HW #5
Math 121 A

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UCB extension

$\frac{2}{2}$

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$\boxed{5.3}$ given $r = e^{-p^2-q^2}$, $p = e^{s}$, $q = e^{-s}$

find $\frac{dr}{ds}$

$$dr = \frac{\partial r}{\partial p}\,dp + \frac{\partial r}{\partial q}\,dq \Rightarrow \frac{dr}{ds} = \frac{\partial r}{\partial p}\frac{dp}{ds} + \frac{\partial r}{\partial q}\frac{dq}{ds}$$

$$\frac{\partial r}{\partial p} = \left(e^{-p^2-q^2}\right)(-2p)$$

$$\frac{\partial r}{\partial q} = e^{(-p^2-q^2)}(-2q)$$

$$\frac{dp}{ds} = e^{s}$$

$$\frac{dq}{ds} = -e^{-s} \quad \checkmark$$

so $$\frac{dr}{ds} = \left(-2pe^{-p^2-q^2}\right)\left(e^{s}\right) + \left(-2qe^{-p^2-q^2}\right)\left(-e^{-s}\right)$$

$$= -2e^{s}p\,\underbrace{e^{-p^2-q^2}}_{r} + 2e^{-s}q\,\underbrace{e^{-p^2-q^2}}_{r}$$

$$= 2r\left[-\underbrace{e^{s}}_{p}p + \underbrace{e^{-s}}_{q}q\right]$$

$$\boxed{\frac{dr}{ds} = 2r\left[-p^2 + q^2\right] \checkmark}$$

→ Any one of these 2 is a solution

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$\boxed{5.4}$ given $x = \ln(u^2 - v^2)$, $u = t^2$, $v = \cos t$

find $\frac{dx}{dt}$.

$$dx = \frac{\partial x}{\partial u}\,du + \frac{\partial x}{\partial v}\,dv$$

$$\frac{dx}{dt} = \frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial x}{\partial v}\frac{dv}{dt}$$

note rule for $\frac{d}{dx}\ln\left(f(x)\,g(x)\right) = \frac{g(x)f'(x) + f(x)\,g'(x)}{f(x)\,g(x)}$

So $\frac{\partial x}{\partial u} = \frac{2u}{u^2 - v^2}$

$$\frac{\partial x}{\partial v} = \frac{-2v}{u^2 - v^2}$$

so $\frac{dx}{dt} = \frac{2u}{u^2 - v^2}(2t) + \left(\frac{-2v}{u^2 - v^2}\right)(-\sin t)$

$$= \frac{4t\,u}{u^2 - v^2} + \frac{2v\,\sin t}{u^2 - v^2} = \boxed{\left(\frac{2}{u^2 - v^2}\right)\left(2tu + v\sin t\right)}$$

I probably need to express everything in terms of t:

$$\frac{dx}{dt} = \left(\frac{2}{t^4 - \cos^2 t}\right)\left(2t(t^2) + \cos t\,\sin t\right)$$

$$\boxed{\frac{dx}{dt} = \frac{4t^3 + 2\,\cos(t)\,\sin(t)}{t^4 - \cos^2(t)}}$$

→ This looks better ☺ ovall in't

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$\boxed{6.3}$ if $x^y = y^x$ find $\frac{dy}{dx}$ at $(2,4)$.

$$y \ln(x) = x \ln(y)$$

$$\frac{\ln(x)}{x} = \frac{\ln(y)}{y}$$

take differentials:

$$\left[\frac{1}{x}\frac{1}{x} + \left(-1 \frac{1}{x^2} \ln(x)\right)\right] dx = \left[\frac{1}{y}\frac{1}{y} + \left(-1 \frac{1}{y^2} \ln(y)\right)\right] dy$$

$$\left[\frac{1}{x^2} - \frac{1}{x^2} \ln(x)\right] dx = \left(\frac{1}{y^2} - \frac{1}{y^2} \ln(y)\right] dy$$

$$\frac{dy}{dx} = \frac{\left(\frac{1-\ln(x)}{x^2}\right)}{\left(\frac{1-\ln(y)}{y^2}\right)} = \frac{y^2(1-\ln(x))}{x^2(1-\ln(y))}$$

at $x=2$, $y=4$ $\Rightarrow$

$$\frac{dy}{dx} = \frac{16(1-\ln(2))}{4(1-\ln(4))} = 4 \frac{(1-\ln(2))}{1-\ln(4)} = -3.1774$$

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$\boxed{6.5}$

if $xy^3 - yx^3 = 6$, find slope at $(1,2)$ and equation of tangent line at this point.

differentiate equation implicitly w.r.t x

$$\left(x\frac{d}{dx}(y^3)+\frac{d}{dx}(x)\,y^3\right)-\left(y\frac{d}{dx}(x^3)+\frac{dy}{dx}x^3\right)=0$$

$$x\left(3y^2\frac{dy}{dx}\right)+y^3-\left(y(3x^2)+x^3\frac{dy}{dx}\right)=0$$

$$3xy^2\frac{dy}{dx}+y^3-3yx^2-x^3\frac{dy}{dx}=0$$

let $x=1,\ y=2 \longrightarrow$

$$3(1)(4)\frac{dy}{dx}+8-3(2)(1)-(1)\frac{dy}{dx}=0$$

$$12\frac{dy}{dx}+8-6-\frac{dy}{dx}=0$$

$\boxed{\frac{dy}{dx} = \frac{-2}{11}}$ ✓ the slope at point $(1,2)$.

equation of Line is $\frac{y-y_1}{x-x_1} = \text{slope}$

so $y-y_1 = \frac{-2}{11}(x-x_1) \Rightarrow y-2 = -\frac{2}{11}(x-1)$

$y-2 = -\frac{2}{11}x + \frac{2}{11} \Rightarrow y+\frac{2}{11}x = \frac{24}{11}$

$\Rightarrow \boxed{2x+11y=24}$

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$\boxed{6.6}$

Find $\frac{d^2y}{dx^2}$ at $(1,2)$. where $xy^3 - yx^3 = 6$

implicit diff of $xy^3 - yx^3 - 6 = 0$ w.r.t. x $\Rightarrow$

$$3xy^2\frac{dy}{dx} + y^3 - 3y\,x^2 - x^3\frac{dy}{dx} = 0$$

implicit diff again $\Rightarrow$

$$\left(3x\frac{d}{dx}\left(y^2\frac{dy}{dx}\right) + 3y^2\frac{dy}{dx}\right) + \frac{d}{dx}\left(y^3\right) - \left(3y\frac{d}{dx}(x^2) + 3x^2\frac{d}{dx}y\right)$$

$$-\left(x^3\frac{d}{dx}\left(\frac{dy}{dx}\right) + 3x^2\frac{dy}{dx}\right) = 0$$

$$\left(3x\left(y^2\frac{d^2y}{dx^2} + 2y\left[\frac{dy}{dx}\right]^2\right) + y^2\frac{dy}{dx}\right) + 3y^2\frac{dy}{dx} - \left(3y(2x) + 3x^2\frac{dy}{dx}\right)$$

$$-\left(x^3\frac{d^2y}{dx^2} + 3x^2\frac{dy}{dx}\right) = 0$$

$$3xy^2y'' + 6xy[y']^2 + 3y^2y' + 3y^2y' - 6yx - 3x^2y' - x^3y'' - 3x^2y' = 0$$

$$y''(3xy^2 - x^3) = 3x^2y' + 3x^2y' + 6yx - 3y^2y' - 3y^2y' - 6xy[y']^2$$

$$y''(3xy^2 - x^3) = y'(3x^2 + 3x^2 - 3y^2 - 3y^2) - 6xy[y']^2 + 6yx$$

at $(1,2)$ $y' = \frac{-2}{11}$ from problem 6.5.

$\Rightarrow$

at $x=1$, $y=2$ $\Rightarrow$

$$y''(3(4)-1) = \frac{-2}{11}\left(6-6(4)\right) - 6(2)\left(-\frac{2}{11}\right)^2 + 6(2)$$

$$y''(11) = \frac{-2}{11}(6-24) - 12\left(\frac{4}{121}\right) + 12$$

$$y''(11) = -\frac{2}{11}(-18) - \frac{48}{121} + 12$$

$$y''(11) = \frac{36}{11} - \frac{48}{121} + 12$$

$$y'' = \frac{3.272 - 0.3967 + 12}{11} = \boxed{1.352}$$

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$\boxed{7.3}$ if $z = xe^{-y}$ and $x = \cosh(t)$, $y = \cos(s)$, find

$$\frac{\partial z}{\partial s}, \quad \frac{\partial z}{\partial t}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial y}\frac{dy}{ds} = (-xe^{-y})(-\sin(s)) = \boxed{\cosh(t)e^{-\cos(s)}\sin(s)}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{dx}{dt} = (e^{-y})(\sinh(t)) = \boxed{e^{-\cos(s)}\sinh t}$$

$\boxed{7.5}$ if $u = x^2y^3z$ and $x = \sin(s+t)$, $y = \cos(s+t)$, $z = e^{st}$, find $\frac{\partial u}{\partial s}$ and $\frac{\partial u}{\partial t}$.

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds} + \frac{\partial u}{\partial z}\frac{dz}{ds}$$

$$= (2xy^3z)\cos(s+t) + (3x^2y^2z)(-\sin(s+t)) + x^2y^3 t e^{st}$$

$$= 2xy^3z\cos(s+t) - 3x^2y^2z\sin(s+t) + x^2y^3 t e^{st}$$

or

$$= 2xy^4z - 3x^3y^2z + x^2y^3tz$$

see next page

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$= (2xy^3z)\cos(s+t) + (3x^2y^2z)(-\sin(s+t)) + (x^2y^3)se^{st}$$

$$= 2xy^3z\underbrace{\cos(s+t)}_{y} - 3x^2y^2z\underbrace{\sin(s+t)}_{x} + x^2y^3s\underbrace{e^{st}}_{z}$$

or

$$\Rightarrow 2xy^4z - 3x^3y^2z + x^2y^3zs$$

⟶ see next page

Note I think I should re write these answers more in terms of 's' and 't' as much as possible instead as I did, so please see next page ⟶

$$\frac{\partial u}{\partial s} = 2\sin(s+t)\cos^4(s+t)\,e^{st} - 3\sin^3(s+t)\cos^2(s+t)\,e^{st}$$
$$+ \sin^2(s+t)\cos^3(s+t)\,t\,e^{st}$$

$$\frac{\partial u}{\partial t} = 2\sin(s+t)\cos^4(s+t)\,e^{st}$$
$$- 3\sin^3(s+t)\cos^2(s+t)\,e^{st} + \sin^2(s+t)\cos^3(s+t)\,s\,e^{st}$$

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$\boxed{7.8}$ if $xs^2+yt^2=1$ and $x^2s+y^2t=xy-4$

find $\frac{\partial x}{\partial s}$, $\frac{\partial x}{\partial t}$, $\frac{\partial y}{\partial s}$, $\frac{\partial y}{\partial t}$ at $(x,y,s,t)=(1,-3,2,-1)$

from first equation, take derivative w.r.t. 's' and then w.r.t. 't' ⇒

$$\left(x\,2s+\frac{\partial x}{\partial s}s^2\right)+\frac{\partial y}{\partial s}t^2=0 \qquad ①$$

$$\frac{\partial x}{\partial t}s^2+\left(2t\,y+t^2\frac{\partial y}{\partial t}\right)=0 \qquad ②$$

From second equation, do the same

$$\left(x^2+s\,2x\frac{\partial x}{\partial s}\right)+2y\frac{\partial y}{\partial s}t=\left(x\frac{\partial y}{\partial s}+y\frac{\partial x}{\partial s}\right) \quad — ③$$

$$2x\frac{\partial x}{\partial t}s+2y\frac{\partial y}{\partial t}t+y^2=\left(x\frac{\partial y}{\partial t}+y\frac{\partial x}{\partial t}\right) \quad — ④$$

4 equations, 4 unknowns.

rewrite ③ and ④

$$\frac{\partial x}{\partial s}(2xs-y)+\frac{\partial y}{\partial s}(2yt-x)=-x^2 \quad — ⑤$$

$$\frac{\partial x}{\partial t}(2xs-y)+\frac{\partial y}{\partial t}(2yt-x)=-y^2 \quad — ⑥$$

rewrite ① and ②

$$\frac{\partial x}{\partial s}(s^2)+\frac{\partial y}{\partial s}(t^2)=-2xs \quad — ⑦$$

$$\frac{\partial x}{\partial t}(s^2)+\frac{\partial y}{\partial t}(t^2)=-2ty \quad — ⑧$$

now solve ⑤ and ⑦ together for $\frac{\partial x}{\partial s}$, $\frac{\partial y}{\partial s}$

and solve ⑥ and ⑧ together for $\frac{\partial x}{\partial t}$, $\frac{\partial y}{\partial t}$

→

solving ⑤ and ⑦ ⟹

$$\underbrace{\begin{pmatrix} 2xs-y & 2yt-x \\ s^2 & t^2 \end{pmatrix}}_{2\times 2} \underbrace{\begin{pmatrix} \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial s} \end{pmatrix}}_{2\times 1} = \underbrace{\begin{pmatrix} -x^2 \\ -2xs \end{pmatrix}}_{2\times 1} \quad \text{———} ⑨$$

solving ⑥ and ⑧ ⟹

$$\begin{pmatrix} 2xs-y & 2yt-x \\ s^2 & t^2 \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \end{pmatrix} = \begin{pmatrix} -y^2 \\ -2yt \end{pmatrix} \quad \text{———} ⑩$$

Solve 9 by Cramer Rule

- says That given $\overbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}^{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \overbrace{\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}^{C}$ Then

$x_i = \dfrac{|A_i|}{|A|}$ where A_i is A matrix with the i^{th} column replaced by C vector.

so
$$\frac{\partial x}{\partial s} = \frac{\begin{vmatrix} -x^2 & 2yt-x \\ -2xs & t^2 \end{vmatrix}}{\begin{vmatrix} 2xs-y & 2yt-x \\ s^2 & t^2 \end{vmatrix}} = \frac{(-x^2t^2)-((2yt-x)(-2xs))}{t^2(2xs-y)-(s^2)(2yt-x)} \boxed{= \frac{x^2(-t^2-2s)+4yxts}{x(2st^2+s^2)+y(-t^2-2ts^2)}}$$

$$\frac{\partial y}{\partial s} = \frac{\begin{vmatrix} 2xs-y & -x^2 \\ s^2 & -2xs \end{vmatrix}}{\begin{vmatrix} 2xs-y & 2yt-x \\ s^2 & t^2 \end{vmatrix}} = \frac{(2xs-y)(-2xs)-(-x^2)(s^2)}{t^2(2xs-y)-(s^2)(2yt-x)} \boxed{= \frac{-3x^2s^2+2xsy}{x(2st^2+s^2)+y(-t^2-2ts^2)}}$$

Similarly, solve ⑩ by Cramer rule to get $\frac{\partial x}{\partial t}$ and $\frac{\partial y}{\partial t}$ ⟹

$$\frac{\partial x}{\partial t} = \frac{\begin{vmatrix} -y^2 & 2yt-x \\ -2yt & t^2 \end{vmatrix}}{\begin{vmatrix} 2xs-y & 2yt-x \\ s^2 & t^2 \end{vmatrix}} = \frac{(-y^2t^2) - (2yt-x)(-2yt)}{x(2st^2+s^2)+y(-t^2-2ts^2)} \quad \text{— } (A)$$

$$= \frac{-y^2t^2 - (-4y^2t^2+2ytx)}{|A|} = \boxed{\frac{3y^2t^2 - 2ytx}{x(2st^2+s^2)+y(-t^2-2ts^2)}}$$

$$\frac{\partial y}{\partial t} = \frac{\begin{vmatrix} 2xs-y & -y^2 \\ s^2 & -2yt \end{vmatrix}}{\begin{vmatrix} 2xs-y & 2yt-x \\ s^2 & t^2 \end{vmatrix}} = \frac{(2xs-y)(-2yt) - (-y^2)(s^2)}{|A|}$$

$$= \boxed{\frac{-4xsyt + 2y^2t + y^2s^2}{x(2st^2+s^2)+y(-t^2-2ts^2)}}$$

now, at $(x, y, s, t) = (1, -3, 2, -1)$ we set

$$\frac{\partial x}{\partial s} = \frac{(1)^2(-(-1)^2 - 2(2)) + 4(-3)(1)(-1)(2)}{(1)(2(2)(-1)^2 + (2)^2) + (-3)(-(-1)^2 - 2(-1)(2)^2)} = \frac{(-1-4)+24}{8-3(-1+8)} = \boxed{\frac{19}{-13}}$$

$$\frac{\partial y}{\partial s} = \frac{-3(1)^2(2)^2 + 2(1)(2)(-3)}{-13} = \frac{-12-12}{-13} = \frac{-24}{-13} = \boxed{\frac{24}{13}}$$

$$\frac{\partial x}{\partial t} = \frac{3(-3)^2(-1)^2 - 2(-3)(-1)(1)}{-13} = \frac{27-6}{-13} = \boxed{\frac{21}{-13}}$$

$$\frac{\partial y}{\partial t} = \frac{-4(1)(2)(-3)(-1) + 2(-3)^2(-1) + (-3)^2(2)^2}{-13} = \frac{-24-18+36}{-13} = \boxed{\frac{6}{13}}$$

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$\boxed{7.15}$ given $x^2u - y^2 v = 1$ and $x+y = uv$

find $\left(\frac{\partial x}{\partial u}\right)_v ; \left(\frac{\partial x}{\partial u}\right)_y$

$\left(\frac{\partial x}{\partial u}\right)_v$ means that $x(u,v)$. i.e x is function of u, v only.

so take derivative w.r.t. u, we get from first equation:

$$\frac{\partial}{\partial u}(x^2u) - \frac{\partial}{\partial u}(y^2 v) = 0$$

$$\left(x^2 + 2x\frac{\partial x}{\partial u}u\right) - \left(2y\frac{\partial y}{\partial u}v\right) = 0$$

$$\frac{\partial x}{\partial u}(2xu) + \frac{\partial y}{\partial u}(-2yv) = -x^2 \qquad \text{——} ①$$

From second equation;

$$\frac{\partial}{\partial u}(x+y) = \frac{\partial}{\partial u}(uv)$$

$$\frac{\partial x}{\partial u} + \frac{\partial y}{\partial u} = v \qquad \text{——} ②$$

Solve ①, ② for $\frac{\partial x}{\partial u}$

from ② $\frac{\partial y}{\partial u} = v - \frac{\partial x}{\partial u}$

plug in ① $\Rightarrow$ $2xu\frac{\partial x}{\partial u} + \left[v - \frac{\partial x}{\partial u}\right](-2yv) = -x^2$

$$2xu\frac{\partial x}{\partial u} - 2v^2y + \frac{\partial x}{\partial u}2yv = -x^2$$

$$\frac{\partial x}{\partial u}\left[2xu + 2yv\right] = -x^2 + 2v^2y$$

$$\boxed{\left(\frac{\partial x}{\partial u}\right)_v = \frac{-x^2 + 2v^2y}{2xu + 2yv}} \longrightarrow$$

note at this point I have the feeling we should be solving this using 'differentials', not taking partial derivatives as I did. even though answer should be the same. (I am a little confused as which method to apply and when).

for example, I redo $\left(\frac{\partial x}{\partial u}\right)_v$ using differentials:

given $\begin{cases} x^2u - y^2v = 1 \\ x + y = uv \end{cases}$

$$\Rightarrow \begin{cases} 2x\,dx\,u + x^2\,du - (2y\,dy\,v + y^2\,dv) = 0 \\ dx + dy = du\,v + u\,dv \end{cases}$$

since $\left(\frac{\partial x}{\partial u}\right)_{v \to \text{means constant}}$. So $dv = 0$

So above equations become

$$\Rightarrow \begin{cases} 2x\,dx\,u + x^2\,du - 2y\,dy\,v = 0 & \text{—①} \\ dx + dy = du\,v & \text{—②} \end{cases}$$

from ① $dy = v\,du - dx$

plug in ② $\Rightarrow 2xu\,dx + x^2\,du - 2y(v\,du - dx)v = 0$

$$dx(2xu + 2yv) = du(-x^2 + 2yv^2)$$

so $\left(\frac{dx}{du}\right)_v = \boxed{\frac{-x^2 + 2v^2y}{2xu + 2yv}}$

This is $\left(\frac{\partial x}{\partial u}\right)$ really since partial derivative.

which is the same I got. Now I do $\left(\frac{\partial x}{\partial u}\right)_y \longrightarrow$

take differential:

$$\begin{cases} 2x\,dx\,u + x^2\,du - 2y\,dy\,v - y^2\,dv = 0 \\ dx + dy = du\,v + u\,dv. \end{cases}$$

since we want $\left(\frac{\partial x}{\partial u}\right)_y \rightarrow$ mean $dy = 0$.

so

$$\Rightarrow \begin{cases} 2x\,dx\,u + x^2\,du - y^2\,dv = 0 & \text{—①} \\ dx = du\,v + u\,dv & \text{—②} \end{cases}$$

from ②, $dv = \dfrac{dx - du\,v}{u}$

plug in ①

$$\Rightarrow 2x\,dx\,u + x^2\,du - y^2\left(\frac{dx - du\,v}{u}\right) = 0$$

$$dx\left(2xu - \frac{y^2}{u}\right) = du\left(-x^2 - y^2\frac{v}{u}\right)$$

so

$$\left(\frac{dx}{du}\right)_y = \frac{-x^2 - y^2\frac{v}{u}}{2xu - \frac{y^2}{u}} \boxed{= \frac{-ux^2 - y^2 v}{2xu^2 - y^2}}$$

$\downarrow$

$\left(\frac{\partial x}{\partial u}\right)_y$ since partial.

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$\boxed{7.19}$

if $z = r + s^2$

$x + y = s^3 + r^3 - 3$

$xy = s^2 - r^2$

find $\left(\frac{\partial x}{\partial z}\right)_s, \left(\frac{\partial x}{\partial z}\right)_r, \left(\frac{\partial x}{\partial z}\right)_y$ at $(r, s, x, y, z) = (-1, 2, 3, 13)$

$\left(\frac{\partial x}{\partial z}\right)_s$ means $x(z,s)$ and we take $ds = 0$ (constant):

$$\Rightarrow \left\{\begin{array}{l} dz = dr + 2s\,ds \\ dx + dy = 3s^2\,ds + 3r^2\,dr \\ x\,dy + y\,dx = 2s\,ds - 2r\,dr \end{array}\right\} A$$

since $ds = 0$, Then we get

$$\left\{\begin{array}{ll} dz = dr & \text{——①} \\ dx + dy = 3r^2\,dr & \text{——②} \\ x\,dy + y\,dx = -2r\,dr & \text{——③} \end{array}\right.$$

From ① sub dz into ③ $\Rightarrow$ $x\,dy + y\,dx = -2r(dz)$. ——④

From ④ $\rightarrow$ $dy = \frac{-2r\,dz - y\,dx}{x}$ ——⑤

From ⑤ sub dy into ② $\Longrightarrow$ $dx + \left(\frac{-2r\,dz - y\,dx}{x}\right) = 3r^2\,dr$ ——⑥

From ① sub dz into ⑥ to eliminate dr

$$x\,dx - 2r\,dz - y\,dx = 3r^2 x\,dz$$

$$dx\,(x - y) = dz\,(3r^2 x + 2r)$$

$$\boxed{\left(\frac{\partial x}{\partial z}\right)_s = \frac{3r^2 x + 2r}{x - y}}$$

now do $\left(\frac{\partial x}{\partial z}\right)_r$

set dr=0 in set of equations maked 'A' in previous page.

$$\Rightarrow \begin{cases} dz = 2s\,ds & \text{———} \textcircled{1} \\ dx+dy = 3s^2 ds & \text{———} \textcircled{2} \\ x\,dy+y\,dx = 2s\,ds & \text{———} \textcircled{3} \end{cases}$$

from ① $ds = \dfrac{dz}{2s}$

replace ds in ② and ③ by above value

$$\Rightarrow \begin{cases} dx+dy = 3s^2\left(\dfrac{dz}{2s}\right) = dx+dy = \dfrac{3}{2}s\,dz & \text{———} \textcircled{4} \\ x\,dy+y\,dx = 2s\left(\dfrac{dz}{2s}\right) = x\,dy+y\,dx = dz & \text{———} \textcircled{5} \end{cases}$$

from ⑤, $dy = \dfrac{dz - y\,dx}{x}$

replace into ④

$$dx + \left(\frac{dz - y\,dx}{x}\right) = \frac{3s\,dz}{2}$$

$$x\,dx + dz - y\,dx = \frac{3s\,dz\,x}{2}$$

$$dx(x-y) = dz\left(\frac{3sx}{2} - 1\right)$$

$$\boxed{\left(\frac{\partial x}{\partial z}\right)_r = \frac{3sx-2}{2(x-y)}}$$

Now do $\left(\frac{\partial x}{\partial z}\right)_y$.

replace $dy=0$ into 'A' equations shown befor.

$$\Rightarrow \begin{cases} dz = dr + 2sds & \text{——— ①} \\ dx = 3s^2ds + 3r^2dr & \text{——— ②} \\ ydx = 2sds - 2rdr & \text{——— ③} \end{cases}$$

from ① $dr = dz - 2sds$

replace dr from ② and ③ $\Rightarrow$

$$\Rightarrow \begin{cases} dx = 3s^2ds + 3r^2(dz - 2sds) \\ ydx = 2sds - 2r(dz - 2sds) \end{cases}$$

$$\Rightarrow \begin{cases} dx = 3s^2ds + 3r^2dz - 6r^2sds & \text{——— ④} \\ ydx = 2sds - 2rdz + 4rsds & \text{——— ⑤} \end{cases}$$

From ⑤ Find ds and replace into ④

$$ds(2s + 4rs) = ydx + 2rdz$$

$$ds = \left(\frac{ydx + 2rdz}{2s + 4rs}\right)$$

plug into ④ $\Rightarrow$ $dx = 3s^2\left(\frac{ydx + 2rdz}{2s + 4rs}\right) + 3r^2dz - 6r^2s\left(\frac{ydx + 2rdz}{2s + 4rs}\right)$

$$dx = \frac{3s^2ydx + 6s^2rdz}{2s + 4rs} + 3r^2dz - \frac{6r^2sydx + 12r^3sdz}{2s + 4rs}$$

$$dx(2s + 4rs) = 3s^2ydx + 6s^2rdz + 6sr^2dz + 12r^2sdz - 6r^2sydx - 12r^3sdz$$

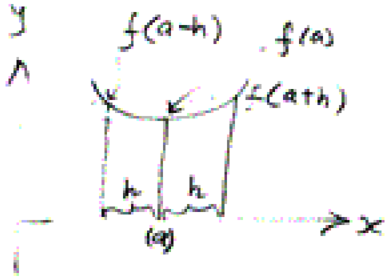
$$dx(2s + 4rs - 3s^2y + 6r^2sy) = dz(6s^2r + 6sr^2)$$

so $$\left(\frac{\partial x}{\partial z}\right)_y = \frac{6s^2r + 6sr^2}{2s + 4rs - 3s^2y + 6r^2sy} = \boxed{\frac{6r(s+r)}{2 + 4r - 3sy + 6r^2y}}$$

Ch4

$\boxed{8.1}$ Use the Taylor series about $x=a$ to verify the familiar "second derivative test" for a maximum or minimum point. That is, show that if $f'(a)=0$ and then $f''(a)>0$ implies a min point at $x=a$ and $f''<0$ implies a maximum point.

let me first look at the min case. Consider diagram:
here $f(a)$ is a min and need to show that this implies $f''(a)>0$.



since it is a min. Then $\dfrac{f(a+h)+f(a-h)}{2} > f(a)$ —— ①

where h is some small distance away from 'a' on either side.

but $f(a+h) = f(a) + h f'(a) + \frac{h^2}{2} f''(a) + \cdots$ ← ignore rest.

and $f(a-h) = f(a) + h f'(a) + \frac{h^2}{2} f''(a)$.

so from ① we get

$$\frac{\left[f(a)+h f'(a)+\frac{h^2}{2}f''(a)\right] + \left[f(a)+h f'(a)+\frac{h^2}{2}f''(a)\right]}{2} > f(a)$$

$$\frac{2f(a)+2h f'(a)+h^2 f''(a)}{2} > f(a)$$

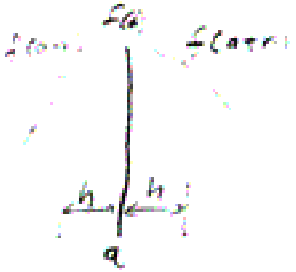
$$f(a)+h f'(a)+\frac{h^2}{2} f''(a) > f(a).$$

$$h f'(a)+\frac{h^2}{2} f''(a) > 0$$

but $f'(a)=0 \Rightarrow \frac{h^2}{2} f''(a)>0$ and since $h\neq 0, \Rightarrow \boxed{f''(a)>0}$

now similarly I show that for a max, it implies $f''(a) < 0$.

here we have $\frac{f(a-h)+f(a+h)}{2} < f(a)$



here $\frac{\left(f(a)+hf'(a)+h^2\frac{f''(a)}{2}\right)+\left(f(a)+hf'(a)+h^2\frac{f''(a)}{2}\right)}{2} < f(a)$

same as before we get

$$hf'(a) + \frac{h^2}{2}f''(a) < 0$$

but $f'(a)$ $\Rightarrow$ $\frac{h^2}{2} f''(a) < 0$

and since $h \neq 0$, $\Longrightarrow$ $f''(a) < 0$ QED)

ch 4

$\boxed{8.2}$ use a 2variable Taylor Series prove the following "second derivative tests" for max or min points of functions of 2 variables. If $f_x = f_y = 0$ at (a,b) then

(a,b) is min point if at (a,b) $f_{xx} > 0$, $f_{yy} > 0$ and $f_{xx} f_{yy} > f_{xy}^2$

(a,b) is max point if at (a,b) $f_{xx} < 0$, $f_{yy} < 0$ and $f_{xx} f_{yy} > f_{xy}^2$

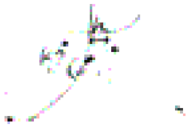
(a,b) is neither min nor max if $f_{xx} f_{yy} < f_{xy}^2$

given $f(x,y)$ at a point, (a,b), expand $f(x,y)$ near (a,b) using Taylor The first 3 terms are

$$f(x,y) = f(a,b) + \left[f_x(a,b)\,h + f_y(a,b)\,k\right] + \frac{1}{2!}\left[f_{xx}(a,b)\,h^2 + 2f_{xy}(a,b)\,hk + f_{yy}(a,b)\,k^2\right]$$

if (a,b) is min, Then $f(x,y)$ at all points close to it are greater than $f(a,b)$.

i.e. $$\frac{f(x,y)_+ + f(x,y)_-}{2} > f(a,b)$$



where $f(x,y)_+$ is one point and $f(x,y)_-$ is another point around (a,b).

Then we have $$\frac{2f(a,b) + 2\left[f_x h + f_y k\right] + \left[f_{xx} h^2 + 2f_{xy} hk + f_{yy} k^2\right]}{2} > f(a,b)$$

now $f_x = f_y = 0$ at (a,b), so above reduces to

$$f(a,b) + \left[f_{xx} h^2 + 2f_{xy} hk + f_{yy} k^2\right] > f(a,b)$$

or $$f_{xx} h^2 + 2f_{xy} hk + f_{yy} k^2 > 0$$

write $f_{xx} = A$, $f_{xy} = B$, $f_{yy} = C$

$$\Rightarrow \quad Ah^2 + 2Bhk + Ck^2 > 0 \qquad \longrightarrow$$

rewrite as $A\left(h+\frac{Bk}{A}\right)^2 + \left(C-\frac{B^2}{A}\right)k^2 > 0$

since $\left(h+\frac{Bk}{A}\right)^2$ is always positive, and k^2 is always positive,

So we have $\left[A \times \text{something positive} + \left(C-\frac{B^2}{A}\right) \times \text{something positive} > 0 \right]$

or $\left[\underbrace{A \times \text{something positive}} + \left(\frac{CA-B^2}{A}\right) \times \text{something positive} > 0 \right]$

This means from looking at A is positive.

and this means $\frac{CA-B^2}{A}$ is positive i.e. $CA-B^2$ is positive

i.e. $CA > B^2$ and since B^2 is always positive (since squared), Then CA is positive, and since A positive, then C is positive

Then

$$\begin{matrix} C>0 \\ A>0 \\ CA>B^2 \end{matrix} \implies \begin{matrix} f_{yy}>0 \\ f_{xx}>0 \\ f_{yy}f_{xx} > f_{xy}^2 \end{matrix}$$

⟶

now to show that (a,b) is a max point if at (a,b), $f_{xx}<0$, $f_{yy}<0$ and $f_{xx}f_{yy} > f_{xy}^2$, same argument is used.

This leads to

$$A\left(h+\frac{BK}{A}\right)^2+\left(C-\frac{B^2}{A}\right)k^2<0$$

again, since $\left(h+\frac{BK}{A}\right)^2$ is always positive, and k^2 is always positive, then

A is negative and $\left(C-\frac{B^2}{A}\right)$ is negative.

i.e $\dfrac{CA-B^2}{A} < 0$

since $A<0$, Then $CA-B^2>0$ to keep expression <0.

so $CA-B^2>0 \Rightarrow CA>B^2$. but B^2 is positive always, so CA must be positive. but $A<0 \Rightarrow C<0$.

i.e

$$\left.\begin{array}{l} A<0 \\ C<0 \\ CA>B^2 \end{array}\right\} \Longrightarrow \begin{array}{l} f_{xx}<0 \\ f_{yy}<0 \\ f_{yy}f_{xx} > f_{xy}^2 \end{array}$$

now need to show that (a,b) is neither a max nor a min if $f_{xx} f_{yy} < f_{xy}^2$.

looking at $$A\left(h+\frac{Bk}{A}\right)^2 + \left(\frac{CA-B^2}{A}\right)k^2 = 0$$

2 cases

assume $\boxed{\begin{matrix}A<0\\C>0\end{matrix}}$, Then since $\left(h+\frac{Bk}{A}\right)^2$ is always positive, Then $\left(\frac{CA-B^2}{A}\right)k^2$ must be positive (to have 0 as total result).

but $k^2>0$, $\Rightarrow \frac{CA-B^2}{A} > 0$.

But $A<0$, Then $CA-B^2<0 \Rightarrow \boxed{CA<B^2}$

assume $\boxed{\begin{matrix}A>0\\C<0\end{matrix}}$ Then since $\left(h+\frac{Bk}{A}\right)^2>0$, Then $\left(\frac{CA-B^2}{A}\right)k^2<0$.

since $A>0 \Rightarrow (CA-B^2)<0$ (since k^2 always positive)

so $\boxed{CA<B^2}$ again.

hence in both cases

$$f_{xx} f_{yy} < f_{xy}^2$$

QED

ch 4

$\boxed{8.3}$ find min and max.

$$f(x,y) = x^2 + y^2 + 2x - 4y + 10.$$

$f_x = 2x + 2$
$f_y = 2y - 4$
$f_{xx} = 2$
$f_{yy} = 2$
$f_{xy} = 0$

at min or max $\left.\begin{array}{l} f_x = 0 = 2x+2 \\ f_y = 0 = 2y-4 \end{array}\right\} \Rightarrow \boxed{\begin{array}{l} x = -1 \\ y = 2 \end{array}}$

now need to find if $(1,2)$ is max or min or saddle.

$f_{xx} > 0$
$f_{yy} > 0$

but $f_{xx} f_{yy} = 4 > \underbrace{f_{xy}}_{\overset{\shortparallel}{0}}{}^{2}$

so $f_{xx} > 0,\ f_{yy} > 0,\ f_{xx} f_{yy} > f_{xy}^2 \Rightarrow$ $\underline{\underline{\text{miN}}}$ point.