e^{\pi(i-\alpha)} + (1-\alpha)e^{\pi(-i-\alpha)}}{(1-\alpha)(1+\alpha)} - \frac{(1+\alpha) + (1-\alpha)}{(1-\alpha)(1+\alpha)} \right) \\ &= \frac{-1}{4\pi} \left(\frac{(1+\alpha)e^{\pi(i-\alpha)} + (1-\alpha)e^{\pi(-i-\alpha)}}{(1-\alpha^2)} - \frac{2}{(1-\alpha^2)} \right) \\ &= \frac{-1}{(1-\alpha^2)4\pi} (e^{\pi(i-\alpha)} + \alpha e^{\pi(i-\alpha)} + e^{\pi(-i-\alpha)} - \alpha e^{\pi(-i-\alpha)} - 2) \\ &= \frac{-1}{(1-\alpha^2)4\pi} (e^{\pi i} e^{-i\pi\alpha} + \alpha e^{\pi i} e^{-i\pi\alpha} + e^{-\pi i} e^{-i\pi\alpha} - \alpha e^{-\pi i} e^{-i\pi\alpha} - 2) \end{aligned}$$

But $e^{\pi i} = -1$ and $e^{-\pi i} = -1$

$$\begin{aligned}
g(\alpha) &= \frac{-1}{(1-\alpha^2)4\pi} \left(-e^{-i\pi\alpha} - \overbrace{\alpha e^{-i\pi\alpha}} - e^{-i\pi\alpha} + \overbrace{\alpha e^{-i\pi\alpha}} - 2 \right) \\
g(\alpha) &= \frac{-1}{(1-\alpha^2)4\pi} \left(-e^{-i\pi\alpha} - e^{-i\pi\alpha} - 2 \right) \\
g(\alpha) &= \frac{1}{(1-\alpha^2)4\pi} \left(e^{-i\pi\alpha} + e^{-i\pi\alpha} + 2 \right) \\
g(\alpha) &= \frac{2e^{-i\pi\alpha} + 2}{(1-\alpha^2)4\pi}
\end{aligned}$$

Hence the exponential fourier transform is

$$g(\alpha) = \frac{e^{-i\pi\alpha} + 1}{(1-\alpha^2)2\pi}$$

Therefore $f(x)$ can be rewritten as

$$\begin{aligned}
f(x) &= \int_{-\infty}^{\infty} g(\alpha) e^{i\alpha x} d\alpha \\
&= \int_{-\infty}^{\infty} \frac{e^{-i\pi\alpha} + 1}{(1-\alpha^2)2\pi} e^{i\alpha x} d\alpha \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 + e^{-i\pi\alpha}}{(1-\alpha^2)} e^{i\alpha x} d\alpha \tag{1}
\end{aligned}$$

Which is the answer required to show.

Part(b)

Now need to show that the above can be written as

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \frac{\cos \alpha x + \cos \alpha(x - \pi)}{1 - \alpha^2} d\alpha$$

From (1)

$$\begin{aligned}
f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 + e^{-i\pi\alpha}}{(1-\alpha^2)} e^{i\alpha x} d\alpha \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\alpha x} + e^{i\alpha x} e^{-i\pi\alpha}}{(1-\alpha^2)} d\alpha \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\alpha x} + e^{i\alpha x - i\pi\alpha}}{(1-\alpha^2)} d\alpha \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\alpha x} + e^{i\alpha(x-\pi)}}{(1-\alpha^2)} d\alpha \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2(\cos \alpha x + \cos \alpha(x - \pi))}{(1-\alpha^2)} d\alpha \\
&= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos \alpha x + \cos \alpha(x - \pi)}{(1-\alpha^2)} d\alpha
\end{aligned}$$

Which is what is required to show.

7 chapter 15, problem 4.3

Problem Find the exponential fourier transform of the given $f(x)$ and write $f(x)$ as a fourier integral.

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \\ 0, & |x| > \pi \end{cases}$$

Solution

Let $F(\alpha)$ be the Fourier transform of $f(x)$ defined as $F(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx$, hence

$$\begin{aligned} F(\alpha) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx \\ 2\pi F(\alpha) &= \int_{-\pi}^0 -e^{-i\alpha x} dx + \int_0^{\pi} e^{-i\alpha x} dx \\ &= -\left[\frac{e^{-i\alpha x}}{-i\alpha} \right]_{-\pi}^0 + \left[\frac{e^{-i\alpha x}}{-i\alpha} \right]_0^{\pi} \\ &= \frac{1}{i\alpha} [e^{-i\alpha x}]_{-\pi}^0 - \frac{1}{i\alpha} [e^{-i\alpha x}]_0^{\pi} \\ &= \frac{1}{i\alpha} [e^0 - e^{i\alpha\pi}] - \frac{1}{i\alpha} [e^{-i\alpha\pi} - e^0] \\ &= \frac{1}{i\alpha} [1 - e^{i\alpha\pi}] - \frac{1}{i\alpha} [e^{-i\alpha\pi} - 1] \\ &= \frac{1}{i\alpha} - \frac{e^{i\alpha\pi}}{i\alpha} - \frac{e^{-i\alpha\pi}}{i\alpha} + \frac{1}{i\alpha} \\ &= \frac{2}{i\alpha} - \frac{1}{i\alpha} (e^{i\alpha\pi} + e^{-i\alpha\pi}) \end{aligned}$$

But $e^{i\alpha\pi} + e^{-i\alpha\pi} = 2 \cos \alpha\pi$. Hence

$$\begin{aligned} 2\pi F(\alpha) &= \frac{2}{i\alpha} - \frac{1}{i\alpha} (2 \cos \alpha\pi) \\ &= \frac{2}{i\alpha} (1 - \cos \alpha\pi) \end{aligned}$$

Hence the Fourier transform of $f(x)$ is

$$\begin{aligned} F(\alpha) &= \frac{1}{2\pi} \left[\frac{2}{i\alpha} (1 - \cos \alpha\pi) \right] \\ &= \frac{1}{\pi\alpha i} (1 - \cos \alpha\pi) \end{aligned}$$

To obtain $f(x)$ given its fourier transform $F(\alpha)$, then we apply the inverse fourier transform

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} F(\alpha) e^{i\alpha x} d\alpha \\ &= \int_{-\infty}^{\infty} \frac{1}{\pi\alpha i} (1 - \cos \alpha\pi) e^{i\alpha x} d\alpha \\ &= \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{1}{\alpha} (1 - \cos \alpha\pi) e^{i\alpha x} d\alpha \end{aligned}$$

8 chapter 15, problem 4.5

Problem Find the exponential fourier transform of the given $f(x)$ and write $f(x)$ as a fourier integral.

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Solution

Let $F(s)$ be the Fourier transform of $f(x)$ defined as $F(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-isx} dx$

$$\begin{aligned} F(s) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-isx} dx \\ 2\pi F(s) &= \int_0^1 e^{-isx} dx \\ &= \left[\frac{e^{-isx}}{-is} \right]_0^1 \\ &= \frac{-1}{is} \left[e^{-isx} \right]_0^1 \\ &= \frac{-1}{is} \left[e^{-is} - e^0 \right] \\ &= \frac{-1}{is} \left[e^{-is} - 1 \right] \end{aligned}$$

Hence the Fourier transform of $f(x)$ is

$$\begin{aligned} F(s) &= \frac{1}{2\pi} \left[\frac{-1}{is} \left[e^{-is} - 1 \right] \right] \\ &= \frac{i}{2\pi s} (e^{-is} - 1) \end{aligned}$$

To obtain $f(x)$ given its fourier transform $F(s)$, then we apply the inverse fourier transform

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} F(s) e^{isx} ds \\ &= \int_{-\infty}^{\infty} \frac{i}{2\pi s} (e^{-is} - 1) e^{isx} ds \\ &= \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{1}{s} (e^{-is} - 1) e^{isx} ds \end{aligned}$$

9 chapter 15, problem 4.7

Problem Find the exponential fourier transform of the given $f(x)$ and write $f(x)$ as a fourier integral.

$$f(x) = \begin{cases} |x| & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

Solution

Let $F(\alpha)$ be the Fourier transform of $f(x)$ defined as $F(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx$

$$\begin{aligned} F(\alpha) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx \\ 2\pi F(\alpha) &= \int_{-1}^0 -x e^{-i\alpha x} dx + \int_0^1 x e^{-i\alpha x} dx \end{aligned}$$

Integrating by parts, $u = x, v = \frac{e^{-i\alpha x}}{-i\alpha}$, hence $\int u dv = uv - \int du v$. The first integral is

$$\begin{aligned}
\int_{-1}^0 -x e^{-iax} dx &= \left[(-x) \frac{e^{-iax}}{-ia} \right]_{-1}^0 + \int_{-1}^0 \frac{e^{-iax}}{-ia} dx \\
&= \left[(x) \frac{e^{-iax}}{ia} \right]_{-1}^0 + \int_{-1}^0 \frac{e^{-iax}}{-ia} dx \\
&= \frac{1}{ia} [0 - (-1) \times e^{ia}] - \frac{1}{ia} \int_{-1}^0 e^{-iax} dx \\
&= \frac{1}{ia} [e^{ia}] - \frac{1}{ia} \left[\frac{e^{-iax}}{-ia} \right]_{-1}^0 \\
&= \frac{1}{ia} [e^{ia}] + \frac{1}{i^2 \alpha^2} [e^{-iax}]_{-1}^0 \\
&= \frac{1}{ia} [e^{ia}] - \frac{1}{\alpha^2} [1 - e^{ia}] \\
&= \frac{e^{ia}}{ia} - \frac{1}{\alpha^2} + \frac{e^{ia}}{\alpha^2}
\end{aligned}$$

And the second integral

$$\begin{aligned}
\int_0^1 x e^{-iax} dx &= \left[x \frac{e^{-iax}}{-ia} \right]_0^1 - \int_0^1 \frac{e^{-iax}}{-ia} dx \\
&= \frac{1}{-ia} [1 \times e^{-ia} - 0] + \frac{1}{ia} \int_0^1 e^{-iax} dx \\
&= \frac{1}{-ia} [e^{-ia}] + \frac{1}{ia} \left[\frac{e^{-iax}}{-ia} \right]_0^1 \\
&= \frac{1}{-ia} [e^{-ia}] - \frac{1}{i^2 \alpha^2} [e^{-iax}]_0^1 \\
&= \frac{1}{-ia} [e^{-ia}] + \frac{1}{\alpha^2} [e^{-ia} - 1] \\
&= \frac{e^{-ia}}{-ia} + \frac{e^{-ia}}{\alpha^2} - \frac{1}{\alpha^2}
\end{aligned}$$

Hence

$$\begin{aligned}
2\pi F(\alpha) &= \left(\frac{e^{ia}}{ia} - \frac{1}{\alpha^2} + \frac{e^{ia}}{\alpha^2} \right) + \left(\frac{e^{-ia}}{-ia} + \frac{e^{-ia}}{\alpha^2} - \frac{1}{\alpha^2} \right) \\
2\pi F(\alpha) &= \frac{e^{ia}}{ia} - \frac{1}{\alpha^2} + \frac{e^{ia}}{\alpha^2} - \frac{e^{-ia}}{ia} + \frac{e^{-ia}}{\alpha^2} - \frac{1}{\alpha^2} \\
&= \frac{e^{ia}}{ia} + \frac{e^{ia}}{\alpha^2} - \frac{e^{-ia}}{ia} + \frac{e^{-ia}}{\alpha^2} - \frac{2}{\alpha^2} \\
&= \frac{1}{\alpha} \left(\frac{e^{ia}}{i} - \frac{e^{-ia}}{i} \right) + \frac{1}{\alpha^2} (e^{ia} + e^{-ia}) - \frac{2}{\alpha^2} \\
&= \frac{1}{\alpha} \left(\frac{e^{ia} - e^{-ia}}{i} \right) + \frac{1}{\alpha^2} (e^{ia} + e^{-ia}) - \frac{2}{\alpha^2}
\end{aligned}$$

But $e^{ia} + e^{-ia} = 2 \cos \alpha$ and $\frac{e^{ia} - e^{-ia}}{i} = 2 \sin \alpha$, Hence the above becomes

$$\begin{aligned}
2\pi F(\alpha) &= \frac{1}{\alpha} (2 \sin \alpha) + \frac{1}{\alpha^2} (2 \cos \alpha) - \frac{2}{\alpha^2} \\
&= \frac{2}{\alpha^2} [\alpha \sin \alpha + \cos \alpha - 1] \\
F(\alpha) &= \frac{1}{\pi \alpha^2} [\alpha \sin \alpha + \cos \alpha - 1]
\end{aligned}$$

Hence the Fourier transform of $f(x)$ is

$$F(\alpha) = \frac{1}{\pi\alpha^2} [\alpha \sin \alpha + \cos \alpha - 1]$$

To obtain $f(x)$ given its fourier transform $F(\alpha)$, then we apply the inverse fourier transform

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} F(\alpha) e^{i\alpha x} d\alpha \\ &= \int_{-\infty}^{\infty} \frac{1}{\pi\alpha^2} [\alpha \sin \alpha + \cos \alpha - 1] e^{i\alpha x} d\alpha \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{\alpha^2} [\alpha \sin \alpha + \cos \alpha - 1] e^{i\alpha x} d\alpha \end{aligned}$$

10 chapter 15, problem 5.1

Problem Show that $g(t) \otimes h(t) = h(t) \otimes g(t)$

Solution

By definition,

$$g(t) \otimes h(t) = \int_0^t g(t - \tau) h(\tau) d\tau \quad (1)$$

Let $u = t - \tau$, $du = -d\tau$, when $\tau = 0$, $u = t$, when $\tau = t$, $u = 0$, Hence The RHS becomes

$$\begin{aligned} \int_0^t g(t - \tau) h(\tau) d\tau &= \int_{u=t}^{u=0} g(u) h(t - u) (-du) \\ &= - \int_{u=t}^{u=0} g(u) h(t - u) du \\ &= \int_{u=0}^{u=t} g(u) h(t - u) du \end{aligned}$$

Since u is a dummy variable of integration, call it anything we want, say τ so above integral becomes

$$\begin{aligned} \int_0^t g(t - \tau) h(\tau) d\tau &= \int_0^t g(\tau) h(t - \tau) d\tau \\ \int_0^t g(t - \tau) h(\tau) d\tau &= \int_0^t h(t - \tau) g(\tau) d\tau \end{aligned} \quad (2)$$

Hence from (2) $g(t) \otimes h(t) = h(t) \otimes g(t)$

11 chapter 15, problem 5.10

Problem

Use convolution integral to find the inverse transform of $\frac{1}{p(p^2+a^2)^2}$

Solution

$$\frac{1}{p(p^2+a^2)^2} = \frac{1}{p} \frac{1}{(p^2+a^2)^2} = GH$$

From Tables using L1 and L17 $g(t) = 1$ and $h(t) = \frac{\sin at - at \cos at}{2a^3}$. Hence the inverse transform of $GH = g(t) \otimes h(t)$. Using L34

$$g(t) \otimes h(t) = \int_0^t g(t-\tau) h(\tau) d\tau \quad (\text{L34})$$

Hence

$$\begin{aligned} g(t) \otimes h(t) &= \int_0^t 1 \times \frac{\sin a(t-\tau) - a(t-\tau) \cos a(t-\tau)}{2a^3} d\tau \\ &= \frac{1}{2a^3} \int_0^t \sin a(t-\tau) - a(t-\tau) \cos a(t-\tau) d\tau \\ &= \frac{1}{2a^3} \left[\int_0^t \sin a(t-\tau) d\tau - at \int_0^t \cos a(t-\tau) d\tau + a \int_0^t \tau \cos a(t-\tau) d\tau \right] \quad (1) \end{aligned}$$

The last integral can be integrated by parts. $u = \tau, v = \frac{\sin a(t-\tau)}{-a}$

$$\begin{aligned} \int_0^t \tau \cos a(t-\tau) d\tau &= \left[\tau \frac{\sin a(t-\tau)}{-a} \right]_0^t - \int_0^t \frac{\sin a(t-\tau)}{-a} d\tau \\ &= \frac{-1}{a} [\tau \sin a(t-\tau)]_0^t + \frac{1}{a} \int_0^t \sin a(t-\tau) d\tau \\ &= \frac{-1}{a} [\tau \sin a(t-\tau)]_0^t + \frac{1}{a} \left[\frac{-\cos a(t-\tau)}{-a} \right]_0^t \\ &= \frac{-1}{a} [\tau \sin a(t-\tau)]_0^t + \frac{1}{a^2} [\cos a(t-\tau)]_0^t \\ &= \frac{-1}{a} [t \sin a(t-t) - 0] + \frac{1}{a^2} [\cos a(t-t) - \cos a(t-0)] \\ &= \frac{-1}{a} [0] + \frac{1}{a^2} [\cos a(0) - \cos a(t)] \\ &= \frac{1}{a^2} [1 - \cos at] \end{aligned}$$

Hence (1) becomes

$$\begin{aligned} g(t) \otimes h(t) &= \frac{1}{2a^3} \left[\int_0^t \sin a(t-\tau) d\tau - at \int_0^t \cos a(t-\tau) d\tau + a \frac{1}{a^2} [1 - \cos at] \right] \\ g(t) \otimes h(t) &= \frac{1}{2a^3} \left[\left[\frac{-\cos a(t-\tau)}{-a\tau} \right]_0^t - at \left[\frac{\sin a(t-\tau)}{-a\tau} \right] + \frac{1}{a} [1 - \cos at] \right] \\ &= \frac{1}{2a^3} \left[\frac{1}{a} [\cos a(t-\tau)]_0^t + t [\sin a(t-\tau)]_0^t + \frac{1}{a} [1 - \cos at] \right] \\ &= \frac{1}{2a^3} \left[\frac{1}{a} [\cos a(t-t) - \cos a(t-0)] + t [\sin a(t-t) - \sin a(t-0)]_0^t + \frac{1}{a} [1 - \cos at] \right] \\ &= \frac{1}{2a^3} \left[\frac{1}{a} [1 - \cos at] + t [-\sin at] + \frac{1}{a} [1 - \cos at] \right] \\ &= \frac{1}{2a^3} \left[\frac{1}{a} [1 - \cos at] - t \sin at + \frac{1}{a} [1 - \cos at] \right] \\ &= \frac{1}{2a^4} (2 - 2 \cos at - at \sin at) \end{aligned}$$

So the inverse Laplace transform of $\frac{1}{p(p^2+a^2)^2}$ is

$$\frac{1}{2a^4} (2 - 2 \cos at - at \sin at)$$

12 chapter 15, problem 5.2

Problem Use L34 and L2 to find the inverse transform of $G(p)H(p)$ when $G(p) = \frac{1}{(p+a)}$ and $H(p) = \frac{1}{(p+b)}$ your result should be L7

Solution

$$\mathcal{L}(e^{-at}) = \frac{1}{p+a} \quad (\text{L2})$$

$$g(t) \otimes h(t) = \int_0^t g(t-\tau) h(\tau) d\tau \quad (\text{L34})$$

Using L2, $g(t) = \mathcal{L}^{-1} \frac{1}{(p+a)} = e^{-at}$, and $h(t) = \mathcal{L}^{-1} \frac{1}{(p+b)} = e^{-bt}$. Now Let $Y(p) = G(p)H(p)$, But

$$G(p)H(p) = \mathcal{L}\{g(t) \otimes h(t)\}$$

Then

$$\begin{aligned} y(t) &= g(t) \otimes h(t) = \int_0^t g(t-\tau) h(\tau) d\tau \\ &= \int_0^t e^{-a(t-\tau)} e^{-b\tau} d\tau \\ &= \int_0^t e^{-at+a\tau} e^{-b\tau} d\tau \\ &= \int_0^t e^{-at} e^{a\tau} e^{-b\tau} d\tau \end{aligned}$$

e^{-at} can be moved outside the integral

$$\begin{aligned} y(t) &= e^{-at} \int_0^t e^{a\tau} e^{-b\tau} d\tau \\ y(t) &= e^{-at} \int_0^t e^{a\tau-b\tau} d\tau \\ y(t) &= e^{-at} \int_0^t e^{\tau(a-b)} d\tau \\ y(t) &= e^{-at} \left[\frac{e^{\tau(a-b)}}{a-b} \right]_0^t \\ y(t) &= \frac{e^{-at}}{a-b} [e^{t(a-b)} - 1] \\ y(t) &= \frac{e^{-at+t(a-b)} - e^{-at}}{a-b} \\ y(t) &= \frac{e^{-bt} - e^{-at}}{a-b} \\ y(t) &= \frac{e^{-at} - e^{-bt}}{b-a} \end{aligned}$$

Which is L7 as required to show.

13 chapter 15, problem 5.22

Problem

Verify Parseval's theorem for $f(x) = e^{-|x|}$ and $g(\alpha) = \text{Fourier transform of } f(x)$

Solution

Parseval theorem says that total energy in a signal equal to the sum of the energies in the harmonics that make up the signal. i.e.

$$\int_{-\infty}^{\infty} |g(\alpha)|^2 d\alpha = \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(x)|^2 dx$$

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(x)|^2 dx &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |e^{-|x|}|^2 dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-2|x|} dx \\ &= \frac{1}{2\pi} \left\{ \int_{-\infty}^0 e^{2x} dx + \int_0^{\infty} e^{-2x} dx \right\} \\ &= \frac{1}{2\pi} \left\{ \left[\frac{e^{2x}}{2} \right]_{-\infty}^0 + \left[\frac{e^{-2x}}{-2} \right]_0^{\infty} \right\} \\ &= \frac{1}{4\pi} \left\{ \left[e^{2x} \right]_{-\infty}^0 - \left[e^{-2x} \right]_0^{\infty} \right\} \\ &= \frac{1}{4\pi} \left\{ \left[e^0 - 0 \right] - \left[0 - e^0 \right] \right\} \\ &= \frac{1}{4\pi} \{1 + 1\} \\ &= \frac{1}{2\pi} \end{aligned} \tag{1}$$

Now we find the Fourier transform for $f(x)$

$$\begin{aligned} g(\alpha) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|x|} e^{-i\alpha x} dx \\ &= \frac{1}{2\pi} \left[\int_{-\infty}^0 e^x e^{-i\alpha x} dx + \int_0^{\infty} e^{-x} e^{-i\alpha x} dx \right] \\ &= \frac{1}{2\pi} \left[\int_{-\infty}^0 e^{x(1-i\alpha)} dx + \int_0^{\infty} e^{x(-1-i\alpha)} dx \right] \\ &= \frac{1}{2\pi} \left(\left[\frac{e^{x(1-i\alpha)}}{1-i\alpha} \right]_{-\infty}^0 + \left[\frac{e^{x(-1-i\alpha)}}{-1-i\alpha} \right]_0^{\infty} \right) \\ &= \frac{1}{2\pi} \left(\frac{1}{1-i\alpha} \left[e^{x(1-i\alpha)} \right]_{-\infty}^0 - \frac{1}{1+i\alpha} \left[e^{x(-1-i\alpha)} \right]_0^{\infty} \right) \\ &= \frac{1}{2\pi} \left(\frac{1}{1-i\alpha} \left[e^{x(1-i\alpha)} \right]_{-\infty}^0 - \frac{1}{1+i\alpha} \left[e^{x(-1-i\alpha)} \right]_0^{\infty} \right) \\ &= \frac{1}{2\pi} \left(\frac{1}{1-i\alpha} \left[1 - e^{-\infty(1-i\alpha)} \right] - \frac{1}{1+i\alpha} \left[e^{\infty(-1-i\alpha)} - 1 \right] \right) \\ &= \frac{1}{2\pi} \left(\frac{1}{1-i\alpha} [1] - \frac{1}{1+i\alpha} [-1] \right) \\ &= \frac{1}{2\pi} \left(\frac{1}{1-i\alpha} + \frac{1}{1+i\alpha} \right) \\ &= \frac{1}{2\pi} \left(\frac{1+i\alpha+1-i\alpha}{(1-i\alpha)(1+i\alpha)} \right) \\ &= \frac{1}{2\pi} \left(\frac{2}{1+\alpha^2} \right) \\ &= \frac{1}{\pi(1+\alpha^2)} \end{aligned}$$

So

$$\begin{aligned}\int_{-\infty}^{\infty} |g(\alpha)|^2 d\alpha &= \int_{-\infty}^{\infty} \left| \frac{1}{\pi(1+\alpha^2)} \right|^2 d\alpha \\ &= \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{1}{(1+\alpha^2)^2} d\alpha\end{aligned}$$

But $\int_{-\infty}^{\infty} \frac{1}{(1+\alpha^2)^2} d\alpha = \frac{\pi}{2}$, Hence

$$\begin{aligned}\int_{-\infty}^{\infty} |g(\alpha)|^2 d\alpha &= \frac{1}{\pi^2} \left(\frac{\pi}{2} \right) \\ &= \frac{1}{2\pi}\end{aligned}\tag{2}$$

Comparing (1) and (2). They are the same. Hence

$$\int_{-\infty}^{\infty} |g(\alpha)|^2 d\alpha = \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(x)|^2 dx$$

was verified for this problem as required.

14 chapter 15, problem 5.4

Problem

Use convolution integral to find the inverse transform of $\frac{1}{(p+a)(p+b)^2}$

Solution

$$\frac{1}{(p+a)(p+b)^2} = \frac{1}{(p+a)} \frac{1}{(p+b)^2}$$

From but from L6

$$\mathcal{L}(t^k e^{-at}) = \frac{k!}{(p+a)^{k+1}}\tag{L2}$$

Hence $\frac{1}{(p+a)} = \mathcal{L}(e^{-at})$ and $\frac{1}{(p+b)^2} = \mathcal{L}(te^{-bt})$. Hence the inverse transform of $\frac{1}{(p+a)} \frac{1}{(p+b)^2} = e^{-at} \otimes te^{-bt}$.

Using L34

$$g(t) \otimes h(t) = \int_0^t g(t-\tau) h(\tau) d\tau\tag{L34}$$

Hence

$$\begin{aligned}e^{-at} \otimes te^{-bt} &= \int_0^t e^{-a(t-\tau)} \tau e^{-b\tau} d\tau \\ &= \int_0^t e^{-at+a\tau} \tau e^{-b\tau} d\tau \\ &= \int_0^t e^{-at} e^{a\tau} \tau e^{-b\tau} d\tau \\ &= e^{-at} \int_0^t e^{a\tau} \tau e^{-b\tau} d\tau \\ &= e^{-at} \int_0^t \tau e^{\tau(a-b)} d\tau\end{aligned}$$

Integrate by parts. $u = \tau, v = \frac{e^{\tau(a-b)}}{a-b}$

$$\begin{aligned}
 e^{-at} \otimes te^{-bt} &= e^{-at} \left\{ \left[\tau \frac{e^{\tau(a-b)}}{a-b} \right]_0^t - \int_0^t \frac{e^{\tau(a-b)}}{a-b} d\tau \right\} \\
 &= e^{-at} \left\{ \left[\tau \frac{e^{\tau(a-b)}}{a-b} \right]_0^t - \frac{1}{a-b} \left[\frac{e^{\tau(a-b)}}{a-b} \right]_0^t \right\} \\
 &= e^{-at} \left(\frac{1}{a-b} \left[\tau e^{\tau(a-b)} \right]_0^t - \frac{1}{(a-b)^2} \left[e^{\tau(a-b)} \right]_0^t \right) \\
 &= e^{-at} \left(\frac{1}{a-b} \left[t e^{t(a-b)} \right] - \frac{1}{(a-b)^2} \left[e^{t(a-b)} - 1 \right] \right) \\
 &= e^{-at} \left(\frac{t e^{ta-tb}}{a-b} - \frac{e^{ta-tb} - 1}{(a-b)^2} \right) \\
 &= \frac{t e^{-tb}}{a-b} - \frac{e^{-tb} - e^{-at}}{(a-b)^2} \\
 &= \frac{(a-b)t e^{-tb} - e^{-tb} + e^{-at}}{(a-b)^2} \\
 &= \frac{((a-b)t - 1)e^{-tb} + e^{-at}}{(a-b)^2}
 \end{aligned}$$

So the inverse laplace transform of $\frac{1}{(p+a)(p+b)^2}$ is

$$\frac{((a-b)t - 1)e^{-tb} + e^{-at}}{(a-b)^2}$$

15 chapter 15, problem 6.2

Problem

Find the inverse laplace transform using 6.6 of the function $\frac{1}{p^4-1}$

Solution

6.6 states that $f(t) =$ sum of all residues of $F(z)e^{zt}$ at all poles. Poles of $F(z) = \frac{1}{z^4-1}$ are at $\pm 1, \pm i$. Hence

$$F(z) = \frac{e^{zt}}{(z-1)(z+1)(z-i)(z+i)}$$

Since each pole is of order 1, we use equation 6.1 page 599 which says

$$\text{Residue of } F(z) \text{ at } z = z_0 \text{ is } \lim_{z \rightarrow z_0} (z - z_0)F(z)$$

Hence sum of residue is

$$\begin{aligned}
R &= \lim_{z \rightarrow +1} \frac{e^{zt}}{(z+1)(z-i)(z+i)} + \lim_{z \rightarrow -1} \frac{e^{zt}}{(z-1)(z-i)(z+i)} \\
&+ \lim_{z \rightarrow +i} \frac{e^{zt}}{(z-1)(z+1)(z+i)} + \lim_{z \rightarrow -i} \frac{e^{zt}}{(z-1)(z+1)(z-i)} \\
&= \frac{e^t}{(1+1)(1-i)(1+i)} + \frac{e^{-t}}{(-1-1)(-1-i)(-1+i)} \\
&+ \frac{e^{it}}{(i-1)(i+1)(i+i)} + \frac{e^{-it}}{(-i-1)(-i+1)(-i-i)} \\
&= \frac{e^t}{4} + \frac{e^{-t}}{-4} + \frac{e^{it}}{-4i} + \frac{e^{-it}}{4i} \\
&= \left(\frac{e^t - e^{-t}}{4} \right) - \frac{1}{2} \left(\frac{e^{it} - e^{-it}}{2i} \right) \\
&= \left(\frac{e^t - e^{-t}}{4} \right) - \frac{1}{2} (\sin t)
\end{aligned}$$

16 chapter 15, problem 6.4

Problem

Find the inverse laplace transform using 6.6 of the function $\frac{p^3}{p^4-16}$

Solution

6.6 states that $f(t) =$ sum of all residues of $F(z)e^{zt}$ at all poles. Poles of $F(z) = \frac{z^3}{z^4-16}$ are at $\pm 2, \pm 2i$, Hence

$$F(z) = \frac{z^3 e^{zt}}{(z-2)(z+2)(z-2i)(z+2i)}$$

Since each pole is of order 1, we use equation 6.1 page 599 which says

$$\text{Residue of } F(z) \text{ at } z = z_0 \text{ is } \lim_{z \rightarrow z_0} (z - z_0)F(z)$$

Hence sum of residue is

$$\begin{aligned}
R &= \lim_{z \rightarrow +2} \frac{z^3 e^{zt}}{(z+2)(z-2i)(z+2i)} + \lim_{z \rightarrow -2} \frac{z^3 e^{zt}}{(z-2)(z-2i)(z+2i)} \\
&+ \lim_{z \rightarrow +2i} \frac{z^3 e^{zt}}{(z-2)(z+2)(z+2i)} + \lim_{z \rightarrow -2i} \frac{z^3 e^{zt}}{(z-2)(z+2)(z-2i)} \\
&= \frac{8e^{2t}}{(2+2)(2-2i)(2+2i)} + \frac{-8e^{-2t}}{(-2-2)(-2-2i)(-2+2i)} \\
&+ \frac{(2i)^3 e^{2it}}{(2i-2)(2i+2)(2i+2i)} + \frac{(-2i)^3 e^{-2it}}{(-2i-2)(-2i+2)(-2i-2i)} \\
&= \frac{8e^{2t}}{(4)8} + \frac{-8e^{-2t}}{(-4)8} + \frac{(-8i)e^{2it}}{-8(4i)} + \frac{(8i)e^{-2it}}{-8(-4i)} \\
&= \frac{e^{2t}}{4} + \frac{e^{-2t}}{4} + \frac{e^{2it}}{4} + \frac{e^{-2it}}{4} \\
&= \frac{e^{2t} + e^{-2t}}{4} + \frac{1}{2} (\cos 2t)
\end{aligned}$$

So inverse Laplace transform of $\frac{p^3}{p^4-16}$ is

$$\frac{e^{2t} + e^{-2t}}{4} + \frac{1}{2}(\cos 2t)$$

17 chapter 15, problem 6.5

Problem

Find the inverse laplace transform using 6.6 of the function $\frac{3p^2}{p^3+8}$

Solution

6.6 states that $f(t) = \text{sum of all residues of } F(z)e^{zt}$ at all poles. To find poles, look at $p^3 + 8 = 0$, hence $p^3 = -8, p = -8^{\frac{1}{3}}$. Let

$$8^{\frac{1}{3}} = re^{i\theta}$$

Then the roots are

$$\begin{aligned} &= 8^{\frac{1}{3}} e^{\frac{i0}{3}}, 8^{\frac{1}{3}} e^{\frac{i(0+2\pi)}{3}}, 8^{\frac{1}{3}} e^{\frac{i(0+4\pi)}{3}} \\ &= 8^{\frac{1}{3}}, 8^{\frac{1}{3}} \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right), 8^{\frac{1}{3}} \left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} \right) \\ &= 2, 2(\cos 120^\circ + i \sin 120^\circ), 2(\cos 240^\circ + i \sin 240^\circ) \\ &= 2, 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right), 2\left(-\frac{1}{2} + i\frac{-\sqrt{3}}{2}\right) \\ &= 2, -1 + i\sqrt{3}, -1 - i\sqrt{3} \end{aligned}$$

Hence

$$p = -2, 1 - i\sqrt{3}, 1 + i\sqrt{3}$$

And

$$\begin{aligned} F(z) &= \frac{3z^2}{z^3 + 8} e^{zt} \\ &= \frac{3z^2}{(z+2)(z-(1-i\sqrt{3}))(z-(1+i\sqrt{3}))} e^{zt} \end{aligned}$$

Since each pole is of order 1, we use equation 6.1 page 599 which says

$$\text{Residue of } F(z) \text{ at } z = z_0 \text{ is } \lim_{z \rightarrow z_0} (z - z_0)F(z)$$

Hence sum of residue is

$$\begin{aligned}
R &= \lim_{z \rightarrow -2} \frac{3z^2}{(z - (1 - i\sqrt{3}))(z - (1 + i\sqrt{3}))} e^{zt} + \lim_{z \rightarrow (1 - i\sqrt{3})} \frac{3z^2}{(z + 2)(z - (1 + i\sqrt{3}))} e^{zt} \\
&+ \lim_{z \rightarrow (1 + i\sqrt{3})} \frac{3z^2}{(z + 2)(z - (1 - i\sqrt{3}))} e^{zt} \\
&= \frac{3(-2)^2}{(-2 - (1 - i\sqrt{3}))(-2 - (1 + i\sqrt{3}))} e^{-2t} + \frac{3(1 - i\sqrt{3})^2}{((1 - i\sqrt{3}) + 2)((1 - i\sqrt{3}) - (1 + i\sqrt{3}))} e^{(1 - i\sqrt{3})t} \\
&+ \frac{3(1 + i\sqrt{3})^2}{((1 + i\sqrt{3}) + 2)((1 + i\sqrt{3}) - (1 - i\sqrt{3}))} e^{(1 + i\sqrt{3})t} \\
&= \frac{12}{(-3 + i\sqrt{3})(-3 - i\sqrt{3})} e^{-2t} + \frac{3(-2 - 2i\sqrt{3})}{(3 - i\sqrt{3})(-2i\sqrt{3})} e^{(1 - i\sqrt{3})t} + \frac{3(-2 + 2i\sqrt{3})}{(3 + i\sqrt{3})(2i\sqrt{3})} e^{(1 + i\sqrt{3})t} \\
&= \frac{12}{12} e^{-2t} + \frac{-6 - 6i\sqrt{3}}{-6i\sqrt{3} - 6} e^{(1 - i\sqrt{3})t} + \frac{-6 + 6i\sqrt{3}}{6i\sqrt{3} - 6} e^{(1 + i\sqrt{3})t} \\
&= e^{-2t} + e^{(1 - i\sqrt{3})t} + e^{(1 + i\sqrt{3})t} \\
&= e^{-2t} + e^t e^{-i\sqrt{3}t} + e^t e^{i\sqrt{3}t} \\
&= e^{-2t} + e^t (e^{i\sqrt{3}t} + e^{-i\sqrt{3}t}) \\
&= e^{-2t} + e^t (2 \cos \sqrt{3}t)
\end{aligned}$$

So inverse Laplace transform of $\frac{3p^2}{p^3+8}$ is

$$e^{-2t} + e^t (2 \cos \sqrt{3}t)$$

18 chapter 15, problem 6.9

Problem

Find the inverse laplace transform using 6.6 of the function $\frac{p}{p^4-1}$

Solution

6.6 states that $f(t) =$ sum of all residues of $F(z)e^{zt}$ at all poles. To find poles, look at $p^4 - 1 = 0$, hence $p^4 = 1$, $p = 1^{\frac{1}{4}}$. Let

$$1^{\frac{1}{4}} = r e^{i\theta}$$

Then roots are

$$\begin{aligned}
&= 1^{\frac{1}{4}} e^{\frac{i0}{4}}, 1^{\frac{1}{4}} e^{\frac{i(0+2\pi)}{4}}, 1^{\frac{1}{4}} e^{\frac{i(0+4\pi)}{4}}, 1^{\frac{1}{4}} e^{\frac{i(0+6\pi)}{4}} \\
&= 1, 1 \left(\cos \frac{2\pi}{4} + i \sin \frac{2\pi}{4} \right), 1 \left(\cos \frac{4\pi}{4} + i \sin \frac{4\pi}{4} \right), 1 \left(\cos \frac{6\pi}{4} + i \sin \frac{6\pi}{4} \right) \\
&= 1, (0 + i), (-1 + i0), (0 - i1) \\
&= 1, i, -1, -i
\end{aligned}$$

Therefore

$$p = 1, i, -1, -i$$

Hence

$$\begin{aligned}
F(z) &= \frac{z}{z^4 - 1} e^{zt} \\
&= \frac{z}{(z - 1)(z - i)(z + 1)(z + i)} e^{zt}
\end{aligned}$$

Since each pole is of order 1, we use equation 6.1 page 599 which says

$$\text{Residue of } F(z) \text{ at } z = z_0 \text{ is } \lim_{z \rightarrow z_0} (z - z_0)F(z)$$

Hence sum of residue is

$$\begin{aligned} R &= \lim_{z \rightarrow 1} \frac{z}{(z-i)(z+1)(z+i)} e^{zt} + \lim_{z \rightarrow i} \frac{z}{(z-1)(z+1)(z+i)} e^{zt} \\ &+ \lim_{z \rightarrow -1} \frac{z}{(z-1)(z-i)(z+i)} e^{zt} + \lim_{z \rightarrow -i} \frac{z}{(z-1)(z-i)(z+1)} e^{zt} \\ &= \frac{1}{(1-i)(1+1)(1+i)} e^t + \frac{i}{(i-1)(i+1)(i+i)} e^{it} \\ &+ \frac{-1}{(-1-1)(-1-i)(-1+i)} e^{-t} + \frac{-i}{(-i-1)(-i-i)(-i+1)} e^{-it} \\ &= \frac{1}{4} e^t + \frac{i}{-4i} e^{it} + \frac{1}{4} e^{-t} + \frac{-i}{4i} e^{-it} \\ &= \frac{1}{4} (e^t + e^{-t}) - \frac{1}{2} (\cos t) \end{aligned}$$

So inverse Laplace transform of $\frac{p}{p^4-1}$ is

$$\frac{1}{4} (e^t + e^{-t}) - \frac{1}{2} (\cos t)$$

19 chapter 15, problem 7.11

Problem

Using the δ function method, Find the response of the following system to a unit impulse.

$$\frac{d^4 y}{dy^4} - y = \delta(t - t_0)$$

Solution

Taking the laplace transform of each side gives (assuming initial conditions for the system are at rest)

$$\begin{aligned} Yp^4 - Y &= e^{-pt_0} \\ Y &= \frac{e^{-pt_0}}{p^4 - 1} \\ Y &= \frac{e^{-pt_0}}{(p^2 - 1)(p^2 + 1)} \end{aligned}$$

Finding the inverse laplace of $\frac{1}{(p-1)(p+1)(p^2+1)} = \frac{1}{(p-1)(p+1)} \frac{1}{(p^2+1)} = GH$. Then $g(t) = \frac{e^t - e^{-t}}{2}$ using L7 and, $h(t) = \sin t$ using L3. Hence the inverse transform is

$$\begin{aligned} g(t) \otimes h(t) &= \int_0^t \frac{e^\tau - e^{-\tau}}{2} \sin(t - \tau) d\tau \\ &= \frac{1}{2} (\sinh t - \sin t) \end{aligned}$$

Using L28 with the result above we get

$$y(t) = \begin{cases} \frac{1}{2} (\sinh(t - t_0) - \sin(t - t_0)) & t > t_0 \\ 0 & t < t_0 \end{cases}$$

Or by expressing sinh using exp, the above becomes

$$y(t) = \begin{cases} \frac{1}{4}(e^{t-t_0} - e^{-t+t_0} - 2 \sin(t - t_0)) & t > t_0 \\ 0 & t < t_0 \end{cases}$$

20 chapter 15, problem 7.7

Problem

Using the δ function method, Find the response of the following system to a unit impulse.
 $y'' + 2y' + y = \delta(t - t_0)$

Solution

Take the laplace transform of each side we get (assume initial conditions for a system at rest)

$$\begin{aligned} Yp^2 + 2Yp + Y &= e^{-pt_0} \\ Y &= \frac{e^{-pt_0}}{p^2 + 2p + 1} \\ Y &= \frac{e^{-pt_0}}{(p + 1)^2} \end{aligned}$$

Using L28 and L6 (for $k = 1$)

$$y(t) = \begin{cases} (t - t_0)e^{-(t-t_0)} & t > t_0 \\ 0 & t < t_0 \end{cases}$$

21 chapter 15, problem 7.9

Problem

Using the δ function method, Find the response of the following system to a unit impulse.
 $y'' + 2y' + 10y = \delta(t - t_0)$

Solution

Taking the laplace transform of each side we get (assume initial conditions for a system at rest)

$$\begin{aligned} Yp^2 + 2Yp + 10Y &= e^{-pt_0} \\ Y &= \frac{e^{-pt_0}}{p^2 + 2p + 10} \\ Y &= \frac{e^{-pt_0}}{(p - a)(p - b)} \end{aligned}$$

Where $a = -1 + 3i$, $b = -1 - 3i$ the roots of $p^2 + 2p + 10$. Using L28 and L7

$$y(t) = \begin{cases} \frac{e^{a(t-t_0)} - e^{b(t-t_0)}}{(-b) - (-a)} & t > t_0 \\ 0 & t < t_0 \end{cases}$$

Replacing values for a, b gives

$$\begin{aligned}
 y(t) &= \frac{e^{a(t-t_0)} - e^{b(t-t_0)}}{a - b} \\
 &= \frac{e^{(-1+3i)(t-t_0)} - e^{(-1-3i)(t-t_0)}}{(-1+3i) - (-1-3i)} \\
 &= \frac{e^{(-1+3i)(t-t_0)} - e^{(-1-3i)(t-t_0)}}{6i} \\
 &= \frac{e^{-t+t_0+3it-3it_0} - e^{-t+t_0-3it+3it_0}}{6i} \\
 &= e^{-t+t_0} \frac{e^{3i(t-t_0)} - e^{-3i(t-t_0)}}{6i} \\
 &= e^{-t+t_0} \left(\frac{1}{3} \right) \left(\frac{e^{3i(t-t_0)} - e^{-3i(t-t_0)}}{2i} \right) \\
 &= \frac{e^{-t+t_0}}{3} \sin 3(t - t_0)
 \end{aligned}$$

Therefore

$$y(t) = \begin{cases} \frac{e^{-t+t_0}}{3} \sin 3(t - t_0) & t > t_0 \\ 0 & t < t_0 \end{cases}$$