HW 12, Math 121 A
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1 chapter 15, problem 4.12

Problem Find the exponential Fourier transform of the given $f(x)$ and write $f(x)$ as a fourier integral.

$$f(x) = \begin{cases} \sin x & |x| < \frac{\pi}{2} \\ \\ 0, & |x| > \frac{\pi}{2} \end{cases}$$

Solution

Let $F(\alpha)$ be the Fourier transform of $f(x)$ defined as $F(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx$

$$F(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx$$
$$2\pi\ F(\alpha) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x\ \ e^{-i\alpha x}\, dx$$

Integration by parts, $u = \sin x, du = \cos x, v = \frac{e^{-i\alpha x}}{-i\alpha}$, hence $\int u\ dv\ = uv - \int du\ v$

$$\begin{aligned} I &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x\ \ e^{-i\alpha x}\, dx \\ &= \left[\sin x\ \frac{e^{-i\alpha x}}{-i\alpha}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\ \ \frac{e^{-i\alpha x}}{-i\alpha}\, dx \\ &= \left[\sin x\ \frac{e^{-i\alpha x}}{-i\alpha}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{i\alpha}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\ \ e^{-i\alpha x}\, dx \end{aligned}$$

Integration by parts the second integral again. $u = \cos x, du = -\sin x$

$$I = \left[\sin x \; \frac{e^{-i\alpha x}}{-i\alpha}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{i\alpha}\left\{\left[\cos x \; \frac{e^{-i\alpha x}}{-i\alpha}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (-\sin x) \; \frac{e^{-i\alpha x}}{-i\alpha} \, dx\right\}$$

$$I = \left[\sin x \; \frac{e^{-i\alpha x}}{-i\alpha}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{i\alpha}\left\{\left[\cos x \; \frac{e^{-i\alpha x}}{-i\alpha}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \left(\frac{1}{i\alpha}\right)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \; e^{-i\alpha x} \, dx\right\}$$

$$I = \left[\sin x \; \frac{e^{-i\alpha x}}{-i\alpha}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{i\alpha}\left[\cos x \; \frac{e^{-i\alpha x}}{-i\alpha}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \frac{1}{i^2\alpha^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \; e^{-i\alpha x} \, dx$$

$$I = \left[\sin x \; \frac{e^{-i\alpha x}}{-i\alpha}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{i\alpha}\left[\cos x \; \frac{e^{-i\alpha x}}{-i\alpha}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{\alpha^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \; e^{-i\alpha x} \, dx$$

But the last integral on the right above is the same as the integral we start with, so

$$I = \left[\sin x \; \frac{e^{-i\alpha x}}{-i\alpha}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{i\alpha}\left[\cos x \; \frac{e^{-i\alpha x}}{-i\alpha}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{\alpha^2} I$$

$$I - \frac{1}{\alpha^2} I = \left[\sin x \; \frac{e^{-i\alpha x}}{-i\alpha}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \frac{i}{\alpha}\left[\cos x \; \frac{e^{-i\alpha x}}{-i\alpha}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$I\left(1 - \frac{1}{\alpha^2}\right) = \left[\sin\left(\frac{\pi}{2}\right) \frac{e^{-i\alpha\frac{\pi}{2}}}{-i\alpha} - \sin\left(-\frac{\pi}{2}\right) \frac{e^{-i\alpha\left(-\frac{\pi}{2}\right)}}{-i\alpha}\right] - \frac{i}{\alpha}\left[\cos\left(\frac{\pi}{2}\right) \frac{e^{-i\alpha\frac{\pi}{2}}}{-i\alpha} - \cos\left(-\frac{\pi}{2}\right) \frac{e^{-i\alpha\left(-\frac{\pi}{2}\right)}}{-i\alpha}\right]$$

$$I\left(\frac{\alpha^2 - 1}{\alpha^2}\right) = \left[\frac{e^{-i\alpha\frac{\pi}{2}}}{-i\alpha} + \frac{e^{i\alpha\left(\frac{\pi}{2}\right)}}{-i\alpha}\right] - \frac{i}{\alpha}(0)$$

$$I = \left(\frac{\alpha^2}{\alpha^2 - 1}\right)\left[\frac{-e^{-i\alpha\frac{\pi}{2}}}{i\alpha} - \frac{e^{i\alpha\frac{\pi}{2}}}{i\alpha}\right]$$

$$I = \left(\frac{\alpha}{\alpha^2 - 1}\right)\frac{-1}{i}\left[e^{i\alpha\frac{\pi}{2}} + e^{-i\alpha\frac{\pi}{2}}\right]$$

$$I = \left(\frac{\alpha i}{\alpha^2 - 1}\right)\left[e^{i\alpha\frac{\pi}{2}} + e^{-i\alpha\frac{\pi}{2}}\right]$$

$$I = \left(\frac{\alpha i}{\alpha^2 - 1}\right) 2\cos\left(\alpha\frac{\pi}{2}\right)$$

Hence the Fourier transform of $f(x)$ is

$$
\begin{aligned}
2\pi\ \mathrm{F}(\alpha) &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x\ \ e^{-i\alpha x}\ dx \\
&= \left(\frac{\alpha i}{\alpha^2-1}\right) 2\cos\left(\alpha\frac{\pi}{2}\right)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
g(\alpha) &= \frac{1}{\pi}\left(\frac{\alpha i}{\alpha^2-1}\right)\cos\left(\alpha\frac{\pi}{2}\right) \\
&= \frac{\alpha\ i\ \cos\left(\alpha\frac{\pi}{2}\right)}{\left(\alpha^2-1\right)\pi}
\end{aligned}
$$

To obtain $f(x)$ given its fourier transform $\mathrm{F}(\alpha)$, then we apply the inverse fourier transform

$$
\begin{aligned}
f(x) &= \int_{-\infty}^{\infty} \mathrm{F}(\alpha)\ e^{i\alpha x}\ d\alpha \\
&= \int_{-\infty}^{\infty} \frac{\alpha\ i\ \cos\left(\alpha\frac{\pi}{2}\right)}{\left(\alpha^2-1\right)\pi}\ \ e^{i\alpha x}\ d\alpha \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\alpha\ i\ \cos\left(\alpha\frac{\pi}{2}\right)}{\alpha^2-1}\ \ e^{i\alpha x}\ d\alpha
\end{aligned}
$$

2 chapter 15, problem 4.18

Problem Find the fourier sin transform of the given $f(x)$ and write $f(x)$ as a fourier integral. Verify the answer is the same as the exponential fourier transform.

$$
f(x) = \begin{cases} x & |x| < 1 \\ \\ 0, & |x| > 1 \end{cases}
$$

Solution

Let $\mathscr{F}_s f(x)$ be the Fourier sin transform of $f(x)$ defined as $g_s(\alpha) = \mathscr{F}_s f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x)\ \sin(\alpha x)\ dx$.

Hence, for the function above we get

$$g_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^1 x \, \sin(\alpha x) \, dx$$

Notice, we integrate from zero, not from -1, since the sin transform is defined only for positive x. Integrating by parts, $u = x, v = \frac{-\cos(\alpha x)}{\alpha}$, hence $\int u \, dv = uv - \int du \, v$

$$\begin{aligned}
g_s(\alpha) &= \sqrt{\frac{2}{\pi}} \left\{ \left[x \left(\frac{-\cos(\alpha x)}{\alpha} \right) \right]_0^1 - \int_0^1 \frac{-\cos(\alpha x)}{\alpha} \, dx \right\} \\
&= \sqrt{\frac{2}{\pi}} \left\{ \frac{-1}{\alpha} [x \, \cos(\alpha x)]_0^1 + \frac{1}{\alpha} \int_0^1 \cos(\alpha x) \, dx \right\} \\
&= \sqrt{\frac{2}{\pi}} \left\{ \frac{-1}{\alpha} [x \, \cos(\alpha x)]_0^1 + \frac{1}{\alpha} \left[\frac{\sin(\alpha x)}{\alpha} \right]_0^1 \right\} \\
&= \sqrt{\frac{2}{\pi}} \frac{1}{\alpha} \left\{ -[\cos(\alpha) - 0] + \frac{1}{\alpha} [\sin(\alpha x)]_0^1 \right\} \\
&= \sqrt{\frac{2}{\pi}} \frac{1}{\alpha} \left\{ -\cos(\alpha) + \frac{1}{\alpha} [\sin(\alpha) - 0] \right\} \\
&= \sqrt{\frac{2}{\pi}} \frac{1}{\alpha} \left(-\cos(\alpha) + \frac{1}{\alpha} \sin(\alpha) \right) \\
&= \sqrt{\frac{2}{\pi}} \frac{1}{\alpha} \left(\frac{1}{\alpha} \sin(\alpha) - \cos(\alpha) \right)
\end{aligned}$$

Hence the Sin Fourier transform of $f(x)$ is

$$g_s(\alpha) = \sqrt{\frac{2}{\pi}} \frac{1}{\alpha} \left(\frac{1}{\alpha} \sin(\alpha) - \cos(\alpha) \right)$$

To obtain $f(x)$ given its sin fourier transform $g_s(\alpha)$, then we apply the inverse sin fourier transform

$$\begin{aligned}
f_s(x) &= \sqrt{\frac{2}{\pi}} \int_0^\infty g_s(\alpha) \, \sin \alpha x \, d\alpha \\
&= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{1}{\alpha} \left(\frac{1}{\alpha} \sin \alpha - \cos \alpha \right) \sin \alpha x \, d\alpha \\
&= \frac{2}{\pi} \int_0^\infty \left(\frac{1}{\alpha^2} \sin \alpha - \frac{1}{\alpha} \cos \alpha \right) \sin \alpha x \, d\alpha \\
&= \frac{2}{\pi} \int_0^\infty \left(\frac{\sin \alpha - \alpha \cos \alpha}{\alpha^2} \right) \sin \alpha x \, d\alpha \qquad \text{(A0)}
\end{aligned}$$

Now we need to show that the above is the same as the inverse fourier transform found for problem 6. From back of the book, the IFT for problem 6 is given as

$$f(x) = \int_{-\infty}^{\infty} \frac{\sin\alpha - \alpha\cos\alpha}{i\pi\alpha^2} e^{i\alpha x} d\alpha$$

Need to convert the above to $f_s(x)$. Since $e^{i\alpha x} = \cos\alpha x + i\sin\alpha x$

$$\begin{aligned}
f(x) &= \int_{-\infty}^{\infty} \frac{\sin\alpha - \alpha\cos\alpha}{i\pi\alpha^2}(\cos\alpha x + i\sin\alpha x) d\alpha \\
&= \int_{-\infty}^{\infty} \frac{(\sin\alpha - \alpha\cos\alpha)\cos\alpha x}{i\pi\alpha^2} + \frac{(\sin\alpha - \alpha\cos\alpha) i\sin\alpha x}{i\pi\alpha^2} d\alpha \\
&= \int_{-\infty}^{\infty} \frac{(\sin\alpha - \alpha\cos\alpha)\cos\alpha x}{i\pi\alpha^2} d\alpha + \int_{-\infty}^{\infty} \frac{(\sin\alpha - \alpha\cos\alpha) i\sin\alpha x}{i\pi\alpha^2} d\alpha \qquad (1)
\end{aligned}$$

Looking at the first integral,

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{\left(\overbrace{\sin\alpha}^{\text{odd}} - \overbrace{\alpha}^{\text{odd}}\overbrace{\cos\alpha}^{\text{even}}\right)\overbrace{\cos\alpha x}^{\text{even}}}{i\pi\overbrace{\alpha^2}^{\text{even}}} d\alpha \\
&= \int_{-\infty}^{\infty} \frac{(\text{odd} - \text{odd}\times\text{even})\times\text{even}}{\text{even}} d\alpha \\
&= \int_{-\infty}^{\infty} \frac{(\text{odd} - \text{odd})\times\text{even}}{\text{even}} d\alpha \\
&= \int_{-\infty}^{\infty} \frac{\text{odd}\times\text{even}}{\text{even}} d\alpha \\
&= \int_{-\infty}^{\infty} \frac{\text{odd}}{\text{even}} d\alpha \\
&= \int_{-\infty}^{\infty} \text{odd}\ d\alpha
\end{aligned}$$

Hence the integral vanishes. Hence (1) becomes

$$f(x) = \int_{-\infty}^{\infty} \frac{(\sin\alpha - \alpha\cos\alpha) i\sin\alpha x}{i\pi\alpha^2} d\alpha \qquad (2)$$

Looking at the above

$$\frac{\left(\overbrace{\sin\alpha}^{\text{odd}} - \overbrace{\alpha}^{\text{odd}}\overbrace{\cos\alpha}^{\text{even}}\right)\overbrace{i\sin\alpha x}^{\text{odd}}}{i\pi\overbrace{\alpha^2}^{\text{even}}} = \frac{(\text{odd} - \text{odd}\times\text{even})\times\text{odd}}{\text{even}}$$
$$= \frac{\text{odd} \quad \times \text{odd}}{\text{even}}$$
$$= \frac{\text{even}}{\text{even}}$$
$$= \text{even}$$

Since the integrand is even, then $\int_{-\infty}^{\infty} = 2\int_0^{\infty}$ Hence (2) becomes

$$f(x) = 2\int_0^{\infty} \frac{(\sin\alpha - \alpha\cos\alpha)i\sin\alpha x}{i\pi\alpha^2}\ d\alpha \tag{3}$$
$$= \frac{2}{\pi}\int_0^{\infty} \frac{(\sin\alpha - \alpha\cos\alpha)}{\alpha^2}\sin\alpha x\ \ d\alpha$$

comparing this to equation (A1) above, we see that

$$f_s(x) = f(x)$$

Which is what we are asked to show.

3 chapter 15, problem 4.20

Problem Find the fourier sin transform of the given $f(x)$ and write $f(x)$ as a fourier integral. Verify the answer is the same as the exponential fourier transform.

$$f(x) = \begin{cases} \sin x & |x| < \frac{\pi}{2} \\ \\ 0, & |x| > \frac{\pi}{2} \end{cases}$$

Solution

Let $\mathscr{F}_s f(x)$ be the Fourier sin transform of $f(x)$ defined as $g_s(\alpha) = \mathscr{F}_s f(x) = \sqrt{\frac{2}{\pi}}\int\limits_0^{\infty} f(x)\ \sin(\alpha x)\,dx$.

Hence, for the function above we get

$$g_s(\alpha) = \sqrt{\frac{2}{\pi}}\int\limits_0^{1} \sin x\ \sin(\alpha x)\,dx$$

Notice, we integrate from zero, not from -1, since the sin transform is defined only for positive x. Since $\sin\beta\sin\gamma = \frac{1}{2}\cos\left(\beta-\gamma\right) - \frac{1}{2}\cos\left(\beta+\gamma\right)$ Then $\sin x\ \sin(\alpha x) = \frac{1}{2}\cos(x-\alpha x) - \frac{1}{2}\cos(x+\alpha x)$. Hence

$$
\begin{aligned}
g_s(\alpha) &= \sqrt{\frac{2}{\pi}}\int_0^1 \frac{1}{2}\cos(x-\alpha x) - \frac{1}{2}\cos(x+\alpha x)\,dx \\
&= \frac{1}{2}\sqrt{\frac{2}{\pi}}\left\{\left[\frac{\sin(x-\alpha x)}{1-\alpha}\right]_0^1 - \left[\frac{\sin(x+\alpha x)}{1+\alpha}\right]_0^1\right\} \\
&= \frac{1}{2}\sqrt{\frac{2}{\pi}}\left\{\left[\frac{\sin(1-\alpha)}{1-\alpha} - 0\right] - \left[\frac{\sin(1+\alpha)}{1+\alpha} - 0\right]_0^1\right\} \\
&= \frac{1}{2}\sqrt{\frac{2}{\pi}}\left(\frac{\sin(1-\alpha)}{1-\alpha} - \frac{\sin(1+\alpha)}{1+\alpha}\right)
\end{aligned}
$$

Hence the Sin Fourier transform of $f(x)$ is

$$
g_s(\alpha) = \frac{1}{2}\sqrt{\frac{2}{\pi}}\left(\frac{\sin(1-\alpha)}{1-\alpha} - \frac{\sin(1+\alpha)}{1+\alpha}\right)
$$

Therefore, to obtain $f(x)$ given its sin fourier transform $g_s(\alpha)$, we apply the inverse sin fourier transform

$$
\begin{aligned}
f_s(x) &= \sqrt{\frac{2}{\pi}}\int_0^\infty g_s(\alpha)\ \sin\alpha x\,d\alpha \\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{2}\sqrt{\frac{2}{\pi}}\left(\frac{\sin(1-\alpha)}{1-\alpha} - \frac{\sin(1+\alpha)}{1+\alpha}\right)\sin\alpha x\,d\alpha \\
&= \frac{1}{\pi}\int_0^\infty \left(\frac{\sin(1-\alpha)}{1-\alpha} - \frac{\sin(1+\alpha)}{1+\alpha}\right)\sin\alpha x\,d\alpha \qquad (1)
\end{aligned}
$$

Now we need to show that the above is the same as the exponential inverse fourier transform found for problem 12. The exponential IFT for problem 12 is

$$
f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\alpha\ i\ \cos\left(\alpha\frac{\pi}{2}\right)}{\alpha^2-1}\ e^{i\alpha x}\,d\alpha \qquad (2)
$$

So Need to show that (1) and (2) are the same. Need to convert the above (1) to $f_s(x)$ in (2).Since $e^{i\alpha x} = \cos\alpha x + i\sin\alpha x$, (2) can be written as

$$\begin{aligned} f(x) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\alpha\, i\, \cos\left(\alpha\frac{\pi}{2}\right)}{\alpha^2-1}\,(\cos\alpha x + i\sin\alpha x)d\alpha \\ &= \frac{1}{\pi}\left[\int_{-\infty}^{\infty} \frac{\alpha\, i\, \cos\left(\alpha\frac{\pi}{2}\right)}{\alpha^2-1}\cos\alpha x\ d\alpha + \int_{-\infty}^{\infty} \frac{\alpha\, i\, \cos\left(\alpha\frac{\pi}{2}\right)}{\alpha^2-1}\, i\sin\alpha x\ d\alpha\right] \end{aligned} \tag{3}$$

Looking at the first integral,

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{\overbrace{\alpha}^{\text{odd}}\ i\ \overbrace{\cos\left(\alpha\frac{\pi}{2}\right)}^{\text{even}}}{\overbrace{\alpha^2}^{\text{even}}\ -1}\overbrace{\cos\alpha x}^{\text{even}}\ d\alpha \\ &= \int_{-\infty}^{\infty} \frac{(\text{odd}\times\ \text{even})\times\text{even}}{\text{even}}\ d\alpha \\ &= \int_{-\infty}^{\infty} \frac{\text{odd}\ \times\text{even}}{\text{even}}\ d\alpha \\ &= \int_{-\infty}^{\infty} \frac{\text{odd}}{\text{even}}\ d\alpha \\ &= \int_{-\infty}^{\infty} \text{odd}\ d\alpha \end{aligned}$$

Hence the integral vanishes. So (3) becomes

$$\begin{aligned} f(x) &= 0 + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\alpha\, i\, \cos\left(\alpha\frac{\pi}{2}\right)}{\alpha^2-1}\, i\sin\alpha x\ d\alpha \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\alpha\, i\, \cos\left(\alpha\frac{\pi}{2}\right)}{\alpha^2-1}\, i\sin\alpha x\ d\alpha \end{aligned}$$

Looking at the above integrand,

$$\begin{aligned} \frac{\overbrace{\alpha}^{\text{odd}}\ i\ \overbrace{\cos\left(\alpha\frac{\pi}{2}\right)}^{\text{even}}}{\overbrace{\alpha^2}^{\text{even}}\ -1}\overbrace{i\sin\alpha x}^{\text{odd}} &= \frac{(\text{odd}\ \times\text{even})\times\text{odd}}{\text{even}} \\ &= \frac{\text{odd}\ \ \times\text{odd}}{\text{even}} \\ &= \frac{\text{even}}{\text{even}} \\ &= \text{even} \end{aligned}$$

Since the integrand is even, then $\int_{-\infty}^{\infty} = 2\int_{0}^{\infty}$. Hence (2) becomes

$$
\begin{aligned}
f(x) &= \frac{2}{\pi}\int_0^\infty \frac{\alpha\, i\ \cos\left(\alpha\frac{\pi}{2}\right)}{\alpha^2-1}\, i\sin\alpha x\ d\alpha \\
&= \frac{-2}{\pi}\int_0^\infty \frac{\alpha\ \ \cos\left(\alpha\frac{\pi}{2}\right)}{\alpha^2-1}\ \sin\alpha x\ d\alpha \\
&= \frac{2}{\pi}\int_0^\infty \frac{\alpha\ \ \cos\left(\alpha\frac{\pi}{2}\right)}{1-\alpha^2}\ \sin\alpha x\ d\alpha
\end{aligned}
$$

But $1-\alpha^2 = (1+\alpha)(1-\alpha)$, therefore

$$
f(x) = \frac{2}{\pi}\int_0^\infty \frac{\alpha\ \ \cos\left(\alpha\frac{\pi}{2}\right)}{(1+\alpha)(1-\alpha)}\ \sin\alpha x\ d\alpha
$$

But $\cos\left(\alpha\frac{\pi}{2}\right)\sin\alpha x = \frac{1}{2}\left(-\sin\left(\alpha\frac{\pi}{2}-\alpha x\right)+\sin\left(\alpha\frac{\pi}{2}+\alpha x\right)\right)$, hence the above becomes

$$
\begin{aligned}
f(x) &= \frac{2}{\pi}\int_0^\infty \frac{\alpha\ \ \frac{1}{2}\left(-\sin\left(\alpha\frac{\pi}{2}-\alpha x\right)+\sin\left(\alpha\frac{\pi}{2}+\alpha x\right)\right)}{(1+\alpha)(1-\alpha)}\ \ d\alpha \\
f(x) &= \frac{1}{\pi}\int_0^\infty \frac{\alpha\ \ \left(-\sin\left(\alpha\frac{\pi}{2}-\alpha x\right)+\sin\left(\alpha\frac{\pi}{2}+\alpha x\right)\right)}{(1+\alpha)(1-\alpha)}\ \ d\alpha \\
f(x) &= \frac{1}{\pi}\int_0^\infty \frac{\alpha\sin\left(\alpha\frac{\pi}{2}+\alpha x\right)}{(1+\alpha)(1-\alpha)} - \frac{\alpha\sin\left(\alpha\frac{\pi}{2}-\alpha x\right)}{(1+\alpha)(1-\alpha)}\ \ d\alpha
\end{aligned}
$$

4 chapter 15, problem 4.21

Problem Find the fourier transform of the given $f(x) = e^{\frac{-x^2}{2\sigma^2}}$

Solution

Let $\mathsf{F}(\alpha)$ be the Fourier sin transform of $f(x)$ defined as

$$
\mathsf{F}(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx
$$

So, for the function above we get

$$
\begin{aligned}
\mathsf{F}(\alpha) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{-x^2}{2\sigma^2}}\, e^{-i\alpha x}\, dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{-x^2}{2\sigma^2}-i\alpha x}\, dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{-x^2-i\alpha x\left(2\sigma^2\right)}{2\sigma^2}}\, dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{-x^2-2i\alpha\sigma^2 x}{2\sigma^2}}\, dx \qquad (1)
\end{aligned}
$$

looking at the exponent $\frac{-x^2-2i\alpha\sigma^2 x}{2\sigma^2}$. completing the square in x gives

$$x^2 + 2i\alpha\sigma^2 x = (x + Z)^2 - Y$$

Solving for Z, Y gives

$$x^2 + 2i\alpha\sigma^2 x = x^2 + 2xZ + Z^2 - Y$$

Therefore $Z = i\alpha\sigma^2, Z^2 - Y = 0$, and $Y = -\alpha^2\sigma^4$. Hence Exponent can be written as

$$
\begin{aligned}
\frac{x^2 + 2i\alpha\sigma^2 x}{-2\sigma^2} &= \frac{\left(x + i\alpha\sigma^2\right)^2 - \left(-\alpha^2\sigma^4\right)}{-2\sigma^2} \\
&= \frac{\left(x + i\alpha\sigma^2\right)^2 + \alpha^2\sigma^4}{-2\sigma^2} \\
&= \frac{\left(x + i\alpha\sigma^2\right)^2}{-2\sigma^2} - \frac{\alpha^2\sigma^4}{2\sigma^2} \\
&= \frac{\left(x + i\alpha\sigma^2\right)^2}{-2\sigma^2} - \frac{\alpha^2\sigma^2}{2}
\end{aligned}
$$

The integral (1) becomes

$$
\begin{aligned}
\mathsf{F}(\alpha) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{\left(x+i\alpha\sigma^2\right)^2}{-2\sigma^2}-\frac{\alpha^2\sigma^2}{2}}\, dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{\left(x+i\alpha\sigma^2\right)^2}{-2\sigma^2}}\, e^{-\frac{\alpha^2\sigma^2}{2}}\, dx
\end{aligned}
$$

Moving $e^{-\frac{\alpha^2\sigma^2}{2}}$ outside the integral because it does not depend on x gives

$$\mathsf{F}(\alpha) = \frac{e^{-\frac{\alpha^2\sigma^2}{2}}}{2\pi} \int_{-\infty}^{\infty} e^{\frac{\left(x+i\alpha\sigma^2\right)^2}{-2\sigma^2}}\ dx$$

Let $y = x + i\alpha\sigma^2$, $dy = dx$ and the limits do not change. Hence we get

$$\mathsf{F}(\alpha) = \frac{e^{-\frac{\alpha^2\sigma^2}{2}}}{2\pi} \int_{-\infty}^{\infty} e^{\frac{y^2}{-2\sigma^2}}\ dy$$

Since the exponential function is raised to a square power, then we can write $\int_{-\infty}^{\infty} e^{y^2} = 2\int_{0}^{\infty} e^{y^2}$ (since even function). Hence above integral becomes

$$\mathsf{F}(\alpha) = \frac{e^{-\frac{\alpha^2\sigma^2}{2}}}{\pi} \int_{0}^{\infty} e^{\frac{y^2}{-2\sigma^2}}\ dy$$

Let $\zeta = \frac{y}{\sqrt{2}\,\sigma}$, then $y = \sqrt{2}\,\sigma\zeta$, and $y^2 = 2\sigma^2\zeta^2$. Hence $d\zeta = \frac{1}{\sqrt{2}\,\sigma}dy$ and the above integral becomes

$$\mathsf{F}(\alpha) = \frac{e^{-\frac{\alpha^2\sigma^2}{2}}}{\pi} \int_{0}^{\infty} e^{-\zeta^2}\ \sqrt{2}\,\sigma\ d\zeta$$

$$\mathsf{F}(\alpha) = \sqrt{2}\,\sigma \frac{e^{-\frac{\alpha^2\sigma^2}{2}}}{\pi} \int_{0}^{\infty} e^{-\zeta^2}\ d\zeta \tag{2}$$

Now from equation 9.5 on page 468

$$\frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-\zeta^2}\ d\zeta = 1$$

$$\int_{0}^{\infty} e^{-\zeta^2}\ d\zeta = \frac{\sqrt{\pi}}{2}$$

Hence (2) becomes

$$F(\alpha) = \sqrt{2}\,\sigma \frac{e^{-\frac{\alpha^2\sigma^2}{2}}}{\pi}\left(\frac{\sqrt{\pi}}{2}\right)$$
$$F(\alpha) = \frac{\sigma}{\sqrt{2\pi}}\, e^{-\frac{\alpha^2\sigma^2}{2}}$$

Which is what we are asked to show.

5 chapter 15, problem 4.23

Problem Show that
$\int_0^\infty \frac{1-\cos\pi\alpha}{\alpha}\sin\alpha\, d\alpha = \frac{\pi}{2}$ and $\int_0^\infty \frac{1-\cos\pi\alpha}{\alpha}\sin\pi\alpha\, d\alpha = \frac{\pi}{4}$

Solution

From problem 17, the Fourier sin transform for $f(x)$ shown in problem 3 is

$$g_s(\alpha) = \frac{\sqrt{2}\,(1-\cos\pi\alpha)}{\sqrt{\pi}\,\alpha}$$

From equation 4.14 page 651, $f(x)$ can be obtained from inverse sin transform is

$$\begin{aligned} f_s(x) &= \sqrt{\frac{2}{\pi}}\int_0^\infty g_s(\alpha)\sin\alpha x\, d\alpha \\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty\left(\frac{\sqrt{2}\,(1-\cos\pi\alpha)}{\sqrt{\pi}\,\alpha}\right)\sin\alpha x\, d\alpha \\ &= \frac{2}{\pi}\int_0^\infty \frac{(1-\cos\pi\alpha)}{\alpha}\sin\alpha x\, d\alpha \qquad (1) \end{aligned}$$

Now, from the definition of $f(x)$, which is

$$f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \\ 0, & |x| > \pi \end{cases}$$

We see that for $x = 1$, $f(x) = 1$, hence substitute in (1) we get

$$\begin{aligned} 1 &= -\frac{2}{\pi}\int_0^\infty \frac{(1-\cos\pi\alpha)}{\alpha}\sin\alpha\, d\alpha \\ \frac{\pi}{2} &= \int_0^\infty \frac{(1-\cos\pi\alpha)}{\alpha}\sin\alpha\, d\alpha \end{aligned}$$

Which is the first result we required to show. For the second result, let $x = \pi$ hence $f(\pi) =$ average value of $f(x)$ at $x = \pi$. Which is given by $\frac{f(\pi_-)+f(\pi_+)}{2} = \frac{1+0}{2} = \frac{1}{2}$. Hence substitute in (1) we get

$$f_s(x = \pi) = \frac{1}{2} = \frac{2}{\pi}\int_0^\infty \frac{(1-\cos \pi\alpha)}{\alpha} \sin \alpha\pi \, d\alpha$$
$$\frac{\pi}{4} = \int_0^\infty \frac{(1-\cos \pi\alpha)}{\alpha} \sin \alpha\pi \, d\alpha$$

Which is the second result we are asked to show.

6 chapter 15, problem 4.25

Problem Show that

(a) represent as an exponential fourier transform the function

$$f(x) = \begin{cases} \sin x & 0 < x < \pi \\ \\ 0, & \text{otherwise} \end{cases}$$

(b) Show that the result can be written as

$$f(x) = \frac{1}{\pi}\int_0^\infty \frac{\cos \alpha x + \cos \alpha(x-\pi)}{1-\alpha^2} d\alpha$$

Solution

The exponential Fourier transform is defined as

$$g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{-i\alpha x} \, dx$$

Applying the function $f(x)$ gives

$$g(\alpha) = \frac{1}{2\pi}\int_0^\pi \sin x \, e^{-i\alpha x} \, dx$$

But

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

Hence the transform can be written as

$$\begin{aligned}
g(\alpha) &= \frac{1}{2\pi}\int_0^{\pi} \frac{e^{ix}-e^{-ix}}{2i}\, e^{-i\alpha x}\,dx \\
&= \frac{1}{4i\pi}\int_0^{\pi} e^{ix-i\alpha x}-e^{-ix-i\alpha x}\ \ dx \\
&= \frac{1}{4i\pi}\int_0^{\pi} e^{x(i-i\alpha)}-e^{x(-i-i\alpha)}\ \ dx \\
&= \frac{1}{4i\pi}\left(\left[\frac{e^{x(i-i\alpha)}}{i-i\alpha}\right]_0^{\pi} - \left[\frac{e^{x(-i-i\alpha)}}{-i-i\alpha}\right]_0^{\pi}\right) \\
&= \frac{1}{4i\pi}\left(\left[\frac{e^{\pi(i-i\alpha)}}{i-i\alpha}-\frac{1}{i-i\alpha}\right]-\left[\frac{e^{\pi(-i-i\alpha)}}{-i-i\alpha}-\frac{1}{-i-i\alpha}\right]\right) \\
&= \frac{1}{4i\pi}\left(\frac{1}{i-i\alpha}\left[e^{\pi(i-i\alpha)}-1\right]-\frac{1}{-i-i\alpha}\left[e^{\pi(-i-i\alpha)}-1\right]\right) \\
&= \frac{1}{4i\pi}\left(\frac{1}{i-i\alpha}\left[e^{\pi(i-i\alpha)}-1\right]+\frac{1}{i+i\alpha}\left[e^{\pi(-i-i\alpha)}-1\right]\right) \\
&= \frac{1}{i}\frac{1}{4i\pi}\left(\frac{1}{1-\alpha}\left[e^{\pi(i-i\alpha)}-1\right]+\frac{1}{1+\alpha}\left[e^{\pi(-i-i\alpha)}-1\right]\right) \\
&= \frac{-1}{4\pi}\left(\frac{e^{\pi(i-i\alpha)}}{1-\alpha}-\frac{1}{1-\alpha}+\frac{e^{\pi(-i-i\alpha)}}{1+\alpha}-\frac{1}{1+\alpha}\right) \\
&= \frac{-1}{4\pi}\left(\frac{(1+\alpha)e^{\pi(i-i\alpha)}+(1-\alpha)e^{\pi(-i-i\alpha)}}{(1-\alpha)(1+\alpha)}-\frac{(1+\alpha)+(1-\alpha)}{(1-\alpha)(1+\alpha)}\right) \\
&= \frac{-1}{4\pi}\left(\frac{(1+\alpha)e^{\pi(i-i\alpha)}+(1-\alpha)e^{\pi(-i-i\alpha)}}{\left(1-\alpha^2\right)}-\frac{2}{\left(1-\alpha^2\right)}\right) \\
&= \frac{-1}{\left(1-\alpha^2\right)4\pi}\left(e^{\pi(i-i\alpha)}+\alpha e^{\pi(i-i\alpha)}+e^{\pi(-i-i\alpha)}-\alpha e^{\pi(-i-i\alpha)}\ -2\right) \\
&= \frac{-1}{\left(1-\alpha^2\right)4\pi}\left(e^{\pi i}\,e^{-i\pi\alpha}+\alpha e^{\pi i}\,e^{-i\pi\alpha}+e^{-\pi i}\,e^{-i\pi\alpha}-\alpha e^{-\pi i}\,e^{-i\pi\alpha}\ -2\right)
\end{aligned}$$

But $e^{\pi i} = -1$ and $e^{-\pi i} = -1$

$$\begin{aligned}
g(\alpha) &= \frac{-1}{\left(1-\alpha^2\right)4\pi}\left(-e^{-i\pi\alpha}-\overbrace{\alpha e^{-i\pi\alpha}}-e^{-i\pi\alpha}+\overbrace{\alpha e^{-i\pi\alpha}}\ -2\right) \\
g(\alpha) &= \frac{-1}{\left(1-\alpha^2\right)4\pi}\left(-e^{-i\pi\alpha}-e^{-i\pi\alpha}-2\right) \\
g(\alpha) &= \frac{1}{\left(1-\alpha^2\right)4\pi}\left(e^{-i\pi\alpha}+e^{-i\pi\alpha}+2\right) \\
g(\alpha) &= \frac{2e^{-i\pi\alpha}+2}{\left(1-\alpha^2\right)4\pi}
\end{aligned}$$

Hence the exponential fourier transform is

$$g(\alpha) = \frac{e^{-i\pi\alpha} + 1}{\left(1-\alpha^2\right)2\pi}$$

Therefore $f(x)$ can be rewritten as

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} g(\alpha)\, e^{i\alpha x}\, d\alpha \\ &= \int_{-\infty}^{\infty} \frac{e^{-i\pi\alpha} + 1}{\left(1-\alpha^2\right)2\pi}\, e^{i\alpha x}\, d\alpha \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1+e^{-i\pi\alpha}}{\left(1-\alpha^2\right)}\, e^{i\alpha x}\, d\alpha \qquad (1) \end{aligned}$$

Which is the answer required to show.

Part(b)

Now need to show that the above can be written as

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty} \frac{\cos\alpha x + \cos\alpha(x-\pi)}{1-\alpha^2} d\alpha$$

From (1)

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1+e^{-i\pi\alpha}}{\left(1-\alpha^2\right)}\, e^{i\alpha x}\, d\alpha \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\alpha x} + e^{i\alpha x}e^{-i\pi\alpha}}{\left(1-\alpha^2\right)}\, d\alpha \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\alpha x} + e^{i\alpha x - i\pi\alpha}}{\left(1-\alpha^2\right)}\, d\alpha \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\alpha x} + e^{i\alpha(x-\pi)}}{\left(1-\alpha^2\right)}\, d\alpha \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2(\cos\alpha x + \cos\alpha(x-\pi))}{\left(1-\alpha^2\right)}\, d\alpha \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos\alpha x + \cos\alpha(x-\pi)}{\left(1-\alpha^2\right)}\, d\alpha \end{aligned}$$

Which is what is required to show.

7 chapter 15, problem 4.3

Problem Find the exponential fourier transform of the given $f(x)$ and write $f(x)$ as a fourier integral.

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \\ 0, & |x| > \pi \end{cases}$$

Solution

Let $F(\alpha)$ be the Fourier transform of $f(x)$ defined as $F(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx$, hence

$$\begin{aligned}
F(\alpha) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx \\
2\pi\, F(\alpha) &= \int_{-\pi}^{0} -\ e^{-i\alpha x}\, dx + \int_{0}^{\pi} e^{-i\alpha x}\, dx \\
&= -\left[\frac{e^{-i\alpha x}}{-i\alpha}\right]_{-\pi}^{0} + \left[\frac{e^{-i\alpha x}}{-i\alpha}\right]_{0}^{\pi} \\
&= \frac{1}{i\alpha}\left[e^{-i\alpha x}\right]_{-\pi}^{0} - \frac{1}{i\alpha}\left[e^{-i\alpha x}\right]_{0}^{\pi} \\
&= \frac{1}{i\alpha}\left[e^{0} - e^{i\alpha\pi}\right] - \frac{1}{i\alpha}\left[e^{-i\alpha\pi} - e^{0}\right]_{0}^{\pi} \\
&= \frac{1}{i\alpha}\left[1 - e^{i\alpha\pi}\right] - \frac{1}{i\alpha}\left[e^{-i\alpha\pi} - 1\right]_{0}^{\pi} \\
&= \frac{1}{i\alpha} - \frac{e^{i\alpha\pi}}{i\alpha} - \frac{e^{-i\alpha\pi}}{i\alpha} + \frac{1}{i\alpha} \\
&= \frac{2}{i\alpha} - \frac{1}{i\alpha}\left(e^{i\alpha\pi} + e^{-i\alpha\pi}\right)
\end{aligned}$$

But $e^{i\alpha\pi} + e^{-i\alpha\pi} = 2\cos\alpha\pi$. Hence

$$\begin{aligned}
2\pi\, F(\alpha) &= \frac{2}{i\alpha} - \frac{1}{i\alpha}(2\cos\alpha\pi) \\
&= \frac{2}{i\alpha}(1 - \cos\alpha\pi)
\end{aligned}$$

Hence the Fourier transform of $f(x)$ is

$$\begin{aligned}
F(\alpha) &= \frac{1}{2\pi}\left[\frac{2}{i\alpha}(1 - \cos\alpha\pi)\right] \\
&= \frac{1}{\pi\alpha i}\,(1 - \cos\alpha\pi)
\end{aligned}$$

To obtain $f(x)$ given its fourier transform $\mathsf{F}(\alpha)$, then we apply the inverse fourier transform

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} \mathsf{F}(\alpha)\, e^{i\alpha x}\, d\alpha \\ &= \int_{-\infty}^{\infty} \frac{1}{\pi\alpha i}\,(1-\cos\alpha\pi)\, e^{i\alpha x}\, d\alpha \\ &= \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{1}{\alpha}(1-\ \cos\alpha\pi)\, e^{i\alpha x}\, d\alpha \end{aligned}$$

8 chapter 15, problem 4.5

Problem Find the exponential fourier transform of the given $f(x)$ and write $f(x)$ as a fourier integral.

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Solution

Let $\mathsf{F}(s)$ be the Fourier transform of $f(x)$ defined as $\mathsf{F}(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx$

$$\begin{aligned} \mathsf{F}(s) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx \\ 2\pi\ \mathsf{F}(s) &= \int_{0}^{1} e^{-isx}\, dx \\ &= \left[\frac{e^{-isx}}{-is}\right]_0^1 \\ &= \frac{-1}{is}\left[e^{-isx}\right]_0^1 \\ &= \frac{-1}{is}\left[e^{-is} - e^{0}\right] \\ &= \frac{-1}{is}\left[e^{-is} - 1\right] \end{aligned}$$

Hence the Fourier transform of $f(x)$ is

$$\begin{aligned} \mathsf{F}(s) &= \frac{1}{2\pi}\left[\frac{-1}{is}\left[e^{-is} - 1\right]\right] \\ &= \frac{i}{2\pi s}\left(e^{-is} - 1\right) \end{aligned}$$

To obtain $f(x)$ given its fourier transform $\mathsf{F}(s)$, then we apply the inverse fourier transform

$$
\begin{aligned}
f(x) &= \int_{-\infty}^{\infty} \mathsf{F}(s)\, e^{isx}\, ds \\
&= \int_{-\infty}^{\infty} \frac{i}{2\pi s} \left(e^{-is} - 1\right) e^{isx}\, ds \\
&= \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{1}{s} \left(e^{-is} - 1\right) e^{isx}\, ds
\end{aligned}
$$

9 chapter 15, problem 4.7

Problem Find the exponential fourier transform of the given $f(x)$ and write $f(x)$ as a fourier integral.

$$
f(x) = \begin{cases} |x| & |x| < 1 \\ \\ 0, & |x| > 1 \end{cases}
$$

Solution

Let $\mathsf{F}(\alpha)$ be the Fourier transform of $f(x)$ defined as $\mathsf{F}(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx$

$$
\begin{aligned}
\mathsf{F}(\alpha) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx \\
2\pi\, \mathsf{F}(\alpha) &= \int_{-1}^{0} -x \;\, e^{-i\alpha x}\, dx \; + \int_{0}^{1} x \;\, e^{-i\alpha x}\, dx
\end{aligned}
$$

Integrating by parts, $u = x, v = \frac{e^{-i\alpha x}}{-i\alpha}$, hence $\int u\, dv \; = uv - \int du\, v$. The first integral is

$$\int_{-1}^{0} -x\ e^{-i\alpha x}\,dx = \left[(-x)\,\frac{e^{-i\alpha x}}{-i\alpha}\right]_{-1}^{0} + \int_{-1}^{0} \frac{e^{-i\alpha x}}{-i\alpha}\,dx$$

$$= \left[(x)\,\frac{e^{-i\alpha x}}{i\alpha}\right]_{-1}^{0} + \int_{-1}^{0} \frac{e^{-i\alpha x}}{-i\alpha}\,dx$$

$$= \frac{1}{i\alpha}\ \left[0-(-1)\times e^{i\alpha}\right] - \frac{1}{i\alpha}\int_{-1}^{0} e^{-i\alpha x}\,dx$$

$$= \frac{1}{i\alpha}\ \left[e^{i\alpha}\right] - \frac{1}{i\alpha}\left[\frac{e^{-i\alpha x}}{-i\alpha}\right]_{-1}^{0}$$

$$= \frac{1}{i\alpha}\ \left[e^{i\alpha}\right] + \frac{1}{i^2\alpha^2}\left[e^{-i\alpha x}\right]_{-1}^{0}$$

$$= \frac{1}{i\alpha}\ \left[e^{i\alpha}\right] - \frac{1}{\alpha^2}\left[1 - e^{i\alpha}\right]$$

$$= \frac{e^{i\alpha}}{i\alpha} - \frac{1}{\alpha^2} + \frac{e^{i\alpha}}{\alpha^2}$$

And the second integral

$$\int_{0}^{1} x\ e^{-i\alpha x}\,dx = \left[x\,\frac{e^{-i\alpha x}}{-i\alpha}\right]_{0}^{1} - \int_{0}^{1} \frac{e^{-i\alpha x}}{-i\alpha}\,dx$$

$$= \frac{1}{-i\alpha}\ \left[1\times e^{-i\alpha} - 0\right] + \frac{1}{i\alpha}\int_{0}^{1} e^{-i\alpha x}\,dx$$

$$= \frac{1}{-i\alpha}\ \left[e^{-i\alpha}\right] + \frac{1}{i\alpha}\left[\frac{e^{-i\alpha x}}{-i\alpha}\right]_{0}^{1}$$

$$= \frac{1}{-i\alpha}\ \left[e^{-i\alpha}\right] - \frac{1}{i^2\alpha^2}\left[e^{-i\alpha x}\right]_{0}^{1}$$

$$= \frac{1}{-i\alpha}\ \left[e^{-i\alpha}\right] + \frac{1}{\alpha^2}\left[e^{-i\alpha} - 1\right]$$

$$= \frac{e^{-i\alpha}}{-i\alpha} + \frac{e^{-i\alpha}}{\alpha^2} - \frac{1}{\alpha^2}$$

Hence

$$
\begin{aligned}
2\pi \ \mathsf{F}(\alpha) &= \left(\frac{e^{i\alpha}}{i\alpha} - \frac{1}{\alpha^2} + \frac{e^{i\alpha}}{\alpha^2}\right) + \left(\frac{e^{-i\alpha}}{-i\alpha} + \frac{e^{-i\alpha}}{\alpha^2} - \frac{1}{\alpha^2}\right)\\
2\pi \ \mathsf{F}(\alpha) &= \frac{e^{i\alpha}}{i\alpha} - \frac{1}{\alpha^2} + \frac{e^{i\alpha}}{\alpha^2} - \frac{e^{-i\alpha}}{i\alpha} + \frac{e^{-i\alpha}}{\alpha^2} - \frac{1}{\alpha^2}\\
&= \frac{e^{i\alpha}}{i\alpha} + \frac{e^{i\alpha}}{\alpha^2} - \frac{e^{-i\alpha}}{i\alpha} + \frac{e^{-i\alpha}}{\alpha^2} - \frac{2}{\alpha^2}\\
&= \frac{1}{\alpha}\left(\frac{e^{i\alpha}}{i} - \frac{e^{-i\alpha}}{i}\right) + \frac{1}{\alpha^2}\left(e^{i\alpha} + e^{-i\alpha}\right) - \frac{2}{\alpha^2}\\
&= \frac{1}{\alpha}\left(\frac{e^{i\alpha} - e^{-i\alpha}}{i}\right) + \frac{1}{\alpha^2}\left(e^{i\alpha} + e^{-i\alpha}\right) - \frac{2}{\alpha^2}
\end{aligned}
$$

But $e^{i\alpha} + e^{-i\alpha} = 2\cos\alpha$ and $\frac{e^{i\alpha}-e^{-i\alpha}}{i} = 2\sin\alpha$, Hence the above becomes

$$
\begin{aligned}
2\pi \ \mathsf{F}(\alpha) &= \frac{1}{\alpha}(2\sin\alpha) + \frac{1}{\alpha^2}(2\cos\alpha) - \frac{2}{\alpha^2}\\
&= \frac{2}{\alpha^2}[\alpha\sin\alpha + \cos\alpha - 1]\\
\mathsf{F}(\alpha) &= \frac{1}{\pi\alpha^2}[\alpha\sin\alpha + \cos\alpha - 1]
\end{aligned}
$$

Hence the Fourier transform of $f(x)$ is

$$
\mathsf{F}(\alpha) = \frac{1}{\pi\alpha^2}[\alpha\sin\alpha + \cos\alpha - 1]
$$

To obtain $f(x)$ given its fourier transform $\mathsf{F}(\alpha)$, then we apply the inverse fourier transform

$$
\begin{aligned}
f(x) &= \int_{-\infty}^{\infty} \mathsf{F}(\alpha)\, e^{i\alpha x}\, d\alpha\\
&= \int_{-\infty}^{\infty} \frac{1}{\pi\alpha^2}[\alpha\sin\alpha + \cos\alpha - 1]\ e^{i\alpha x}\, d\alpha\\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{\alpha^2}[\alpha\sin\alpha + \cos\alpha - 1]\ e^{i\alpha x}\, d\alpha
\end{aligned}
$$

10 chapter 15, problem 5.1

Problem Show that $g(t) \circledast h(t) = h(t) \circledast g(t)$

Solution

By definition,

$$g(t) \circledast h(t) = \int_0^t g(t-\tau)\, h(\tau)\, d\tau \tag{1}$$

Let $u = t - \tau$, $du = -d\tau$, when $\tau = 0$, $u = t$, when $\tau = t$, $u = 0$, Hence The RHS becomes

$$\begin{aligned}
\int_0^t g(t-\tau)\, h(\tau)\, d\tau &= \int_{u=t}^{u=0} g(u)\, h(t-u)\, (-du) \\
&= -\int_{u=t}^{u=0} g(u)\, h(t-u)\, du \\
&= \int_{u=0}^{u=t} g(u)\, h(t-u)\, du
\end{aligned}$$

Since u is a dummy variable of integration, call it anything we want, say τ so above integral becomes

$$\begin{aligned}
\int_0^t g(t-\tau)\, h(\tau)\, d\tau &= \int_0^t g(\tau)\, h(t-\tau)\, d\tau \\
\int_0^t g(t-\tau)\, h(\tau)\, d\tau &= \int_0^t h(t-\tau)\, g(\tau)\ \ d\tau
\end{aligned} \tag{2}$$

Hence from (2) $g(t) \circledast h(t) = \ h(t) \circledast g(t)$

11 chapter 15, problem 5.10

Problem

Use convolution integral to find the inverse transform of $\frac{1}{p\left(p^2+a^2\right)^2}$

Solution

$$\frac{1}{p\left(p^2+a^2\right)^2} = \frac{1}{p}\frac{1}{\left(p^2+a^2\right)^2} = GH$$

From Tables using L1 and L17 $g(t) = 1$ and $h(t) = \frac{\sin at - at\ \cos at}{2a^3}$. Hence the inverse transform of $GH = \ g(t) \circledast h(t)$. Using L34

$$g(t) \circledast h(t) = \int_0^t g(t-\tau)\, h(\tau)\, d\tau \tag{L34}$$

Hence

$$g(t) \circledast h(t) = \int_0^t 1 \times \frac{\sin a(t-\tau) - a(t-\tau)\ \cos a(t-\tau)}{2a^3}\quad d\tau$$
$$= \frac{1}{2a^3}\int_0^t \sin a(t-\tau) - a(t-\tau)\ \cos a(t-\tau)\, d\tau$$
$$= \frac{1}{2a^3}\left[\int_0^t \sin a(t-\tau)\, d\tau - at\int_0^t\ \cos a(t-\tau)d\tau + a\int_0^t \tau\cos a(t-\tau)\, d\tau\right] \tag{1}$$

The last integral can be integrated by parts. $u = \tau$, $v = \frac{\sin a(t-\tau)}{-a}$

$$\int_0^t \tau\cos a(t-\tau)\, d\tau = \left[\tau\frac{\sin a(t-\tau)}{-a}\right]_0^t - \int_0^t \frac{\sin a(t-\tau)}{-a}\, d\tau$$
$$= \frac{-1}{a}[\tau\sin a(t-\tau)]_0^t + \frac{1}{a}\int_0^t \sin a(t-\tau)\, d\tau$$
$$= \frac{-1}{a}[\tau\sin a(t-\tau)]_0^t + \frac{1}{a}\left[\frac{-\cos a(t-\tau)}{-a}\right]_0^t$$
$$= \frac{-1}{a}[\tau\sin a(t-\tau)]_0^t + \frac{1}{a^2}[\ \cos a(t-\tau)]_0^t$$
$$= \frac{-1}{a}[t\sin a(t-t) - 0] + \frac{1}{a^2}[\ \cos a(t-t) - \cos a(t-0)]$$
$$= \frac{-1}{a}[0] + \frac{1}{a^2}[\ \cos a(0) - \cos a(t)]$$
$$= \frac{1}{a^2}[\ 1 - \cos at]$$

Hence (1) becomes

$$g(t) \circledast h(t) = \frac{1}{2a^3}\left[\int_0^t \sin a(t-\tau)\, d\tau - at\int_0^t\ \cos a(t-\tau)d\tau + a\frac{1}{a^2}[\ 1-\cos at]\right]$$
$$g(t) \circledast h(t) = \frac{1}{2a^3}\left[\left[\frac{-\cos a(t-\tau)}{-a\tau}\right]_0^t - at\left[\frac{\sin a(t-\tau)}{-a\tau}\right] + \frac{1}{a}[\ 1-\cos at]\right]$$
$$= \frac{1}{2a^3}\left[\frac{1}{a}[\cos a(t-\tau)]_0^t + t\ [\ \sin a(t-\tau)]_0^t + \frac{1}{a}[\ 1-\cos at]\right]$$
$$= \frac{1}{2a^3}\left[\frac{1}{a}[\cos a(t-t) - \cos a(t-0)] + t\ [\ \sin a(t-t) - \sin a(t-0)]_0^t + \frac{1}{a}[\ 1-\cos at]\right]$$
$$= \frac{1}{2a^3}\left[\frac{1}{a}[1-\cos at] + t\ [\ -\sin at] + \frac{1}{a}[\ 1-\cos at]\right]$$
$$= \frac{1}{2a^3}\left[\frac{1}{a}[1-\cos at] - t\quad \sin at + \frac{1}{a}[\ 1-\cos at]\right]$$
$$= \frac{1}{2a^4}(2 - 2\cos at - at\quad \sin at)$$

So the inverse Laplace transform of $\frac{1}{p\left(p^2+a^2\right)^2}$ is

$$\frac{1}{2a^4}(2 - 2\cos at - at \quad \sin at)$$

12 chapter 15, problem 5.2

Problem Use L34 and L2 to find the inverse transform of $G\left(p\right)H(p)$ when $G(p) = \frac{1}{\left(p+a\right)}$ and $H(p) = \frac{1}{\left(p+b\right)}$ your result should be L7

Solution

$$\mathscr{L}\left(e^{-at}\right) = \frac{1}{p+a} \tag{L2}$$

$$g(t) \circledast h(t) = \int_0^t g(t-\tau)\, h(\tau)\, d\tau \tag{L34}$$

Using L2, $g(t) = \mathscr{L}^{-1}\frac{1}{\left(p+a\right)} = e^{-at}$, and $h(t) = \mathscr{L}^{-1}\frac{1}{\left(p+b\right)} = e^{-bt}$. Now Let $Y(p) = G(p)H(p)$, But

$$G(p)H(p) = \mathscr{L}\left\{\, g(t) \circledast h(t) \,\right\}$$

Then

$$\begin{aligned} y(t) = g(t) \circledast h(t) &= \int_0^t g(t-\tau)\, h(\tau)\, d\tau \\ &= \int_0^t e^{-a(t-\tau)} \;\; e^{-b\tau}\, d\tau \\ &= \int_0^t e^{-at+a\tau} \;\; e^{-b\tau}\, d\tau \\ &= \int_0^t e^{-at}\, e^{a\tau} \;\; e^{-b\tau}\, d\tau \end{aligned}$$

e^{-at} can be moved outside the integral

$$y(t) = e^{-at} \int_0^t e^{a\tau} \ e^{-b\tau} \, d\tau$$

$$y(t) = e^{-at} \int_0^t e^{a\tau - b\tau} \quad d\tau$$

$$y(t) = e^{-at} \int_0^t e^{\tau(a-b)} \quad d\tau$$

$$y(t) = e^{-at} \left[\frac{e^{\tau(a-b)}}{a-b} \right]_0^t$$

$$y(t) = \frac{e^{-at}}{a-b} \left[e^{t(a-b)} - 1 \right]$$

$$y(t) = \frac{e^{-at+t(a-b)} \ - e^{-at}}{a-b}$$

$$y(t) = \frac{e^{-bt} \ - e^{-at}}{a-b}$$

$$y(t) = \frac{e^{-at} \ - e^{-bt}}{b-a}$$

Which is L7 as required to show.

13 chapter 15, problem 5.22

Problem

Verify Parseval's theorem for $f(x) = e^{-|x|}$ and $g(\alpha)$ =Fourier transform of $f(x)$

Solution

Parseval theorem says that total energy in a signal equal to the sum of the energies in the harmonics that make up the signal. i.e.

$$\int_{-\infty}^{\infty} \left| g(\alpha) \right|^2 \, d\alpha = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| f(x) \right|^2 \, dx$$

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|f(x)\right|^2\,dx &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|e^{-|x|}\right|^2\,dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-2|x|}\,dx \\
&= \frac{1}{2\pi}\left\{\int_{-\infty}^{0} e^{2x}\,dx + \int_{0}^{\infty} e^{-2x}\,dx\right\} \\
&= \frac{1}{2\pi}\left\{\left[\frac{e^{2x}}{2}\right]_{-\infty}^{0} + \left[\frac{e^{-2x}}{-2}\right]_{0}^{\infty}\right\} \\
&= \frac{1}{4\pi}\left\{\left[e^{2x}\right]_{-\infty}^{0} - \left[e^{-2x}\right]_{0}^{\infty}\right\} \\
&= \frac{1}{4\pi}\left\{\left[e^{0}-0\right] - \left[0-e^{0}\right]\right\} \\
&= \frac{1}{4\pi}\{1+1\} \\
&= \frac{1}{2\pi} \qquad (1)
\end{aligned}
$$

Now we find the Fourier transform for $f(x)$

$$
\begin{aligned}
g(\alpha) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|x|}e^{-ix\alpha}dx \\
&= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{x}e^{-ix\alpha}dx + \int_{0}^{\infty} e^{-x}e^{-ix\alpha}dx\right] \\
&= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{x(1-i\alpha)}\,dx + \int_{0}^{\infty} e^{\,x(-1-i\alpha)}\,dx\right] \\
&= \frac{1}{2\pi}\left(\left[\frac{e^{x(1-i\alpha)}}{1-i\alpha}\right]_{-\infty}^{0} + \left[\frac{e^{x(-1-i\alpha)}}{-1-i\alpha}\right]_{0}^{\infty}\right) \\
&= \frac{1}{2\pi}\left(\frac{1}{1-i\alpha}\left[e^{x(1-i\alpha)}\right]_{-\infty}^{0} - \frac{1}{1+i\alpha}\left[e^{x(-1-i\alpha)}\right]_{0}^{\infty}\right) \\
&= \frac{1}{2\pi}\left(\frac{1}{1-i\alpha}\left[e^{x(1-i\alpha)}\right]_{-\infty}^{0} - \frac{1}{1+i\alpha}\left[e^{x(-1-i\alpha)}\right]_{0}^{\infty}\right) \\
&= \frac{1}{2\pi}\left(\frac{1}{1-i\alpha}\left[1-e^{-\infty(1-i\alpha)}\right] - \frac{1}{1+i\alpha}\left[e^{\infty(-1-i\alpha)}-1\right]\right) \\
&= \frac{1}{2\pi}\left(\frac{1}{1-i\alpha}[1] - \frac{1}{1+i\alpha}[-1]\right) \\
&= \frac{1}{2\pi}\left(\frac{1}{1-i\alpha} + \frac{1}{1+i\alpha}\right) \\
&= \frac{1}{2\pi}\left(\frac{1+i\alpha+1-i\alpha}{(1-i\alpha)(1+i\alpha)}\right) \\
&= \frac{1}{2\pi}\left(\frac{2}{1+\alpha^2}\right) \\
&= \frac{1}{\pi\left(1+\alpha^2\right)}
\end{aligned}
$$

So

$$
\begin{aligned}
\int_{-\infty}^{\infty}\left|g(\alpha)\right|^2 d\alpha &= \int_{-\infty}^{\infty}\left|\frac{1}{\pi\left(1+\alpha^2\right)}\right|^2 d\alpha \\
&= \frac{1}{\pi^2}\int_{-\infty}^{\infty}\frac{1}{\left(1+\alpha^2\right)^2}d\alpha
\end{aligned}
$$

But $\int_{-\infty}^{\infty}\frac{1}{\left(1+\alpha^2\right)^2}d\alpha = \frac{\pi}{2}$, Hence

$$
\begin{aligned}
\int_{-\infty}^{\infty}\left|g(\alpha)\right|^2 d\alpha &= \frac{1}{\pi^2}\left(\frac{\pi}{2}\right) \\
&= \frac{1}{2\pi} \qquad (2)
\end{aligned}
$$

Comparing (1) and (2). They are the same. Hence

$$\int_{-\infty}^{\infty} \left|g(\alpha)\right|^2 d\alpha = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|f(x)\right|^2 dx$$

was verified for this problem as required.

14 chapter 15, problem 5.4

Problem

Use convolution integral to find the inverse transform of $\frac{1}{(p+a)(p+b)^2}$

Solution

$$\frac{1}{(p+a)(p+b)^2} = \frac{1}{(p+a)}\frac{1}{(p+b)^2}$$

From but from L6

$$\mathscr{L}\left(t^k e^{-at}\right) = \frac{k!}{(p+a)^{k+1}} \tag{L2}$$

Hence $\frac{1}{(p+a)} = \mathscr{L}\left(e^{-at}\right)$ and $\frac{1}{(p+b)^2} = \mathscr{L}\left(e^{-bt}\right)$. Hence the inverse transform of $\frac{1}{(p+a)}\frac{1}{(p+b)^2} = e^{-at} \circledast te^{-bt}$. Using L34

$$g(t) \circledast h(t) = \int_0^t g(t-\tau)\, h(\tau)\, d\tau \tag{L34}$$

Hence

$$\begin{aligned}
e^{-at} \circledast te^{-bt} &= \int_0^t e^{-a(t-\tau)}\ \tau e^{-b\tau}\ d\tau \\
&= \int_0^t e^{-at+a\tau}\ \tau e^{-b\tau}\ d\tau \\
&= \int_0^t e^{-at}e^{a\tau}\ \tau e^{-b\tau}\ d\tau \\
&= e^{-at}\int_0^t e^{a\tau}\ \tau e^{-b\tau}\ d\tau \\
&= e^{-at}\int_0^t \tau e^{\tau(a-b)}\ \ d\tau
\end{aligned}$$

Integrate by parts. $u=\tau$, $v=\frac{e^{\tau(a-b)}}{a-b}$

$$
\begin{aligned}
e^{-at}\circledast te^{-bt} &= e^{-at}\left\{\left[\tau\frac{e^{\tau(a-b)}}{a-b}\right]_0^t-\int_0^t\frac{e^{\tau(a-b)}}{a-b}\,d\tau\right\}\\
&= e^{-at}\left\{\left[\tau\frac{e^{\tau(a-b)}}{a-b}\right]_0^t-\frac{1}{a-b}\left[\frac{e^{\tau(a-b)}}{a-b}\right]_0^t\right\}\\
&= e^{-at}\left(\frac{1}{a-b}\left[\tau e^{\tau(a-b)}\right]_0^t-\frac{1}{(a-b)^2}\left[e^{\tau(a-b)}\right]_0^t\right)\\
&= e^{-at}\left(\frac{1}{a-b}\left[te^{t(a-b)}\right]-\frac{1}{(a-b)^2}\left[e^{t(a-b)}-1\right]\right)\\
&= e^{-at}\left(\frac{te^{ta-tb}}{a-b}-\frac{e^{ta-tb}-1}{(a-b)^2}\right)\\
&= \frac{te^{-tb}}{a-b}-\frac{e^{-tb}-e^{-at}}{(a-b)^2}\\
&= \frac{(a-b)te^{-tb}-e^{-tb}+e^{-at}}{(a-b)^2}\\
&= \frac{((a-b)t-1)e^{-tb}+e^{-at}}{(a-b)^2}
\end{aligned}
$$

So the inverse laplace transform of $\frac{1}{(p+a)(p+b)^2}$ is

$$\frac{((a-b)t-1)\,e^{-tb}+e^{-at}}{(a-b)^2}$$

15 chapter 15, problem 6.2

Problem

Find the inverse laplace transform using 6.6 of the function $\frac{1}{p^4-1}$

Solution

6.6 states that $f(t)=$ sum of all residues of $F(z)e^{zt}$ at all poles. Poles of $F(z)=\frac{1}{z^4-1}$are at $\pm 1,\pm i$. Hence

$$F(z)=\frac{e^{zt}}{(z-1)(z+1)(z-i)(z+i)}$$

Since each pole is of order 1, we use equation 6.1 page 599 which says

$$\text{Residue of } F(z) \text{ at } z=z_0 \text{ is } \lim_{z\to z_0}(z-z_0)F(z)$$

Hence sum of residue is

$$\begin{aligned}R &= \lim_{z\to+1}\frac{e^{zt}}{(z+1)(z-i)(z+i)}+\lim_{z\to-1}\frac{e^{zt}}{(z-1)(z-i)(z+i)}\\&+\lim_{z\to+i}\frac{e^{zt}}{(z-1)(z+1)(z+i)}+\lim_{z\to-i}\frac{e^{zt}}{(z-1)(z+1)(z-i)}\end{aligned}$$

$$\begin{aligned}&=\frac{e^{t}}{(1+1)(1-i)(1+i)}+\frac{e^{-t}}{(-1-1)(-1-i)(-1+i)}\\&+\frac{e^{it}}{(i-1)(i+1)(i+i)}+\frac{e^{-it}}{(-i-1)(-i+1)(-i-i)}\end{aligned}$$

$$\begin{aligned}&=\frac{e^{t}}{4}+\frac{e^{-t}}{-4}+\frac{e^{it}}{-4i}+\frac{e^{-it}}{4i}\\&=\left(\frac{e^{t}-e^{-t}}{4}\right)-\frac{1}{2}\left(\frac{e^{it}-e^{-it}}{2i}\right)\\&=\left(\frac{e^{t}-e^{-t}}{4}\right)-\frac{1}{2}(\sin t)\end{aligned}$$

16 chapter 15, problem 6.4

Problem

Find the inverse laplace transform using 6.6 of the function $\frac{p^3}{p^4-16}$

Solution

6.6 states that $f(t)=$ sum of all residues of $F(z)e^{zt}$ at all poles. Poles of $F(z)=\frac{z^3}{z^4-16}$ are at $\pm 2,\pm 2i$, Hence

$$F(z)=\frac{z^3e^{zt}}{(z-2)(z+2)(z-2i)(z+2i)}$$

Since each pole is of order 1, we use equation 6.1 page 599 which says

$$\text{Residue of } F(z) \text{ at } z=z_0 \text{ is } \lim_{z\to z_0}(z-z_0)F(z)$$

Hence sum of residue is

$$R = \lim_{z\to+2}\frac{z^3e^{zt}}{(z+2)(z-2i)(z+2i)} + \lim_{z\to-2}\frac{z^3e^{zt}}{(z-2)(z-2i)(z+2i)} + \lim_{z\to+2i}\frac{z^3e^{zt}}{(z-2)(z+2)(z+2i)} + \lim_{z\to-2i}\frac{z^3e^{zt}}{(z-2)(z+2)(z-2i)}$$

$$= \frac{8e^{2t}}{(2+2)(2-2i)(2+2i)} + \frac{-8e^{-2t}}{(-2-2)(-2-2i)(-2+2i)} + \frac{(2i)^3e^{2it}}{(2i-2)(2i+2)(2i+2i)} + \frac{(-2i)^3e^{-2it}}{(-2i-2)(-2i+2)(-2i-2i)}$$

$$\begin{aligned}
&= \frac{8e^{2t}}{(4)\,8} + \frac{-8e^{-2t}}{(-4)8} + \frac{(-8i)e^{2it}}{-8(4i)} + \frac{(8i)e^{-2it}}{-8(-4i)} \\
&= \frac{e^{2t}}{4} + \frac{e^{-2t}}{4} + \frac{e^{2it}}{4} + \frac{e^{-2it}}{4} \\
&= \frac{e^{2t}+e^{-2t}}{4} + \frac{1}{2}(\cos 2t)
\end{aligned}$$

So inverse Laplace transform of $\frac{p^3}{p^4-16}$ is

$$\frac{e^{2t}+e^{-2t}}{4} + \frac{1}{2}(\cos 2t)$$

17 chapter 15, problem 6.5

Problem

Find the inverse laplace transform using 6.6 of the function $\frac{3p^2}{p^3+8}$

Solution

6.6 states that $f(t)$ = sum of all residues of $F(z)e^{zt}$ at all poles. To find poles, look at $p^3+8=0$, hence $p^3=-8$, $p=-8^{\frac{1}{3}}$. Let

$$8^{\frac{1}{3}} = re^{i\theta}$$

Then the roots are

$$
\begin{aligned}
&= 8^{\frac{1}{3}} e^{\frac{i0}{3}}, 8^{\frac{1}{3}} e^{\frac{i(0+2\pi)}{3}}, 8^{\frac{1}{3}} e^{\frac{i(0+4\pi)}{3}} \\
&= 8^{\frac{1}{3}}\ , 8^{\frac{1}{3}}(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}), 8^{\frac{1}{3}}(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}) \\
&= 2\ , 2(\cos 120^0 + i\sin 120^0), 2(\cos 240^0 + i\sin 240^0) \\
&= 2\ , 2(-\frac{1}{2} + i\frac{\sqrt{3}}{2}), 2(-\frac{1}{2} + i\frac{-\sqrt{3}}{2}) \\
&= 2, -1 + i\sqrt{3}\ , -1 - i\sqrt{3}
\end{aligned}
$$

Hence

$$p = -2, 1 - i\sqrt{3}\ , 1 + i\sqrt{3}$$

And

$$
\begin{aligned}
F(z) &= \frac{3z^2}{z^3+8} e^{zt} \\
&= \frac{3z^2}{(z+2)\left(z - \left(1 - i\sqrt{3}\right)\right)\left(z - \left(1 + i\sqrt{3}\right)\right)} e^{zt}
\end{aligned}
$$

Since each pole is of order 1, we use equation 6.1 page 599 which says

$$\text{Residue of } F(z) \text{ at } z = z_0 \ \text{ is } \lim_{z\to z_0} \ (z - z_0)F(z)$$

Hence sum of residue is

$$R = \lim_{z\to -2} \frac{3z^2}{\left(z-\left(1-i\sqrt{3}\right)\right)\left(z-\left(1+i\sqrt{3}\right)\right)}e^{zt} + \lim_{z\to\left(1-i\sqrt{3}\right)} \frac{3z^2}{(z+2)\left(z-\left(1+i\sqrt{3}\right)\right)}e^{zt}$$
$$+ \lim_{z\to\left(1+i\sqrt{3}\right)} \frac{3z^2}{(z+2)\left(z-\left(1-i\sqrt{3}\right)\right)}e^{zt}$$
$$= \frac{3(-2)^2}{\left(-2-\left(1-i\sqrt{3}\right)\right)\left(-2-\left(1+i\sqrt{3}\right)\right)}e^{-2t} + \frac{3\left(1-i\sqrt{3}\right)^2}{\left(\left(1-i\sqrt{3}\right)+2\right)\left(\left(1-i\sqrt{3}\right)-\left(1+i\sqrt{3}\right)\right)}e^{\left(1-i\sqrt{3}\right)t}$$
$$+ \frac{3\left(1+i\sqrt{3}\right)^2}{\left(\left(1+i\sqrt{3}\right)+2\right)\left(\left(1+i\sqrt{3}\right)-\left(1-i\sqrt{3}\right)\right)}e^{\left(1+i\sqrt{3}\right)t}$$
$$= \frac{12}{\left(-3+i\sqrt{3}\right)\left(-3-i\sqrt{3}\right)}e^{-2t} + \frac{3\left(-2-2i\sqrt{3}\right)}{\left(3-i\sqrt{3}\right)\left(-2i\sqrt{3}\right)}e^{\left(1-i\sqrt{3}\right)t} + \frac{3\left(-2+2i\sqrt{3}\right)}{\left(3+i\sqrt{3}\right)\left(2i\sqrt{3}\right)}e^{\left(1+i\sqrt{3}\right)t}$$
$$= \frac{12}{12}e^{-2t} + \frac{-6-6i\sqrt{3}}{-6i\sqrt{3}-6}e^{\left(1-i\sqrt{3}\right)t} + \frac{-6+6i\sqrt{3}}{6i\sqrt{3}-6}e^{\left(1+i\sqrt{3}\right)t}$$
$$= e^{-2t} + e^{\left(1-i\sqrt{3}\right)t} + e^{\left(1+i\sqrt{3}\right)t}$$
$$= e^{-2t} + e^{t}e^{-i\sqrt{3}t} + e^{t}e^{i\sqrt{3}t}$$
$$= e^{-2t} + e^{t}\left(e^{i\sqrt{3}t} + e^{-i\sqrt{3}t}\right)$$
$$= e^{-2t} + e^{t}\left(2\cos\sqrt{3}\,t\right)$$

So inverse Laplace transform of $\frac{3p^2}{p^3+8}$ is

$$e^{-2t} + e^{t}\left(2\cos\sqrt{3}\,t\right)$$

18 chapter 15, problem 6.9

Problem

Find the inverse laplace transform using 6.6 of the function $\frac{p}{p^4-1}$

Solution

6.6 states that $f(t) =$ sum of all residues of $F(z)e^{zt}$ at all poles. To find poles, look at $p^4-1=0$, hence $p^4=1$, $p=1^{\frac{1}{4}}$. Let

$$1^{\frac{1}{4}} = re^{i\theta}$$

Then roots are

$$
\begin{aligned}
&= 1^{\frac{1}{4}} e^{\frac{i0}{4}}, 1^{\frac{1}{4}} e^{\frac{i(0+2\pi)}{4}}, 1^{\frac{1}{4}} e^{\frac{i(0+4\pi)}{4}}, ^{\frac{1}{4}} e^{\frac{i(0+6\pi)}{4}} \\
&= 1\ , 1\left(\cos\frac{2\pi}{4} + i\sin\frac{2\pi}{4}\right), 1\left(\cos\frac{4\pi}{4} + i\sin\frac{4\pi}{4}\right), 1\left(\cos\frac{6\pi}{4} + i\sin\frac{6\pi}{4}\right) \\
&= 1\ , (0+i), (-1+i\,0), (0-i\,1) \\
&= 1, i, -1, -i
\end{aligned}
$$

Therefore

$$p = 1, i, -1, -i$$

Hence

$$
\begin{aligned}
F(z) &= \frac{z}{z^4-1} e^{zt} \\
&= \frac{z}{(z-1)(z-i)(z+1)(z+i)} e^{zt}
\end{aligned}
$$

Since each pole is of order 1, we use equation 6.1 page 599 which says

$$\text{Residue of } F(z) \text{ at } z = z_0 \text{ is } \lim_{z\to z_0} (z-z_0)F(z)$$

Hence sum of residue is

$$
\begin{aligned}
R &= \lim_{z\to 1} \frac{z}{(z-i)(z+1)(z+i)} e^{zt} + \lim_{z\to i} \frac{z}{(z-1)(z+1)(z+i)} e^{zt} \\
&\quad + \lim_{z\to -1} \frac{z}{(z-1)(z-i)(z+i)} e^{zt} + \lim_{z\to -i} \frac{z}{(z-1)(z-i)(z+1)} e^{zt} \\
&= \frac{1}{(1-i)(1+1)(1+i)} e^{t} + \frac{i}{(i-1)(i+1)(i+i)} e^{it} \\
&\quad + \frac{-1}{(-1-1)(-1-i)(-1+i)} e^{-t} + \frac{-i}{(-i-1)(-i-i)(-i+1)} e^{-it} \\
&= \frac{1}{4} e^{t} + \frac{i}{-4i} e^{it} + \frac{1}{4} e^{-t} + \frac{-i}{4i} e^{-it} \\
&= \frac{1}{4}\left(e^{t} + e^{-t}\right) - \frac{1}{2}(\cos t)
\end{aligned}
$$

So inverse Laplace transform of $\frac{p}{p^4-1}$ is

$$\frac{1}{4}\left(e^{t} + e^{-t}\right) - \frac{1}{2}(\cos t)$$

19 chapter 15, problem 7.11

Problem

Using the δ function method, Find the response of the following system to a unit impulse. $\frac{d^4y}{dy^4} - y = \delta(t - t_0)$

Solution

Taking the laplace transform of each side gives (assuming initial conditions for the system are at rest)

$$\begin{aligned} Yp^4 - Y &= e^{-pt_0} \\ Y &= \frac{e^{-pt_0}}{p^4 - 1} \\ Y &= \frac{e^{-pt_0}}{\left(p^2 - 1\right)\left(p^2 + 1\right)} \end{aligned}$$

Finding the inverse laplace of $\frac{1}{(p-1)(p+1)(p^2+1)} = \frac{1}{(p-1)(p+1)}\frac{1}{(p^2+1)} = GH$. Then $g(t) = \frac{e^t - e^{-t}}{2}$ using L7 and, $h(t) = \sin t$ using L3. Hence the inverse transform is

$$\begin{aligned} g(t) \circledast h(t) &= \int_0^t \frac{e^{\tau} - e^{-\tau}}{2} \sin(t - \tau)\, d\tau \\ &= \frac{1}{2}(\sinh t - \sin t) \end{aligned}$$

Using L28 with the result above we get

$$y(t) = \begin{cases} \frac{1}{2}(\sinh(t - t_0) - \sin(t - t_0)) & t > t_0 \\ 0 & t < t_0 \end{cases}$$

Or by expressing sinh using exp, the above becomes

$$y(t) = \begin{cases} \frac{1}{4}\left(e^{t - t_0} - e^{-t + t_0} - 2\sin(t - t_0)\right) & t > t_0 \\ 0 & t < t_0 \end{cases}$$

20 chapter 15, problem 7.7

Problem

Using the δ function method, Find the response of the following system to a unit impulse. $y'' + 2y' + y = \delta(t - t_0)$

Solution

Take the laplace transform of each side we get (assume initial conditions for a system at rest)

$$\begin{aligned} Yp^2 + 2Yp + Y = e^{-pt_0} \\ Y &= \frac{e^{-pt_0}}{p^2 + 2p + 1} \\ Y &= \frac{e^{-pt_0}}{\left(p + 1\right)^2} \end{aligned}$$

Using L28 and L6 (for $k = 1$)

$$y(t) = \begin{cases} (t - t_0)e^{-(t-t_0)} & t > t_0 \\ 0 & t < t_0 \end{cases}$$

21 chapter 15, problem 7.9

Problem

Using the δ function method, Find the response of the following system to a unit impulse. $y'' + 2y' + 10y = \delta(t - t_0)$

Solution

Taking the laplace transform of each side we get (assume initial conditions for a system at rest)

$$\begin{aligned} Yp^2 + 2Yp + 10Y = e^{-pt_0} \\ Y &= \frac{e^{-pt_0}}{p^2 + 2p + 10} \\ Y &= \frac{e^{-pt_0}}{\left(p - a\right)\left(p - b\right)} \end{aligned}$$

Where $a = -1 + 3i,\ b = -1 - 3i$ the roots of $p^2 + 2p + 10$. Using L28 and L7

$$y(t) = \begin{cases} \frac{e^{a\left(t-t_0\right)} - e^{b\left(t-t_0\right)}}{(-b)-(-a)} & t > t_0 \\ 0 & t < t_0 \end{cases}$$

Replacing values for a, b gives

$$\begin{aligned}
y(t) &= \frac{e^{a(t-t_0)} - e^{b(t-t_0)}}{a-b} \\
&= \frac{e^{(-1+3i)(t-t_0)} - e^{(-1-3i)(t-t_0)}}{(-1+3i)-(-1-3i)} \\
&= \frac{e^{(-1+3i)(t-t_0)} - e^{(-1-3i)(t-t_0)}}{6i} \\
&= \frac{e^{-t+t_0+3it-3it_0} - e^{-t+t_0-3it+3it_0}}{6i} \\
&= e^{-t+t_0}\,\frac{e^{3i(t-t_0)} - e^{-3i(t-t_0)}}{6i} \\
&= e^{-t+t_0}\left(\frac{1}{3}\right)\left(\frac{e^{3i(t-t_0)} - e^{-3i(t-t_0)}}{2i}\right) \\
&= \frac{e^{-t+t_0}}{3}\sin 3(t-t_0)
\end{aligned}$$

Therefore

$$y(t) = \begin{cases} \frac{e^{-t+t_0}}{3}\sin 3(t-t_0) & t > t_0 \\ 0 & t < t_0 \end{cases}$$