Math 121 A

HW # 10

$\frac{2}{2}$

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UCB extension

Ch 7, problem 4.12.

Find average value of $\cos^2 \frac{7\pi}{2}x$ over $(0, \frac{8}{7})$.

by definition, this is $I = \frac{1}{8/7}\int_0^{8/7} \cos^2 \frac{7\pi}{2}x \, dx$

$$\cos^2 \alpha = \frac{1+\cos(2\alpha)}{2}$$

so $\cos^2\left(\frac{7\pi}{2}x\right) = \frac{1+\cos(7\pi x)}{2}$ ✓

so $I = \frac{7}{8}\int_0^{8/7} \frac{1+\cos(7\pi x)}{2}dx = \frac{7}{16}\int_0^{8/7} 1+\cos(7\pi x)\,dx$

$$= \frac{7}{16}\left[x + \frac{1}{7\pi}\sin(7\pi x)\right]_0^{8/7} \checkmark$$

$$= \frac{7}{16}\left[\frac{8}{7} + \frac{1}{7\pi}\sin\left(7\pi\frac{8}{7}\right) - \left(0 + \frac{1}{7\pi}\sin(0)\right)\right]$$

$$= \frac{7}{16}\left[\frac{8}{7} + \frac{1}{7\pi}\sin(8\pi) - 0\right] = \frac{7}{16}\left[\frac{8}{7} + 0\right] = \left(\frac{7}{16}\right)\left(\frac{8}{7}\right) = \boxed{\frac{1}{2}}$$

Chapter 7
problem 4.13.

Using 4.3 and a method similar to 4.5 to 4.7 show that

$$\int_a^b \sin^2 kx\, dx = \int_a^b \cos^2 kx\, dx = \frac{1}{2}(b-a)$$

if $k(b-a)$ is multiple of π.

eq. 4.3 is   Av. $f(x)$ over $(a,b) = \frac{1}{b-a}\int_a^b f(x)\,dx$

$$\sin^2(kx) = \frac{1-\cos(2kx)}{2}$$ ✓

so $$\int_a^b \sin^2(kx)dx = \frac{1}{2}\int_a^b 1-\cos(2kx)\,dx = \frac{1}{2}\left[x - \frac{1}{2k}\sin(2kx)\right]_a^b$$

$$= \frac{1}{2}\left[b - \frac{1}{2k}\sin(2kb) - a + \frac{1}{2k}\sin(2ka)\right]$$

$$= \frac{1}{2}\left[(b-a) + \frac{1}{2k}\left(\sin(2ka) - \sin(2kb)\right)\right]$$ ✓

but $\sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$

so $$\int_a^b \sin^2(kx)dx = \frac{1}{2}\left[(b-a) + \frac{1}{2k}\left(2\sin\left(\frac{2ka-2kb}{2}\right)\cos\left(\frac{2ka+2kb}{2}\right)\right)\right]$$

$$\int_a^b \sin^2(kx)dx = \frac{1}{2}\left[(b-a) + \frac{1}{k}\left(\sin(k(a-b))\cos(k(a+b))\right)\right] \quad (1)$$ ✓

similarly

$$\int_a^b \cos^2(kx)\,dx = \frac{1}{2}\int_a^b 1+\cos(2kx)\,dx$$

$$= \frac{1}{2}\left[x + \frac{1}{2k}\sin(2kx)\right]_a^b \longrightarrow$$

$$= \frac{1}{2}\left[b + \frac{1}{2k}\sin(2kb) - a - \frac{1}{2k}\sin(2ka) \right]$$

$$= \frac{1}{2}\left[(b-a) + \frac{1}{2k}\left(\sin(2kb) - \sin(2ka) \right) \right]$$

but $\sin(\alpha) - \sin(\beta) = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$

so

$$= \frac{1}{2}\left[(b-a) + \frac{1}{2k}\left(2\sin\left(\frac{2kb-2ka}{2}\right)\cos\left(\frac{2kb+2ka}{2}\right) \right) \right]$$

$$= \frac{1}{2}\left[(b-a) + \frac{1}{k}\left(\sin(k(b-a))\cos(k(a+b)) \right) \right] \quad (2)$$

looking at (1) and (2) above we see that

if $k(b-a)$ is multiple of π, we get

$\sin k(b-a) \longrightarrow 0$, hence we get simplification to:

$$\int_a^b \sin^2 kx\, dx = \frac{1}{2}(b-a)$$

and $\int_a^b \cos^2 kx\, dx = \frac{1}{2}(b-a)$

if $k(b-a) = n\pi$

or $k = \frac{n\pi}{(b-a)}$

n integer

chapter 7
problem 4.15

evaluate $\int_{-\frac{1}{4}}^{\frac{11}{4}} \cos^2 \pi x \, dx$

$a = -\frac{1}{4}$, $b = \frac{11}{4}$

the limit is $\left[\frac{11}{4} + \frac{1}{4}\right] = \frac{12}{4} = 3$

but from problem 13 we found that

$$\int_a^b \cos^2 kx = \frac{1}{2}(b-a)$$

if $k(b-a) = n\pi$ for n integer.

here $(b-a) = 3$

and $k = \pi$.

hence $k(b-a)$ is a multiple of π. so we can use result from problem 13.

i.e.

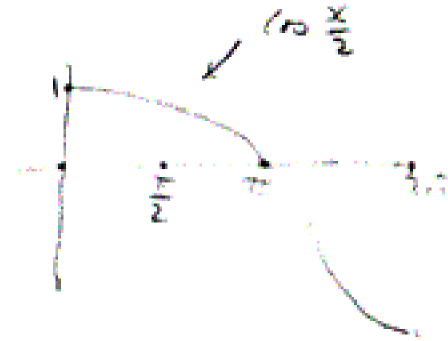
$$\int_{-\frac{1}{4}}^{\frac{11}{4}} \cos^2 \pi x \, dx = \frac{1}{2}(b-a) = \frac{1}{2}\left(\frac{11}{4} + \frac{1}{4}\right)$$

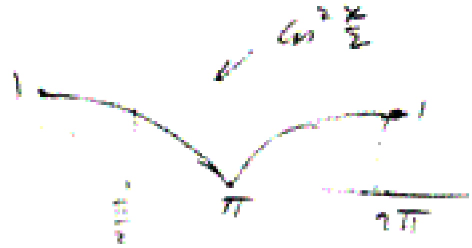
$$= \boxed{\frac{3}{2}}$$

HW#10

ch 7, 45 Find average value of

$\cos^2 \frac{x}{2}$ on $(0, \frac{\pi}{2})$





by definition, average value of $f(x)$ over (a, b) is

$$\frac{1}{(b-a)} \int_a^b f(x)\,dx = \frac{1}{\pi/2} \int_0^{\pi/2} \cos^2 \frac{x}{2}\,dx.$$

but $\cos^2(a) = \frac{1+\cos(2a)}{2}$. so $\cos^2(\frac{x}{2}) = \frac{1+\cos(x)}{2}$.

so $I = \frac{2}{\pi} \int_0^{\pi/2} \frac{1+\cos(x)}{2}\,dx = \frac{1}{\pi} \int_0^{\pi/2} 1+\cos(x)\,dx$

$= \frac{1}{\pi} \left[x + \sin(x) \right]_0^{\pi/2} = \frac{1}{\pi} \left[(\frac{\pi}{2} + 1) - (0+0) \right] = \frac{1}{\pi} \left[\frac{\pi}{2} + 1 \right]$

$= \boxed{\frac{1}{2} + \frac{1}{\pi}}$

chapter 7

problem 5.2

Given function over interval $-\pi < x < \pi$, sketch it over several periods. Expand the periodic function in a sin-cosine Fourier series.

$$f(x) = \begin{cases} 1 & 0 < x < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x < \pi \end{cases}$$

expand over several periods

now to expand in Fourier series

$$a_n = \frac{\int_{-\pi}^{\pi} f(x) \cos nx \, dx}{\int_{-\pi}^{\pi} \cos^2 nx \, dx} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \qquad n \geq 0$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} 0 \cos nx + \int_{0}^{\pi/2} 1 \cdot \cos nx \, dx + \int_{\pi/2}^{\pi} 0 \cos nx \, dx \right]$$

$$= \frac{1}{\pi} \int_{0}^{\pi/2} \cos nx \, dx \quad \begin{cases} \frac{1}{\pi} \left[\frac{1}{n} \sin nx \right]_0^{\pi/2} = \frac{1}{\pi} \left[\frac{1}{n} \sin \frac{n\pi}{2} \right] = \frac{1}{\pi n} \sin \frac{n\pi}{2} \quad (n \neq 0) \\ \frac{1}{\pi} \left[x \right]_0^{\pi/2} = \frac{1}{\pi} \left[\frac{\pi}{2} \right] = \frac{1}{2} \quad (n = 0) \end{cases}$$

$$a_n = \begin{cases} \frac{1}{n\pi} \sin \frac{n\pi}{2} & n \neq 0 \\ \frac{1}{2} & n = 0 . \end{cases}$$

so the cosin series is

$$= \frac{1}{4} + \left(\frac{1}{\pi} \sin \frac{\pi}{2}\right) \cos x + \left(\frac{1}{2\pi} \sin \frac{2\pi}{2}\right) \cos 2x + \left(\frac{1}{3\pi} \sin \frac{3\pi}{2}\right) \cos 3x + \left(\frac{1}{4\pi} \sin \frac{4\pi}{2}\right) \cos 4x$$

$$+ \left(\frac{1}{5\pi} \sin \frac{5\pi}{2}\right) \cos 5\pi + \ldots$$

$$\boxed{= \frac{1}{4} + \frac{1}{\pi} \cos x - \frac{1}{3\pi} \cos 3x + \frac{1}{5\pi} \cos 5\pi - \ldots\ldots} \qquad (1)$$

now to find the 'sin' series part

$$b_n = \frac{\int_{-\pi}^{\pi} f(x) \sin nx \, dx}{\int_{-\pi}^{\pi} \sin^2 nx \, dx} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \qquad n > 0 .$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} 0 \sin nx + \int_{0}^{\pi/2} 1 \sin nx \, dx + \int_{\pi/2}^{\pi} 0 \cdot dx \right]$$

$$= \frac{1}{\pi} \left[-\frac{1}{n} \cos nx \Big|_{0}^{\pi/2} \right] = -\frac{1}{n\pi} \left[\cos \frac{n\pi}{2} - \cos 0 \right] =$$

$$= -\frac{1}{n\pi} \left[\cos \frac{n\pi}{2} - 1 \right] = \frac{1}{n\pi} - \frac{1}{n\pi} \cos \frac{n\pi}{2}$$

so sin series is

$$b_n = \left(\frac{1}{\pi} - \frac{1}{\pi} \cos \frac{\pi}{2}\right) \sin x + \left(\frac{1}{2\pi} - \frac{1}{2\pi} \cos \pi\right) \sin 2x + \left(\frac{1}{3\pi} - \frac{1}{3\pi} \cos \frac{3\pi}{2}\right) \sin 3x +$$

$$\left(\frac{1}{4\pi} - \frac{1}{4\pi} \cos \frac{4\pi}{2}\right) \sin 4x + \left(\frac{1}{5\pi} - \frac{1}{5\pi} \cos \frac{5\pi}{2}\right) \sin 5x + \ldots \longrightarrow$$

$$b_n = \frac{1}{\pi}\sin x + \left(\frac{1}{2\pi}+\frac{1}{2\pi}\right)\sin 2x + \left(\frac{1}{3\pi}\right)\sin 3x + \left(\frac{1}{4\pi}-\frac{1}{4\pi}\right)\sin 4x$$

$$+ \left(\frac{1}{5\pi}\right)\sin 5x + \cdots$$

$$= \frac{1}{\pi}\sin x + \frac{2\sin 2x}{2\pi} + \frac{\sin 3x}{3\pi} + \frac{\sin 5x}{5\pi} + \cdots$$

$$\boxed{= \frac{1}{\pi}\left(\sin x + \frac{2\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right)} \quad (2)$$

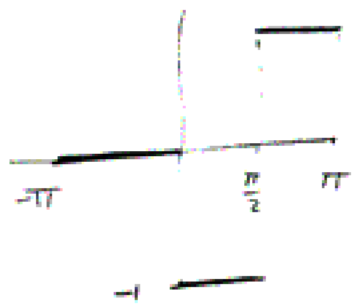
so $f(x) = (1) + (2)$

as shown above.

problem 5, chapter 7, section 5.

find Fourier Series for

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ -1 & 0 < x < \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < x < \pi \end{cases}$$



for extend to 2π period:

to find fourier series.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\left[\int_0^{\pi/2}(-1)\cos nx\, dx + \int_{\pi/2}^{\pi}\cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[-\int_0^{\pi/2}\cos nx\, dx + \int_{\pi/2}^{\pi}\cos nx\, dx\right]$$

for $n=0$ $\quad a_0 = \frac{1}{\pi}\left[\left(-\frac{\pi}{2}\right) + \left(\frac{\pi}{2}\right)\right] = 0$

for $n \neq 0$

$$a_n = \frac{1}{\pi}\left[-\left[\frac{1}{n}\sin nx\right]_0^{\pi/2} + \left[\frac{1}{n}\sin nx\right]_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}\left[\sin\frac{n\pi}{2} - 0\right] + \frac{1}{n}\left[\sin n\pi - \sin\frac{n\pi}{2}\right]\right]$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}\left(\sin\frac{n\pi}{2}\right) + \frac{1}{n}\left(-\sin\frac{n\pi}{2}\right)\right] = \frac{-2}{n\pi}\sin\frac{n\pi}{2}$$

so looking at few terms:

$a_1 = \quad -\frac{2}{\pi}$

$a_2 = \quad \frac{-2}{2\pi}(0) = 0$

$a_3 = \quad \frac{-2}{3\pi}\sin\left(\frac{3}{2}\pi\right) = +\frac{2}{3\pi}$

so a series is

$$\boxed{-\frac{2}{\pi}\cos x + \frac{2}{3\pi}\cos 3x - \frac{2}{5\pi}\cos 5x + \cdots}$$

notice, no a_0 term.

now to find the sin series:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\left(\int_0^{\pi/2} -\sin nx\, dx + \int_{\pi/2}^{\pi} \sin nx\, dx\right)$$

$$= \frac{1}{\pi}\left(-\left(-\frac{\cos nx}{n}\right)_0^{\pi/2} + \left(-\frac{\cos nx}{n}\right)_{\pi/2}^{\pi} \right)$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\left[\cos nx\right]_0^{\pi/2} - \frac{1}{n}\left[\cos nx\right]_{\pi/2}^{\pi} \right)$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\left[\cos\frac{n\pi}{2} - 1\right] - \frac{1}{n}\left[\cos n\pi - \cos\frac{n\pi}{2}\right] \right)$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\left[\cos\frac{n\pi}{2} - 1 - \cos n\pi + \cos\frac{n\pi}{2} \right) \right)$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\left(2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right)\right] = \frac{1}{n\pi}\left(2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right)$$

looking at few terms;

$\longrightarrow$

$n=1 \quad \frac{1}{\pi}\left[2\cos\frac{\pi}{2}-1-\cos\pi\right]=\frac{1}{\pi}\left[-1+1\right]=0$

$n=2 \quad \frac{1}{2\pi}\left[2\cos\pi-1-\cos 2\pi\right]=\frac{1}{2\pi}\left[-2-1-1\right]=\frac{1}{2\pi}\left[-5\right]=-\frac{5}{2\pi}$

$n=3 \quad \frac{1}{3\pi}\left[2\cos\frac{3\pi}{2}-1-\cos 3\pi\right]=\frac{1}{3\pi}\left[0-1+1\right]=0$

$n=4 \quad \frac{1}{4\pi}\left[2\cos 2\pi-1-\cos 4\pi\right]=\frac{1}{4\pi}\left[2-1-1\right]=0$

$n=5 \quad \frac{1}{5\pi}\left[2\cos\frac{5\pi}{2}-1-\cos 5\pi\right]=\frac{1}{5\pi}\left[0-1+1\right]=0$

$n=6 \quad \frac{1}{6\pi}\left[2\cos\frac{6\pi}{2}-1-\cos 6\pi\right]=\frac{1}{6\pi}\left[-2-1-1\right]=-\frac{5}{6\pi}$

So sine series is

$$-\frac{5}{2\pi}\sin 2x-\frac{5}{6\pi}\sin 6x-\frac{5}{10\pi}\sin 10x-\dots$$

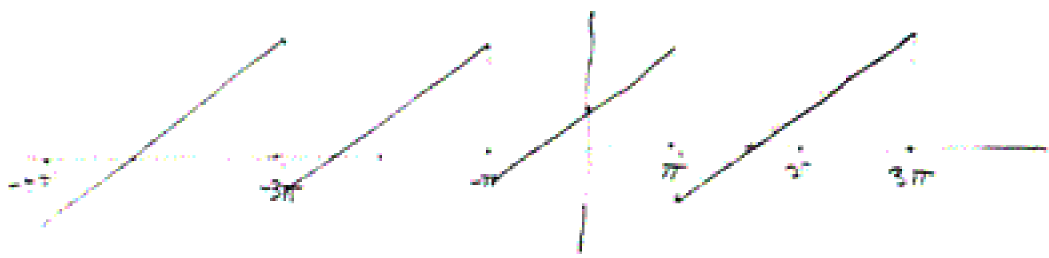
hence fourier series is

$$\boxed{\left(-\frac{2}{\pi}\cos x+\frac{2}{3\pi}\cos 3x-\frac{2}{5\pi}\cos 5x+\dots\right) + \left(-\frac{5}{2\pi}\sin 2x-\frac{5}{6\pi}\sin 6x-\frac{5}{10\pi}\sin 10x-\dots\right)}$$

problem 8 chapter 7, section 5

find fourier series for $f(x) = 1+x \quad -\pi < x < \pi$

extend to make it periodic over 2π:



to find a_n

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (1+x)\cos nx \, dx .$$

$$n=0 \Rightarrow a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(1+x)\,dx = \frac{1}{\pi}\left[2\pi + \frac{1}{2}\left[x^2\right]_{-\pi}^{\pi}\right]$$

$$a_0 = \frac{1}{\pi}\left[2\pi + \frac{1}{2}\left(\pi^2 - (-\pi)^2\right)\right] = \frac{1}{\pi}\left[2\pi + \frac{1}{2}\left(\pi^2 - \pi^2\right)\right] = 2$$

now, for a_n, $n>1$,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(1+x)\cos nx \, dx = \frac{1}{\pi}\left[\underbrace{\int_{-\pi}^{\pi}\cos nx}_{=0 \text{ since over one period}} + \underbrace{\int_{-\pi}^{\pi} x \cos nx \, dx}_{=0 \text{ since odd over period}}\right]$$

x odd, $\cos nx$ even $\rightarrow$ odd

hence $a_n = 0 \quad n>1$. $\longrightarrow$

to find b_n

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (1+x) \sin nx \, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \cancel{\sin nx} + \int_{-\pi}^{\pi} x \sin nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx$$

integrate by parts. let $u = x$, $dv = \sin nx$.

$du = 1$, $V = -\frac{1}{n}\cos nx$

$$\Rightarrow \int_{-\pi}^{\pi} x \sin nx \, dx = \left[-x\frac{\cos nx}{n}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} 1 \cdot \frac{1}{n}\cos nx \, dx$$

$$= -\frac{1}{n}\Big[x\cos nx\Big]_{-\pi}^{\pi} + \frac{1}{n}\underbrace{\int_{-\pi}^{\pi} \cos nx \, dx}_{=0}$$

$$= -\frac{1}{n}\Big[\pi\cos n\pi - (-\pi)\cos(-n\pi)\Big] = -\frac{1}{n}\Big[\pi\cos n\pi + \pi\cos(n\pi)\Big]$$

$$= -\frac{1}{n}\Big[2\pi\cos n\pi\Big] = -\frac{2\pi}{n}\cos n\pi$$

for few values of n.

$n = 1 \Rightarrow -2\pi\cos\pi = 2\pi$

$n = 2 \Rightarrow -\pi\cos 2\pi = -\pi$

$n = 3 \Rightarrow -\frac{2}{3}\pi\cos 3\pi = \frac{2}{3}\pi$

$n = 4 \Rightarrow -\frac{1}{2}\pi\cos 4\pi = -\frac{1}{2}\pi$

so Fourier Series

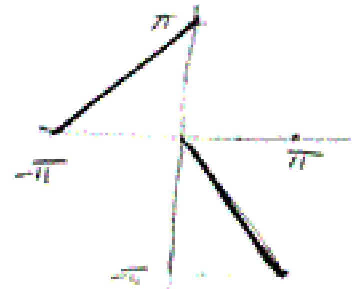
$$f(x) = 1 + \frac{1}{\pi}\left(2\pi\sin x - \pi\sin 2x + \frac{2}{3}\pi\sin 3x - \frac{1}{2}\pi\sin 4x \ldots\right)$$

$$\boxed{f(x) = 1 + 2\sin x - \sin 2x + \frac{2}{3}\sin 3x - \frac{1}{2}\sin 4x \ldots}$$

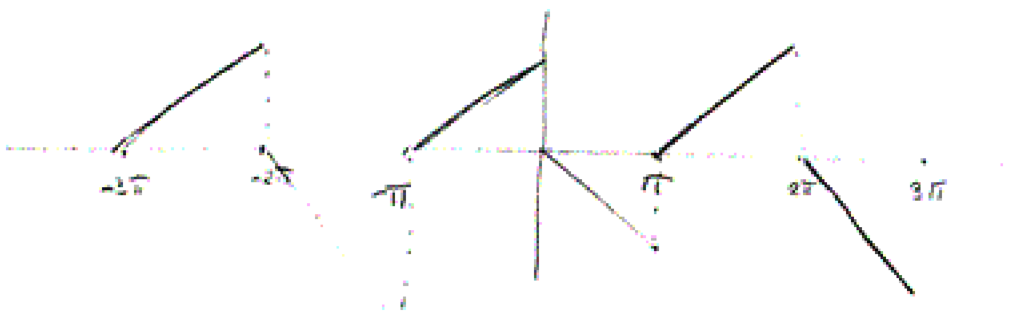
problem 10, section 5, chapter 7

Find fourier series for

$$f(x) = \begin{cases} x+\pi & -\pi < x < 0 \\ -x & 0 < x < \pi \end{cases}$$



extend it



to find a_n

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

for $n=0$, $\quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} x+\pi\, dx + \int_{0}^{\pi} -x\, dx\right]$

$$= \frac{1}{\pi}\left[\left[\frac{x^2}{2}\right]_{-\pi}^{0} + \pi\left[x\right]_{-\pi}^{0} - \left[\frac{x^2}{2}\right]_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{2}\left(0 - (-\pi)^2\right) + \pi\left[0 - -\pi\right] - \frac{1}{2}\left[\pi^2\right]\right] = \frac{1}{\pi}\left(\frac{1}{2}\left(-\pi^2\right) + \pi^2 - \frac{\pi^2}{2}\right)$$

$$= \frac{1}{\pi}\left(-\frac{\pi^2}{2} + \pi^2 - \frac{\pi^2}{2}\right) = 0$$

for $n>0$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(x+\pi)\cos nx\,dx - \int_0^\pi x\cos nx\,dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^0 x\cos nx\,dx + \int_{-\pi}^0 \pi\cos nx\,dx - \int_0^\pi x\cos nx\,dx\right]$$

$$\int_{-\pi}^0 x\cos nx\,dx \Rightarrow \quad u=x \quad , \quad dv=\cos nx \,.$$

$$\Rightarrow \left[x\frac{\sin nx}{n}\right]_{-\pi}^0 - \int_{-\pi}^0 \frac{\sin nx}{n}\,dx = \frac{1}{n}\left[-\pi\sin(-n\pi)\right] - \frac{1}{n}\int_{-\pi}^0 \sin nx\,dx$$

$$= -\frac{\pi}{n}\left[\sin(-n\pi)\right] + \frac{1}{n}\left[\frac{1}{n}\cos nx\right]_{-\pi}^0$$

$= 0$ for all n

$$= \frac{\pi}{n}\sin(n\pi) + \frac{1}{n^2}\left[\cos 0 - \cos(-n\pi)\right] = \cancel{\frac{\pi}{n}\sin n\pi} + \frac{1}{n^2}\left[1-\cos n\pi\right]$$

$$\int_0^\pi x\cos nx\,dx = \left[x\frac{\sin nx}{n}\right]_0^\pi - \frac{1}{n}\int_0^\pi \sin nx\,dx$$

$$= \frac{1}{n}\left[\cancel{\pi\sin n\pi}\right] + \frac{1}{n}\frac{1}{n}\left(\cos nx\right)_0^\pi = \frac{1}{n^2}\left(\cos n\pi - 1\right)$$

$$= \frac{1}{n^2}\left(\cos n\pi - 1\right)$$

$$\int_{-\pi}^0 \pi\cos nx\,dx = \pi\left[\frac{1}{n}\sin nx\right]_{-\pi}^0 = \frac{\pi}{n}\left[\sin 0 - \cancel{\sin \pi n}\right]$$

so $a_n = \dfrac{1}{\pi}\left[\dfrac{1}{n^2} - \dfrac{1}{n^2}\cos n\pi - \left(\dfrac{1}{n^2}\cos n\pi - \dfrac{1}{n^2}\right)\right]$

$$= \frac{1}{\pi}\left[\frac{2}{n^2} - \frac{2}{n^2}\cos n\pi\right]$$

so for $n=1$, $\quad a_1 = \dfrac{1}{\pi}\left[2+2\right] = \dfrac{4}{\pi}$

$n=2 \quad a_2 = \dfrac{1}{\pi}\left[\dfrac{2}{4} - \dfrac{2}{4}\right] = 0$ $\rightarrow$

$n=3 \qquad a_3 = \frac{1}{\pi}\left[\frac{2}{9} - \frac{2}{9}(-1)\right] = \frac{1}{\pi}\left[\frac{4}{9}\right]$

$n=4 \qquad a_4 = \frac{1}{\pi}\left[\frac{2}{16} - \frac{2}{16}\right] = 0$

so the cosine series is

$$\boxed{\frac{4}{\pi}\cos x + \frac{4}{9\pi}\cos 3x + \frac{4}{25}\cos 5x + \dots}$$

to find the b_n terms.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} x\sin nx\,dx + \int_{-\pi}^{0} \pi\sin nx\,dx - \int_{0}^{\pi} x\sin nx\,dx\right]$$

$$\int_{-\pi}^{0} x\sin nx\,dx = -\left[\frac{x\cos nx}{n}\right]_{-\pi}^{0} + \frac{1}{n}\int_{-\pi}^{0}\cos nx\,dx$$

$$= -\frac{1}{n}\left[0 - (-\pi)\cos n(-\pi)\right] + \frac{1}{n}\quad\frac{1}{n}\left[\sin nx\right]_{-\pi}^{0}$$

$$= -\frac{1}{n}\left[\pi\cos n\pi\right] + \frac{1}{n^2}\left[0 - \sin -n\pi\right]$$

$$= -\frac{1}{n}\left[\pi\cos n\pi\right] + \frac{1}{n^2}\left[\sin n\pi\right]$$

$$\int_{-\pi}^{0}\pi\sin nx\,dx = -\frac{\pi}{n}\left[\cos n\pi\right]_{-\pi}^{0} = -\frac{\pi}{n}\left[1 - \cos n\pi\right]$$

$\longrightarrow$

$$\int_0^{\pi} x \sin nx \, dx = \left[-\frac{x \cos nx}{n}\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi} \cos nx \, dx$$

$$= -\frac{1}{n}\left[\pi \cos n\pi\right] + \frac{1}{n}\frac{1}{n}\left[\sin nx\right]_0^{\pi}$$

$$= -\frac{\pi}{n}\left[\cos n\pi\right] + \frac{1}{n^2}\left[0\right] = -\frac{\pi}{n}\cos n\pi$$

so $b_n = \cancel{-\frac{\pi}{n}\cos n\pi} + \frac{1}{n^2}\sin n\pi - \frac{\pi}{n} + \cancel{\frac{\pi}{n}\cos n\pi} + \frac{\pi}{n}\cos n\pi$

$$= \underbrace{\frac{1}{n^2}\sin n\pi}_{=0} - \frac{\pi}{n} + \frac{\pi}{n}\cos n\pi .$$

look at few n values:

$b_1 = -\pi + \pi \cos \pi = -\pi - \pi = -2\pi$

$b_2 = -\frac{\pi}{2} + \frac{\pi}{2}\cos 2\pi = -\frac{\pi}{2} + \frac{\pi}{2} = 0$

$b_3 = -\frac{\pi}{3} + \frac{\pi}{3}\cos 3\pi = -\frac{\pi}{3} - \frac{\pi}{3} = -\frac{2}{3}\pi$

so sin series is

$$\boxed{-2\sin x - \frac{2}{3}\sin 3x - \frac{2}{5}\sin 5x - \cdots}$$

hence Fourier series is

$$\left(\frac{4}{\pi}\cos x + \frac{4}{9\pi}\cos 3x + \frac{4}{25}\cos 5x + \cdots\right)$$
$$-\left(2\sin x + \frac{2}{3}\sin 3x + \frac{2}{5}\sin 5x + \cdots\right)$$

problems 5, section 6, chapter 7

use Dirichlet theorem to find the value to which Fourier series converges at $x = 0, \pm\frac{\pi}{2}, \pm\pi, \pm 2\pi$ for

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ -1 & 0 < x < \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < x < \pi \end{cases}$$

at $x = 0$, it will converge to $\dfrac{f(0_-) + f(0_+)}{2} = \dfrac{0-1}{2} = \boxed{-\frac{1}{2}}$

at $x = -\frac{\pi}{2}$, $\dfrac{f(-\frac{\pi}{2}_-) + f(-\frac{\pi}{2}_+)}{2} = \dfrac{-1+1}{2} = \boxed{0}$

at $x = \frac{\pi}{2}$, $\dfrac{f(\frac{\pi}{2}_-) + f(\frac{\pi}{2}_+)}{2} = \dfrac{-1+1}{2} = \boxed{0}$

at $x = -\pi$, $\dfrac{f(-\pi_-) + f(-\pi_+)}{2} = \dfrac{1+0}{2} = \boxed{\frac{1}{2}}$

at $x = \pi$, $\boxed{\frac{1}{2}}$

$x = 2\pi$, $\boxed{-\frac{1}{2}}$

$x = -2\pi$, $\boxed{-\frac{1}{2}}$

problem 4, section 7, chapter 7.

expand the following function in Fourier series of complex exponentials e^{inx} in interval $(-\pi, \pi)$ and verify using Euler formula that answer is equal to one found in section 5.

$$f(x) = \begin{cases} -1 & -\pi < x < \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < x < \pi \end{cases}$$

$$f(x) = c_0 + c_1 e^{ix} + c_{-1} e^{-ix} + c_2 e^{2ix} + c_{-2} e^{-2ix} + \cdots$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$$

Find c_n.

$$c_n = \frac{1}{2\pi} \left[\int_{-\pi}^{\frac{\pi}{2}} -e^{-inx}\, dx + \int_{\frac{\pi}{2}}^{\pi} e^{-inx}\, dx \right]$$

$$= \frac{1}{2\pi} \left[-\frac{1}{-in} \left[e^{-inx} \right]_{-\pi}^{\frac{\pi}{2}} + \frac{1}{-in} \left[e^{-inx} \right]_{\frac{\pi}{2}}^{\pi} \right]$$

$$= \frac{1}{2\pi} \frac{1}{in} \left[\left(e^{-in\frac{\pi}{2}} - e^{+in\pi} \right) - \left(e^{-in\pi} - e^{-in\frac{\pi}{2}} \right) \right]$$

$$= \frac{1}{2\pi i n} \left[e^{-in\frac{\pi}{2}} + e^{-in\frac{\pi}{2}} - e^{-in\pi} - e^{+in\pi} \right]$$

notice few relationships on $e^{in\pi}$

$$e^{-in\pi} = \cos n\pi - i\sin n\pi = \cos n\pi = -1 \text{ for odd } \pm n$$
$$= 1 \text{ for even } \pm n, 0$$

$$e^{in\pi} = \cos n\pi + i\sin n\pi = \cos n\pi = - \text{ for odd } \pm n$$
$$= 1 \text{ for even } \pm n, 0$$

$$e^{-in\frac{\pi}{2}} = \cos n\frac{\pi}{2} - i\sin\frac{n\pi}{2} = -i \text{ for odd } n \quad 1, 5, 9, \ldots \quad -3, -7, -11, \ldots$$
$$= i \text{ for odd } n \quad 3, 7, 11, \ldots \quad -1, -5, -9, \ldots$$
$$= -1 \text{ for even } \pm n \quad 2, 6, 10, \ldots$$
$$= 1 \text{ for even } \pm n \quad 4, 8, 12, \ldots, 0$$

$$e^{in\frac{\pi}{2}} = \cos n\frac{\pi}{2} + i\sin\frac{n\pi}{2} = i \text{ for odd } n \quad 1, 5, 9, \ldots \quad -3, -7, -11, \ldots$$
$$= -i \text{ for odd } n \quad 3, 7, 11, \ldots \quad -1, -5, -9, \ldots$$
$$= -1 \text{ for even } n \quad 2, 6, 10, \ldots$$
$$= 1 \text{ for even } n \quad 0, 4, 8, 12, \ldots$$

So C_n becomes

$$C_0 = \frac{1}{2\pi}\left[\int_{-\pi}^{\frac{\pi}{2}} -dx + \int_{\frac{\pi}{2}}^{\pi} dx\right] = \frac{1}{2\pi}\left[-\left[x\right]_{-\pi}^{\frac{\pi}{2}} + \left[x\right]_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[-\left(\frac{\pi}{2}+\pi\right) + \left(\pi - \frac{\pi}{2}\right)\right] = \frac{1}{2\pi}\left[-\pi\right] = \boxed{-\frac{1}{2}}$$

$\longrightarrow$

So $f(x)$ is

positive n

$$+\frac{1}{2\pi i}\left(\frac{1}{1}\left(e^{-i\frac{\pi}{2}}+e^{-i\frac{\pi}{2}}-e^{-i\pi}-e^{i\pi}\right)e^{ix}+\frac{1}{2}\left(e^{-i\pi}+e^{-i\pi}-e^{-2i\pi}-e^{i\pi}\right)e^{i2x} + \cdots\right)$$

$$+\frac{1}{2\pi i}\left(-1\left(e^{i\frac{\pi}{2}}+e^{i\frac{\pi}{2}}-e^{i\pi}-e^{-i\pi}\right)e^{-ix}+\left(-\frac{1}{2}\right)\left(e^{i\pi}+e^{i\pi}-e^{2i\pi}-e^{-2i\pi}\right)e^{-2ix} \cdots\right)$$

negative n

$$=\frac{1}{2\pi i}\left((-i-i+1+1)e^{ix}+\left(\frac{1}{2}\right)(-1-1-1-1)e^{i2x}+\cdots\right)$$

$$+\frac{1}{2\pi i}\left(-1(-i-i+1+1)e^{-ix}+\left(-\frac{1}{2}\right)(-1-1-1-1)e^{-2ix}+\cdots\right)$$

$$=\frac{1}{2\pi i}\left((-2i+2)e^{ix}-2e^{i2x}+\cdots\right)$$

$$+\frac{1}{2\pi i}\left((2i-2)e^{-ix}+2e^{-2ix}+\cdots\right)$$

$$=\frac{1}{2\pi i}\left(\left(-2ie^{ix}+2e^{ix}-2e^{i2x}+\cdots\right)+\left(2ie^{-ix}-2e^{-ix}+2e^{-i2x}+\cdots\right)\right)$$

$\longrightarrow$

Collect Terms

$$= \frac{1}{2\pi i}\left((2ie^{-ix} - 2ie^{ix}) + (2e^{ix} - 2e^{-ix}) + (2e^{-2ix} - 2e^{i2x}) \ldots \right.$$

$$= \frac{1}{\pi}\left(\underbrace{(e^{-ix} - e^{ix})}_{-2\cos x} + \underbrace{\left(\frac{e^{ix} - e^{-ix}}{i}\right)}_{2\sin x} + \underbrace{\left(\frac{e^{-2ix} - e^{i2x}}{i}\right)}_{2\sin 2x} + \ldots \right.$$

$$= \frac{1}{\pi}\left(-2\cos x + 2\sin x + 2\sin 2x \quad . \quad . \right)$$

So $\boxed{f(x) = \frac{1}{2} + \frac{2}{\pi}\left(-\cos x + \sin x + \sin 2x \right)}$

$\frac{1}{2} \to c_0$

this matches result of # 5.4 by the Fourier series. Verified using mathematica.

7.7 chapter 7 $\frac{2}{2}$ Name? Nasser Abbasi

$$f(x) = \frac{1}{2}a_0 + \sum_1^\infty a_n \cos nx + \sum_1^\infty b_n \sin nx = \sum_{-\infty}^{\infty} C_n e^{inx}$$

$$= \frac{1}{2}a_0 + \left(a_1 \cos nx + a_2 \cos 2nx + \cdots\right) + \left(b_1 \sin x + b_2 \sin 2x + \cdots\right)$$

$$= \cdots + C_{-1} e^{-inx} + C_0 + C_1 e^{inx} + \cdots$$

so $C_0 = \frac{1}{2} a_0$

expand $e^{-inx} = \cos nx - i \sin nx$

$e^{inx} = \cos nx + i \sin nx$

so $= \frac{1}{2}a_0 + \left(a_1 \cos nx + \cdots \right) + \left(b_1 \sin x + b_2 \sin 2x \cdots\right)$

$= \cdots + C_{-1}\left(\cos x - i \sin nx\right) + C_0 + C_1\left(\cos x + i \sin nx\right) + \cdots$

$+ \cos x \left(C_{-1} + C_1\right) + \sin x \left(-i C_{-1} + i C_1\right)$

so $a_1 = C_{-1} + C_1$

$b_1 = i\left(-C_{-1} + C_1\right)$

$a_n = C_{-n} + C_n$

$b_n = i\left(-C_{-n} + C_n\right)$

problem 12, section 7, chapter 7.

Show that if a real $f(x)$ is expanded in complex exp. fourier series $\sum_{-\infty}^{\infty} c_n e^{inx}$, then $c_{-n} = \overline{c_n}$ where $\overline{c_n}$ means complex conjugate of c_n.

$$\boxed{c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx} \quad \text{--- (1)}$$

$$\boxed{\text{so } c_{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{inx}\, dx} \quad (2)$$

from (1), $\overline{c_n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(x)}\;\overline{e^{-inx}}\, dx$, but $f(x)$ is real. so $\overline{f(x)} = f(x)$

so $\overline{c_n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, \overline{e^{-inx}}\, dx.$

but $\overline{e^{-inx}} = e^{inx}$

so $$\boxed{\overline{c_n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{inx}\, dx} \quad \text{--- (3)}$$

compare (2) and (3)

$$\Rightarrow \boxed{c_{-n} = \overline{c_n}}$$

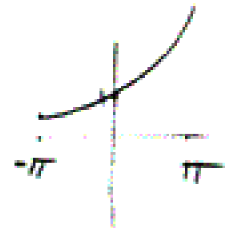
Problem 12, section 8, Chapter 7

sketch several periods of function and find fourier series.

(a) $f(x) = e^x \qquad -\pi < x < \pi$

(b) $f(x) = e^x \qquad 0 < x < 2\pi$

(a)



$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx\, dx$$

$$n=0 \quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\, dx = \frac{1}{\pi}\left[e^x\right]_{-\pi}^{\pi} = \boxed{\frac{1}{\pi}\left[e^{\pi} - e^{-\pi}\right]}$$

$$n \neq 0 \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx\, dx.$$

integration by parts:

let $u = \cos nx \quad du = -n \sin nx$
$dv = e^x$
$\Rightarrow a_n = \frac{1}{\pi}\left[\left[\cos nx\, e^x\right]_{-\pi}^{\pi} - n\int \sin nx\, e^x\, dx\right]$

$$I = \frac{1}{\pi}\left[\left(\cos nx\, e^x\right)_{-\pi}^{\pi} - n\left[\left(\sin nx\, e^x\right)_{-\pi}^{\pi} - n\underbrace{\int \cos nx\, e^x\, dx}_{I}\right]\right]$$

$$I = \frac{1}{\pi}\left[\left(\cos nx\, e^x\right)_{-\pi}^{\pi} - n\left(\sin nx\, e^x\right)_{-\pi}^{\pi} + n^2 I\right]$$

$$I - \frac{n^2}{\pi} I = \frac{1}{\pi}\left(\cos nx\, e^x\right)_{-\pi}^{\pi} - \frac{n}{\pi}\left(\sin nx\, e^x\right)_{-\pi}^{\pi} \longrightarrow$$

$$a_n\left(1-\frac{n^2}{\pi}\right) = \frac{1}{\pi}\left(\cos n\pi\, e^{\pi} - \cos n\pi\, e^{-\pi}\right) - \frac{n}{\pi}\left(\sin n\pi\, e^{\pi} + \sin n\pi\, e^{-\pi}\right)$$

$$a_n = \frac{1}{\left(\frac{\pi - n^2}{\pi}\right)}\,\frac{1}{\pi}\left[\left(\cos n\pi\, e^{\pi} - \cos n\pi\, e^{-\pi}\right) - n\left(\sin n\pi\, e^{\pi} + \sin n\pi\, e^{-\pi}\right)\right]$$

$$a_n = \left(\frac{1}{\pi - n^2}\right)\left[\cos n\pi\left(e^{\pi} - e^{-\pi}\right) - n\left(\overset{=0}{\cancel{\sin n\pi}}\left(e^{\pi} + e^{-\pi}\right)\right)\right]$$

$$= \frac{1}{\pi - n^2}\cos n\pi\left(e^{\pi} - e^{-\pi}\right) = \boxed{\frac{e^{\pi} - e^{-\pi}}{\pi - n^2}\cos n\pi}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx\, e^{x}\, dx = \frac{1}{\pi}\left[\left(\sin nx\, e^{x}\right)_{-\pi}^{\pi} - n\int \cos nx\, e^{x}\right]$$

$$= \frac{1}{\pi}\left[\left(\sin n\pi\, e^{\pi} + \sin n\pi\, e^{-\pi}\right) - n\left[\left(\cos nx\, e^{x}\right)_{-\pi}^{\pi} + n\int \sin nx\, e^{x}\, dx\right]\right]$$

$$= \frac{1}{\pi}\left[\sin n\pi\left(e^{\pi} - e^{-\pi}\right) - n\left[\left(\cos n\pi\, e^{\pi} - \cos n\pi\, e^{-\pi}\right) + n\, I\right]\right]$$

$$I = \sin n\pi\left(\frac{e^{\pi} - e^{-\pi}}{\pi}\right) - \frac{n}{\pi}\left(\cos n\pi\left(e^{\pi} - e^{-\pi}\right)\right) - \frac{n^2}{\pi} I$$

$$I + \frac{n^2}{\pi} I = \frac{e^{\pi} - e^{-\pi}}{\pi}\left(\sin n\pi - n\cos n\pi\right)$$

$$b_n = \frac{1}{\left(1 + \frac{n^2}{\pi}\right)}\,\frac{e^{\pi} - e^{-\pi}}{\pi}\left(\overset{=0}{\cancel{\sin n\pi}} - n\cos n\pi\right)$$

(3

$$b_n = \left(\frac{e^{\pi} - e^{-\pi}}{\pi + n^2}\right) (-n \cos n\pi)$$

so $a_n =$

$n=1$ $\left(\frac{e^{\pi} - e^{-\pi}}{\pi - 1}\right)(-1)$

$n=2$ $\frac{e^{\pi} - e^{-\pi}}{\pi - 4}$

$n=3$ $\frac{e^{\pi} - e^{-\pi}}{\pi - 9}(-1)$ etc.

b_n:

$b_1 = \frac{e^{\pi} - e^{-\pi}}{\pi + 1}(1)$

$\checkmark$

$b_2 = \frac{e^{\pi} - e^{-\pi}}{\pi + 4}(-2)$

$$\text{so } f(x) = \frac{1}{2\pi}\left(e^{\pi} - e^{-\pi}\right) + \left(-\frac{(e^{\pi} - e^{-\pi})}{\pi - 1}\cos x + \frac{e^{\pi} - e^{-\pi}}{\pi - 4}\cos 2x + \cdots\right)$$

$$+ \left(\frac{e^{\pi} - e^{-\pi}}{\pi + 1}\sin x - 2\,\frac{e^{\pi} - e^{-\pi}}{\pi + 4}\sin 2x + \cdots\right.$$

$$f(x) = e^{\pi} - e^{-\pi}\left[\left(\frac{1}{2\pi} - \frac{1}{\pi - 1}\cos x + \frac{1}{\pi - 4}\cos 2x + \cdots\right)\right.$$

$$\left. + \left(\frac{1}{\pi + 1}\sin x - \frac{2}{\pi + 4}\sin 2x + \cdots\right)\right]$$