HW # 3

Math 121 A

NASSER ABBASI

UCB extension.

$\frac{2}{2}$

ch 2

$\boxed{9.2}$ express in form $x+iy$.

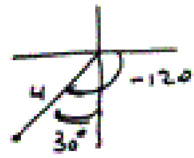
$e^{i\frac{\pi}{2}}$

hence $x = 0$
$y = 1$

so $e^{i\frac{\pi}{2}} = \boxed{0 + i}$

$\boxed{9.12}$ $4e^{-\frac{8}{3}i\pi}$

length $= 4$ angle $-\frac{8}{3}\pi = -120°$



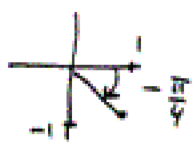
so $x = -4\cos 60° = -4\left(\frac{1}{2}\right) = -2$

$y = -4\cos 30° = -4\frac{\sqrt{3}}{2} = -2\sqrt{3}$

so $z = \boxed{-2-2\sqrt{3}\,i}$

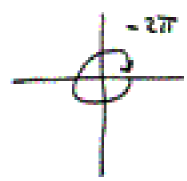
$\boxed{9.19}$ $z_1 = (1-i)^8$

$z = 1-i \Rightarrow r = \sqrt{2}$
$\theta = -\frac{\pi}{4}$



so $z = \sqrt{2}\, e^{-\frac{\pi}{4}i}$

so $z^8 = \left(\sqrt{2}\, e^{-\frac{\pi}{4}i}\right)^8 = 2^4 e^{-2\pi i} = 16 e^{-2\pi i}$



so $x = 16$
$y = 0$

i.e. $\boxed{16 + 0i}$

Ch 2

$\boxed{9.24}$ express in $x+iy$ form

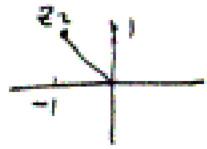
$$\frac{(1-i\sqrt{3})^{21}}{(i-1)^{38}}$$

$z_1 = 1 - i\sqrt{3}$

so $r = \sqrt{1+3} = 2$

$\theta = -60^\circ = -\frac{\pi}{3}$

$z_1 = 2e^{-\frac{\pi}{3}i}$



$z_2 = i - 1$

so $r_2 = \sqrt{2}$

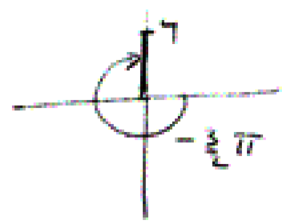
$\theta_2 = 90^\circ + 45^\circ = \frac{3}{4}\pi$

so $z_2 = \sqrt{2}\, e^{\frac{3}{4}i\pi}$

so $$z = \frac{z_1^{21}}{z_2^{38}} = \frac{(2e^{-\frac{\pi}{3}i})^{21}}{(\sqrt{2}\, e^{\frac{3}{4}i\pi})^{38}} = \frac{2^{21}\, e^{-\frac{21}{3}i\pi}}{2^{19}\, e^{\frac{3}{2}\pi i\,(19)}}$$

$$= 2^{2}\, e^{-7i\pi - \frac{57}{2}\pi i} = 4e^{\left(\frac{-14-57}{2}\right)i\pi} = 4e^{-\frac{71}{2}i\pi}$$

$$= 4e^{-35\frac{1}{2}i\pi} = 4e^{(-34\pi - \frac{3}{2}\pi)i} = 4e^{-\frac{3}{2}\pi i}$$



so $x = 0$

$y = 4$

the number is $\boxed{0 + 4i}$

ch 2

$\boxed{9.27}$ show that for any real y, $|e^{iy}|=1$. hence show that $|e^z|=e^x$ for every complex z.

$$|e^{iy}| = |\cos y + i\sin y| = \sqrt{\cos^2 y+\sin^2 y} = \sqrt{1} = 1$$

$$e^z = e^{x+iy} \quad \text{where } z=x+iy.$$

hence $|e^z| = |e^{x+iy}| = |e^x e^{iy}| = |e^x|\,|e^{iy}|$

but $|e^x| = e^x$ since real x.

and $|e^{iy}|=1$ from above.

hence $e^z = e^x$

$\boxed{9.28}$ show that absolute value of a product of two complex numbers is equal to the product of the abs values.

let the two complex numbers be z_1, z_2

we need to show that $|z_1 z_2| = |z_1|\,|z_2|$.

write z as $re^{i\theta}$.

so $|z_1 z_2| = |r_1 e^{i\theta_1}\, r_2 e^{i\theta_2}| = |r_1 r_2\, e^{i(\theta_1+\theta_2)}|$

$= r_1 r_2$ since this is the length of $z_1 z_2$.

$$|z_1|\;|z_2| = \underset{\downarrow \atop r_1}{|r_1 e^{i\theta_1}|}\;\underset{\downarrow \atop r_2}{|r_2 e^{i\theta_2}|} = r_1\, r_2$$

hence $|z_1 z_2| = |z_1|\,|z_2|$.

Now, show that abs value of quotient of two complex numbers is the quotient of the abs. values $\Rightarrow$

we need to show that $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

$$\left|\frac{z_1}{z_2}\right| = \left|\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}}\right| = \left|\frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}\right| = \frac{r_1}{r_2}$$

$$\frac{|z_1|}{|z_2|} = \frac{|r_1 e^{i\theta_1}|}{|r_2 e^{i\theta_2}|} = \frac{r_1}{r_2} \qquad \text{QED.}$$

$\boxed{10.18}$ Find all values of roots and plot them

$\sqrt{i}$

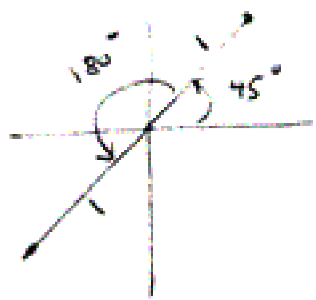
$\theta = \frac{\pi}{2}$, $r = 1$

so $z = e^{i\frac{\pi}{2}}$

so $z^{1/2} = 1^{1/2}\left(e^{i\frac{\pi}{2}}\right)^{1/2} = e^{i\left(\frac{\pi}{2} + 2\pi k\right)\frac{1}{2}}$ $\qquad k = 0, 1$

so roots $= e^{i\left(\frac{\pi}{4}\right)}$, $e^{i\left(\frac{\pi}{4} + \frac{2\pi}{2}\right)}$

$= e^{i\frac{\pi}{4}}$, $e^{i\left(\frac{5}{4}\pi\right)}$



we need to show that $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

$$\left|\frac{z_1}{z_2}\right| = \left|\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}}\right| = \left|\frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}\right| = \frac{r_1}{r_2}$$

$$\frac{|z_1|}{|z_2|} = \frac{|r_1 e^{i\theta_1}|}{|r_2 e^{i\theta_2}|} = \frac{r_1}{r_2} \qquad \text{QED.}$$

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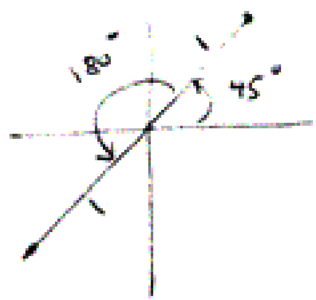
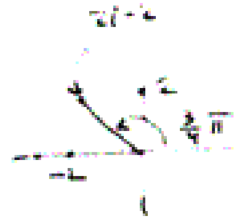


ch 2

$\boxed{10 \cdot 22}$ Find roots of

$$\sqrt[3]{2i-2}$$



$$r = \sqrt{2^2+2^2} = 2\sqrt{2}$$

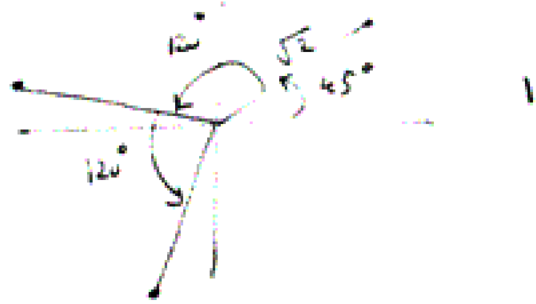
$$\theta = \frac{3}{4}\pi$$

so $$\left(2\sqrt{2}\, e^{i\left(\frac{3}{4}\pi\right)}\right)^{1/3} = \sqrt{2}\, e^{i\left(\frac{3}{4}\pi + 2\pi k\right)^{1/3}} \quad k=0,1,2$$

so roots

$$\sqrt{2}\, e^{i\left(\frac{1}{4}\pi\right)}, \quad \sqrt{2}\, e^{i\left(\frac{\pi}{4} + \frac{2\pi}{3}\right)}, \quad \sqrt{2}\, e^{i\left(\frac{\pi}{4} + \frac{4\pi}{3}\right)}$$

each root is $\frac{2\pi}{3} = 120^\circ$ away from previous root.



Ch 2

$\boxed{10.28}$ Find formula for $\sin 3\theta$ and $\cos 3\theta$.

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i\sin n\theta.$$

so put $n=3$ here.

$$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$$
$$= (\cos\theta + i\sin\theta)^2(\cos\theta + i\sin\theta)$$
$$= (\cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta)(\cos\theta + i\sin\theta)$$
$$= \cos^3\theta + i\cos^2\theta\sin\theta - \sin^2\theta\cos\theta - i\sin^3\theta$$
$$+ 2i\cos^2\theta\sin\theta - 2\cos\theta\sin^2\theta$$

$$= \cos^3\theta - 3\sin^2\theta\cos\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

so by equating real parts to real parts and imaginary parts to imaginary parts we get

$$\cos 3\theta = \cos^3\theta - 3\sin^2\theta\cos\theta$$
$$\sin 3\theta = -\sin^3\theta + 3\cos^2\theta\sin\theta$$

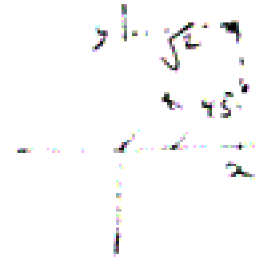
ch 2

$\boxed{11.5}$ Find in $x+iy$ form

$$Z = e^{i\frac{\pi}{4} + \frac{\ln 2}{2}} = \left(e^{\ln 2}\right)^{1/2} e^{i\frac{\pi}{4}}$$

but $e^{\ln x} = x$

so $z = 2^{1/2} e^{i\frac{\pi}{4}} = \sqrt{2}\, e^{i\frac{\pi}{4}}$



so $x = \sqrt{2} \cos\frac{\pi}{4} = \sqrt{2}\,\frac{1}{\sqrt{2}} = 1$

$y = \sqrt{2} \sin\frac{\pi}{4} = \sqrt{2}\,\frac{1}{\sqrt{2}} = 1$

so $z = \boxed{1+i}$

Ch 2

$\boxed{11.6}$ Find in $x+iy$ form

$\sin i$

since $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

Then $\sin i = \dfrac{e^{i^2} - e^{-i^2}}{2i} = \dfrac{e^{-1} - e^{1}}{2i}$

$= \dfrac{\frac{1}{e} - e}{2i} \dfrac{(-2i)}{(-2i)} = \dfrac{-2i\left(\frac{1}{e} - e\right)}{4}$

$= -\frac{i}{2}\left(\frac{1}{e} - e\right)$

so $x = 0$
$y = -\frac{1}{2}\left(\frac{1}{e} - e\right) \approx 1.1752$

so $\sin i = \boxed{1.1752\, i}$

ch 2
$\boxed{11.11}$ $\int_{-\pi}^{\pi} \cos 2x \cos 3x \, dx$

$$= \int_{-\pi}^{\pi} \left(\frac{e^{2xi} + e^{-2xi}}{2} \right) \left(\frac{e^{3xi} + e^{-3xi}}{2} \right) dx$$

$$= \frac{1}{4} \int_{-\pi}^{\pi} e^{5xi} + e^{-xi} + e^{xi} + e^{-5xi} \, dx$$

$$= \frac{1}{4} \left(\int_{-\pi}^{\pi} e^{5xi} dx + \int_{-\pi}^{\pi} e^{-xi} dx + \int_{-\pi}^{\pi} e^{xi} dx + \int_{-\pi}^{\pi} e^{-5xi} dx \right)$$

But $\int_{-\pi}^{\pi} e^{nxi} \, dx = \frac{1}{ni} \left[e^{nxi} \right]_{-\pi}^{\pi} = \frac{1}{ni} \left[e^{n\pi i} - e^{-n\pi i} \right]$

and $e^{n\pi i} = \cos n\pi + i \underbrace{\cancel{\sin n\pi}}_{=0}$.

and $e^{-n\pi i} = \cos -n\pi + i \sin -n\pi = \cos n\pi - i \underbrace{\cancel{\sin n\pi}}_{=0}$

So $e^{n\pi i} = e^{-n\pi i}$

So $\int_{-\pi}^{\pi} e^{nxi} \, dx = 0$ for any integer $n \neq 0$.

hence $\frac{1}{4} \left(\int_{-\pi}^{\pi} e^{5xi} dx + \ldots\ldots\ldots \right) = \frac{1}{4} (0 + 0 + 0 + 0) = 0$

Ch 2

$\boxed{11\cdot 18}$ evaluate $\int e^{(a+ib)x}\,dx$ to show that

$$\int e^{ax}\sin bx\,dx = e^{ax}\frac{(a\sin bx - b\cos bx)}{a^2+b^2}.$$

$$\int e^{ax}\sin bx\,dx = \int e^{ax}\left(\frac{e^{bix}-e^{-bix}}{2i}\right)dx$$

$$= \frac{1}{2i}\int e^{ax}e^{bix} - e^{ax}e^{-bix}\,dx$$

$$= \frac{1}{2i}\int e^{x(a+bi)} - e^{x(a-bi)}\,dx = \frac{1}{2i}\left[\frac{1}{a+bi}e^{x(a+bi)} - \frac{1}{a-bi}e^{x(a-bi)}\right]$$

$$= \frac{1}{2i}\left[\frac{(a-bi)e^{x(a+ib)} - (a+bi)e^{x(a-bi)}}{(a+bi)(a-bi)}\right]$$

$$= \frac{1}{2i}\left[\frac{(a-bi)e^{ax}e^{xib} - (a+bi)e^{ax}e^{-xbi}}{a^2+b^2}\right]$$

$$= \frac{e^{ax}}{a^2+b^2}\left[\frac{(a-bi)e^{xib} - (a+bi)e^{-xbi}}{2i}\right]$$

$$= \frac{e^{ax}}{a^2+b^2}\left[\frac{ae^{xib} - bie^{xib} - ae^{-xbi} - bie^{-xbi}}{2i}\right]$$

$$= \frac{e^{ax}}{a^2+b^2}\left[\frac{a(e^{ixb}-e^{-xbi})}{2i} - \frac{bi(e^{xib}+e^{-xbi})}{2i}\right]$$

$$= \frac{e^{ax}}{a^2+b^2}\left[a\sin xb - b\cos bx\right]$$

ch 2

$\boxed{14.6}$ Find one value of following in $x+iy$ form

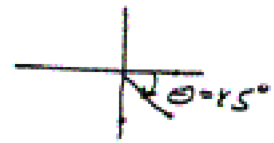
$$\ln\left(\frac{1-i}{\sqrt{2}}\right)$$

$\ln w$ is a multivalued function. we are asked to find one value.

first express $\frac{1-i}{\sqrt{2}}$ in polar.

$z = \frac{1-i}{\sqrt{2}}$, $r = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2}+\frac{1}{2}} = \sqrt{\frac{1}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}$

$\theta = -\frac{\pi}{4}$.



so $\ln\left(\frac{1}{2}\sqrt{2}\, e^{-i\frac{\pi}{4}}\right)$

$= \mathrm{Ln}\, \frac{\sqrt{2}}{2} + \ln e^{-i\left(\frac{\pi}{4}\right)}$

$= \mathrm{Ln}\, \frac{\sqrt{2}}{2} - i\left(\frac{\pi}{4} \pm 2n\pi\right)$

So $\ln w =$ $\mathrm{Ln}\frac{\sqrt{2}}{2} - i\frac{\pi}{4}$, $\mathrm{Ln}\frac{\sqrt{2}}{2} - i\frac{9}{4}\pi$, $\mathrm{Ln}\frac{\sqrt{2}}{2} - i\frac{17}{4}\pi$, etc...

pick the first one

$$\ln\left(\frac{1-i}{\sqrt{2}}\right) = \boxed{\mathrm{Ln}\frac{\sqrt{2}}{2} - i\frac{\pi}{4}}$$

ch 2

$\boxed{14.9}$ Find the value of $(-1)^i$ in the form $x+iy$.

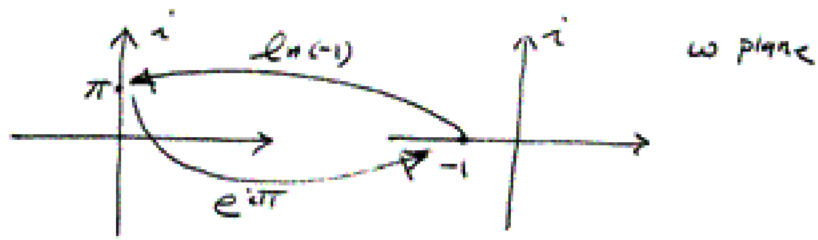
$$(-1)^i = e^{i\ln(-1)}$$

since $e^{i\ln(-1)} = (e^{\ln(-1)})^i = (-1)^i$

$$\text{so } (e^{\ln(-1)})^i = (e^{\pi i})^i = e^{-\pi} \overset{??}{=} -1$$

notice that $\ln(-1)$ is not defined in the real line.

but in complex plane, $\ln(-1) = i\pi$ (or $i(\pi \pm 2n\pi)$)



This is because $e^{i\pi} = e^{z} =$ ~~[illegible]~~ at $e^{0+i\pi} = e^{0} e^{i\pi}$

$$= 1\,(\cos\pi + i\sin\pi) = \underline{\underline{-1}}$$

So since $e^{i\pi} = -1$ Then by definition, $\ln(-1) = i\pi$

$$\text{so } (-1)^i = e^{i\ln(-1)} = e^{i(i(\pi \pm 2n\pi))} = e^{-(\pi \mp 2n\pi)}$$

$$= e^{-\pi} \text{ or } e^{-3\pi} \text{ or } e^{-4\pi}, \ldots \text{ or } e^{-\pi}, e^{\pi}, e^{3\pi}, \ldots$$

$\downarrow$

pick this $= \boxed{-1}$??

see next page

let $\omega = (-1)^i$

so $\ln(\omega) = \ln(-1)^i$

$= i \ln(-1)$

$= i \ln(e^{i(\pi \pm 2\pi k)})$

~~$= i \ln(e^{i})$~~

$\ln(\omega) = i(i(\pi \pm 2\pi k))$

$\ln(\omega) = -(\pi \pm 2\pi k)$

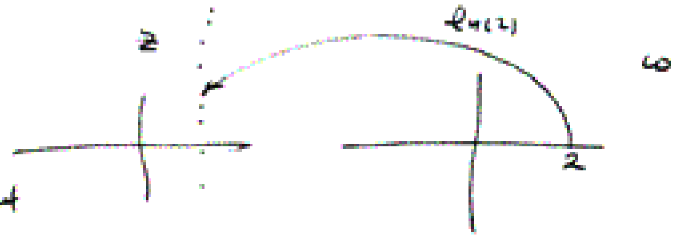
so $\omega = e^{-(\pi \pm 2\pi k)}$

so $\omega = e^{-\pi}$ or $e^{-\pi+2\pi}$ or $e^{-\pi-2\pi}$ or ...

$$\boxed{\omega = e^{-\pi} \text{ or } e^{\pi} \text{ or } e^{-3\pi} \text{ or } e \ldots}$$

ch 4 express in $x+iy$:

(14.14) $\cos(\pi + i \ln 2)$



first find $\ln(2)$, a multivalued function in complex plane.

let $z = \ln(2)$.

$$e^z = e^{\ln(2)} = e^{\ln(2e^{i(0\pm 2n\pi)})} = e^{\ln(2) + \ln e^{i(0\pm 2n\pi)}}$$

so $z = \ln(2) + i(0 \pm 2n\pi)$

so $\cos(\pi + i \ln 2) = \cos(\pi + i(\ln(2) + i(0\pm 2n\pi)))$

$$= \cos(\pi + i\ln(2) - (0 \pm 2n\pi)) \qquad n = 0,1,2,\dots$$

$$= \cos(\pi \mp 2n\pi + i\ln(2))$$

$$= \cos(\pi(1\mp 2n) + i\ln(2)) \qquad n=0,1,2,\dots$$

let $z = \pi(1\mp 2n) + i\ln(2)$.

so $\cos z = \dfrac{e^{iz} + e^{-iz}}{2} = \dfrac{e^{i(\pi(1\mp 2n) + i\ln(2))} + e^{-i(i\ln(2) + \pi(1\mp 2n))}}{2}$

$$= \frac{e^{i\pi(1\mp 2n)} e^{-\ln(2)} + e^{-i(\pi(1\mp 2n))} e^{\ln(2)}}{2}$$

now $e^{i\pi(1\mp 2n)} = \cos \pi(1\mp 2n) + i \sin(\pi(1\mp 2n)) \qquad n=0,1,2,\dots$

$= -1$ for any n

also $e^{-i(\pi(1\mp 2n))} = \cos -\pi(1\mp 2n) - i\sin(\pi(1\mp 2n))$

$= \cos \pi(1\mp 2n) - i\sin(\pi(1\mp 2n)) = -1$ for any n

so $\cos z = \dfrac{-e^{-\ln(2)} - e^{\ln(2)}}{2} = -\dfrac{1}{2}\left(e^{\ln(2)} + e^{-\ln(2)}\right)$

$$= \boxed{-1.25} \text{ or } \boxed{-\tfrac{5}{4}} \Rightarrow -\tfrac{5}{4} + 0i$$

ch 2

$\boxed{14.22}$ find in $x+iy$ form

$$\sin\left(i \ln\left(\frac{\sqrt{3}+i}{2} \right) \right)$$

let $z = \frac{\sqrt{3}+i}{2}$. $|z| = \sqrt{\frac{3}{4}+\frac{1}{4}} = 1$

so $z = 1\left(\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\right)$ i.e. $\cos\theta = \frac{\sqrt{3}}{2}$, $\sin\theta = \frac{1}{2}$

so $\theta = 30^\circ = \frac{\pi}{6}$

so $z = 1 e^{i\frac{\pi}{6}}$

so $e^{\ln(1\cdot e^{i\frac{\pi}{6}})} = e^{\mathrm{Ln}(1) + \ln(e^{i\pi/6})} = e^{\mathrm{Ln}(1) + i(\pi/6 \pm 2n\pi)}$

so $\sin\left(i \ln\left(\frac{\sqrt{3}+i}{2}\right)\right) = \sin\left(i\left(\mathrm{Ln}(1) + i(\pi/6 \pm 2n\pi)\right)\right)$

$\mathrm{Ln}(1) = 0$

so $= \sin(-(\pi/6 \pm 2n\pi))$ $n = 0, 1, 2, \ldots$ $-1, -2, \ldots$

pick $n=0$

$= \sin\left(-\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = \boxed{-\frac{1}{2}}$

or $\boxed{-\frac{1}{2} + 0\,i}$

ch 2 15

$\boxed{14.24}$ (b) show that $(a^b)^c$ can have more values than a^{bc}.

$(i^i)^i$ and i^{-1}

so need to show that $(i^i)^i$ can have more values than i^{-1}.

$$(i^i)^i = ((e^{\ln i})^i)^i = ((e^{\ln(1 e^{i(\frac{\pi}{2}\pm 2n\pi)})})^i)^i$$

$$= ((e^{\ln(1) + \ln e^{i(\frac{\pi}{2}\pm 2n\pi)}})^i)^i \qquad n=0,1,2,\ldots$$

$$= ((e^{\ln(1)}\, e^{\ln e^{i(\pi/2 \pm 2n\pi)}})^i)^i$$

$$= ((e^{i(\frac{\pi}{2}\pm 2n\pi)})^i)^i = (e^{-(\frac{\pi}{2}\pm 2n\pi)})^i$$

$$= e^{-i(\frac{\pi}{2}\pm 2n\pi)}$$

$$= \cos\left(\tfrac{\pi}{2}\pm 2n\pi\right) - i\sin\left(\tfrac{\pi}{2}\pm 2n\pi\right).$$

but $\cos\frac{\pi}{2}\pm 2n\pi = 0$ for any n.

so $(i^i)^i = -i\sin(\frac{\pi}{2}\pm 2n\pi)$

which is $-i\sin(\frac{\pi}{2})$, $-i\sin(\frac{\pi}{2}+2\pi)$, $-i\sin(\frac{\pi}{2}+4\pi)$, ...

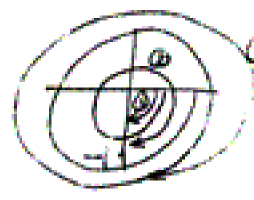
$-i\sin(\frac{\pi}{2}-2\pi)$, $-i\sin(\frac{\pi}{2}-4\pi)$, ...

while $i^{-1} = -i\sin(\frac{\pi}{2})$

These can be seen better if plotted.

$i^{-1} = -i$

one value. while $-i\sin(\frac{\pi}{2}+2n\pi)$



these for different n.

ch 2
$\boxed{15.3}$ Find in $x+iy$ form.

$\cosh^{-1}(1/2)$.

$\cosh z = \frac{1}{2}$

$\frac{e^{z}+e^{-z}}{2} = \frac{1}{2} \Rightarrow e^{z}+e^{-z} = 1$

let $u = e^{z} \Rightarrow u + u^{-1} = 1$ multiply by u

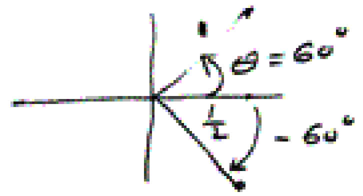
$u^2 + 1 - u = 0$

$u^2 - u + 1 = 0 \Rightarrow u = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where $a=1$, $b=-1$, $c=1$

so $u = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$.

$|u| = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1$

$u = 1\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2} i\right)$ $\theta = \frac{\pi}{3}$ or $-\frac{\pi}{3}$



so $z = \operatorname{Ln}(1) + i\left(\pm\frac{\pi}{3} \pm 2n\pi\right)$

$= \operatorname{Ln}(1) + i\left(\pm\frac{\pi}{3} \pm 2n\pi\right) = \boxed{i\left(\pm\frac{\pi}{3} + 2n\pi\right)}$ $n = 0, 1, 2, \ldots$

So $\cosh^{-1}(z)$ is multivalued function like $\operatorname{Ln}(z)$ is.

Ch 2 [illegible] show that $\tan z$ can never take value of $\pm i$

assume $\tan z = i$

i.e $\frac{\sin z}{\cos z} = i$

i.e $$\frac{\frac{e^{iz}-e^{-iz}}{2i}}{\frac{e^{iz}+e^{-iz}}{2}} = i$$

i.e $$-i\frac{(e^{iz}-e^{-iz})}{e^{iz}+e^{-iz}} = i$$

i.e $$\frac{-e^{iz}+e^{-iz}}{e^{iz}+e^{-iz}} = 1$$

i.e $-e^{iz}+e^{-iz}-e^{iz}-e^{-iz}=0$

i.e $-2e^{iz}=0$

$e^{iz}=0$

$\Longrightarrow$ iz has no value which can make $e^{iz}=0$.

$e^{x}=0$ has no solution.

hence $\tan z$ can not be i.

if I let $\tan z = -i$, then my difference is that I get

$e^{iz}-e^{-iz}+e^{iz}-e^{-iz}=0$

or $-2e^{-iz}=0$

or $e^{-iz}=0$ $\Longrightarrow$ again, not possible

QED

ch 2
$\boxed{16.11}$ prove that $\cos\theta + \cos 3\theta + \cdots + \cos(2n-1)\theta = \dfrac{\sin 2n\theta}{2\sin\theta}$

and $\sin\theta + \sin 3\theta + \cdots + \sin(2n-1)\theta = \dfrac{\sin^2 n\theta}{\sin\theta}$.

write $e^{i\theta} + e^{i3\theta} + \cdots + e^{i(2n-1)\theta} = \dfrac{a(1-r^n)}{1-r}$

where $a = e^{i\theta}$, $r = e^{i2\theta}$. and use Euler relationship.

$$(\cos\theta + i\sin\theta) + (\cos 3\theta + i\sin 3\theta) + \cdots (\cos(2n-1)\theta + i\sin(2n-1)\theta) = \frac{e^{i\theta}(1 - e^{i2n\theta})}{1 - e^{i2\theta}}$$

so $(\cos\theta + \cos 3\theta + \cdots + \cos(2n-1)\theta) + i(\sin\theta + \sin 3\theta + \cdots + \sin(2n-1)\theta) =$

looking at RHS: denominator is:

$$1 - e^{i2\theta} = \underbrace{e^{i\theta}e^{-i\theta}}_{1} - \underbrace{e^{i\theta}e^{i\theta}}_{e^{i2\theta}} = e^{i\theta}(e^{-i\theta} - e^{i\theta})$$

$$= \frac{(2i)}{(2i)} e^{i\theta}(e^{-i\theta} - e^{i\theta})$$

$$= (2i)\, e^{i\theta} \frac{(e^{-i\theta} - e^{i\theta})}{2i}$$

$$= (-2i)\, e^{i\theta} \frac{(e^{i\theta} - e^{-i\theta})}{2i} = (-2i) e^{i\theta} \sin\theta$$

numerator is: $e^{i\theta}(1 - e^{i2n\theta}) = e^{i\theta}\Big[\underbrace{e^{in\theta}e^{-in\theta}}_{1} - \underbrace{(e^{in\theta}e^{in\theta})}_{e^{i2n\theta}}\Big]$

$$= e^{i\theta}\left[e^{in\theta}(e^{-in\theta} - e^{in\theta})\right]$$

$$= e^{i\theta}\left[\frac{(2i)}{(2i)} e^{in\theta}(e^{-in\theta} - e^{in\theta})\right] = e^{i\theta}\left[(-2i)e^{in\theta}\frac{(e^{in\theta} - e^{-in\theta})}{2i}\right]$$

$$= e^{i\theta}\left[-2i\, e^{in\theta} \sin n\theta\right]$$

$$\text{whole RHS} = \frac{e^{i\theta}\left[-2i\, e^{in\theta}\sin n\theta\right]}{-2i\, e^{i\theta}\sin\theta} = e^{in\theta}\frac{\sin n\theta}{\sin\theta}$$

$$= (\cos n\theta + i\sin n\theta)\frac{\sin n\theta}{\sin\theta} = \frac{\cos n\theta \sin n\theta}{\sin\theta} + i\,\frac{\sin n\theta \sin n\theta}{\sin\theta} \Rightarrow$$

now equate real parts and imaginary parts ⇒

$$\cos\theta + \cos 3\theta + \cdots + \cos(2n-1)\theta = \frac{\cos n\theta \sin n\theta}{\sin\theta} \qquad (1)$$

and $$\sin\theta + \sin 3\theta + \cdots + \sin(2n-1)\theta = \frac{\sin n\theta \sin n\theta}{\sin\theta} \qquad (2)$$

$$\sin n\theta \cos n\theta = \frac{1}{2}\sin 2n\theta$$

so (1) becomes $$\boxed{\cos\theta + \cos 3\theta + \cdots = \frac{\sin 2n\theta}{2\sin\theta}}$$

and (2) is $$\boxed{\sin\theta + \sin 3\theta + \cdots + \sin(2n-1)\theta = \frac{\sin^2 n\theta}{\sin\theta}}$$

Ch 2

$\boxed{17.14}$ Find the circle of convergence of series

$$\sum \frac{(z-2i)^n}{n}$$

$$\rho_n = \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(z-2i)^{n+1}}{n+1}}{\frac{(z-2i)^n}{n}}\right| = \left|\frac{(z-2i)\,n}{n+1}\right|$$

$$\rho = \lim_{n\to\infty} \rho_n = |z-2i|$$

so converges for $|z-2i| < 1$

let $z = x+iy$.

then we want $|x+iy-2i| < 1$

$$|x+i(y-2)| < 1$$

$$\sqrt{x^2+(y-2)^2} < 1$$

$$x^2+(y-2)^2 < 1$$

$$x^2+y^2+4-4y < 1$$

$$x^2+y^2-4y < -3 \qquad ①$$

equation of circle can be written as

$$(x-x_0)^2+(y-y_0)^2 = r^2$$

So x^2+y^2-4y can be written as $(x-0)^2+(y-2)^2-4$

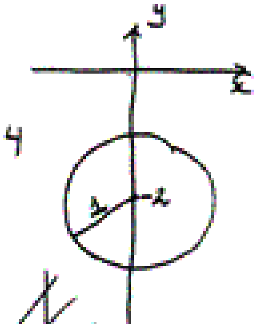
so ① becomes $(x-0)^2+(y-2)^2-4 < -3$

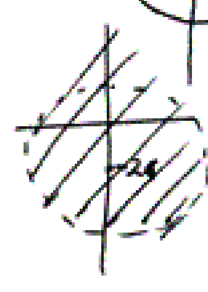
i.e. $(0,+2)$ $\qquad (x-0)^2+(y-2)^2 < +1$

So center is $(0,-2)$

and radius $r^2 < +1$

$\Rightarrow r < 1$





ch 2

$\boxed{17.17}$ verify $\arcsin z = -i \ln(iz \pm \sqrt{1-z^2})$

let $\arcsin z = w$

so $\sin w = z$

$$z = \frac{e^{iw} - e^{-iw}}{2i}$$

now need to find w in terms of to get answer required.

e^{iw} is a complex number. let $e^{iw} = \alpha$

$$\text{so } z = \frac{\alpha - \alpha^{-1}}{2i} = \frac{\alpha^2 - 1}{2i\alpha}$$

so $z(2i\alpha) = \alpha^2 - 1$

$\alpha^2 - 1 - \alpha(2iz) = 0 \implies \alpha = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{2iz \pm \sqrt{-4z^2 - 4(-1)}}{2}$$

$$= \frac{2iz \pm 2\sqrt{1-z^2}}{2}$$

so $\alpha = iz \pm \sqrt{1-z^2}$

but $\alpha = e^{iw}$

so $e^{iw} = iz \pm \sqrt{1-z^2}$

so $\ln e^{iw} = \ln(iz \pm \sqrt{1-z^2})$

$iw = \ln(iz \pm \sqrt{1-z^2})$

$w = \frac{1}{i} \ln(iz \pm \sqrt{1-z^2})$

$$\boxed{w = -i \ln(iz \pm \sqrt{1-z^2})}$$

ch 2

$\boxed{17.23}$ verify $\cos iz = \cosh z$.

$$\cos iz = \frac{e^{i(iz)} + e^{-i(iz)}}{2} = \frac{e^{-z} + e^{+z}}{2} = \cosh(z).$$

$\boxed{17.24}$ verify $\cosh iz = \cos z$.

$$\cosh(iz) = \frac{e^{-(iz)} + e^{(iz)}}{2} = \frac{e^{iz} + e^{-iz}}{2} = \cos(z)$$

$\boxed{17.30}$ write series for $e^{x(1+i)}$. write $1+i$ in the $re^{i\theta}$ form and obtain the powers of $(1+i)$. Then show for example that $e^x \cos x$ series has no x^2 term, no x^6 term, etc. and a similar result for $e^x \sin x$. Find a formula for the general term for each series.

power series for $e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

so $e^{x(1+i)} = 1 + (x+ix) + \frac{(x+ix)^2}{2!} + \frac{(x+ix)^3}{3!} + \cdots$

but I need to rewrite using $e^{i\theta}$. so above is not useful.

$(1+i) = \sqrt{2}\, e^{i\frac{\pi}{4}}$

so $e^{x(1+i)} = e^{x\sqrt{2} e^{i\frac{\pi}{4}}} = 1 + x\sqrt{2} e^{i\frac{\pi}{4}} + \frac{x^2 (\sqrt{2} e^{i\frac{\pi}{4}})^2}{2!} + \cdots$

$$= 1 + x\sqrt{2} e^{i\frac{\pi}{4}} + \frac{x^2}{2!} 2 e^{i\frac{\pi}{2}} + \frac{x^3}{3!} 2\sqrt{2}\, e^{i\frac{3}{4}\pi} + \cdots \frac{(x\sqrt{2} e^{i\frac{\pi}{4}})^n}{n!}$$

now $e^{x(1+i)} = e^x e^{xi} = e^x(\cos x + i \sin x)$

So $e^x(\cos x + i\sin x) = \sum_{n=0}^{\infty} \frac{(x\sqrt{2} e^{i\frac{\pi}{4}})^n}{n!}$ $\Rightarrow$

ch 2

$\boxed{17.23}$ verify $\cos iz = \cosh z$.

$$\cos iz = \frac{e^{i(iz)} + e^{-i(iz)}}{2} = \frac{e^{-z} + e^{+z}}{2} = \cosh(z).$$

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So $e^{x(1+i)} = e^{x\sqrt{2}e^{i\frac{\pi}{4}}} = 1 + x\sqrt{2}e^{i\frac{\pi}{4}} + \frac{x^2 (\sqrt{2}e^{i\frac{\pi}{4}})^2}{2!} + \cdots$

$$= 1 + x\sqrt{2}e^{i\frac{\pi}{4}} + \frac{x^2}{2!} 2e^{i\frac{\pi}{2}} + \frac{x^3}{3!} 2\sqrt{2}\, e^{i\frac{3}{4}\pi} + \cdots \frac{(x\sqrt{2}e^{i\frac{\pi}{4}})^n}{n!}$$

now $e^{x(1+i)} = e^x e^{xi} = e^x(\cos x + i \sin x)$

So $e^x(\cos x + i\sin x) = \sum_{n=0}^{\infty} \frac{(x\sqrt{2}e^{i\frac{\pi}{4}})^n}{n!}$ $\Rightarrow$

$$e^x\cos x + i\, e^x \sin x = \sum_0^\infty \frac{x^n\, 2^{\frac{n}{2}}\, e^{i\frac{n\pi}{4}}}{n!}$$

$$e^x\cos x + i\, e^x \sin x = \sum_0^\infty \frac{x^n\, 2^{\frac{n}{2}}}{n!}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)$$

$$= \sum_0^\infty \left(\frac{(x\sqrt{2})^n}{n!}\cos\frac{n\pi}{4}\right) + i\left(\frac{(x\sqrt{2})^n}{n!}\sin\frac{n\pi}{4}\right)$$

Compare real parts and imaginary parts.

$$\boxed{\begin{aligned} e^x\cos x &= \sum_0^\infty \frac{(x\sqrt{2})^n}{n!}\cos\frac{n\pi}{4} \\ e^x\sin x &= \sum_0^\infty \frac{(x\sqrt{2})^n}{n!}\sin\frac{n\pi}{4} \end{aligned}}$$

$n=2,\ \theta=90°$
$n=6,\ \theta=270$
$n=10,\ \theta=90°+360$
etc....

now when $n=2, 6, 10 \ldots$ ~~etc~~, then $\cos\frac{n\pi}{4}=0$

hence $e^x\cos x$ is represented by series with no $n=2, 6, 10, \ldots$ i.e with no $x^2, x^6, x^{10} \ldots$

Similarly, looking at the $e^x\sin x$ series and asking when will $\sin\frac{n\pi}{4}$ be zero. this happen at $\theta = 0, 180, 360$, etc... i.e at $n = 0, 4, 8, 12 \ldots$

So The $e^x\sin x$ series has no $x^4, x^8, x^{12}, \ldots$ term

QED

Ch 2

$\boxed{17.32}$ Use series you know to show that

$$\sum_{n=0}^{\infty} \frac{(1+i\pi)^n}{n!} = -e$$

$$-e = e(-1)$$

$$= e(\cos\pi + i\sin\pi)$$

$$= e\, e^{i\pi} \quad \text{, using } \text{~~[illegible]~~ Euler formula}$$

$$= e^{1+i\pi}$$

$$= 1 + (1+i\pi) + \frac{(1+i\pi)^2}{2!} + \frac{(1+i\pi)^3}{3!} + \cdots$$

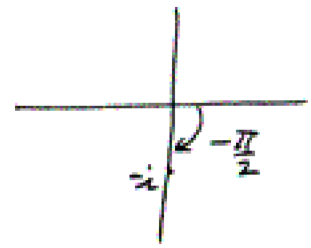
$$= \sum_{0}^{\infty} \frac{(1+i\pi)^n}{n!}$$

Ch 2

11.1 Find one or more values of each of the following complex number

$(-i)^i$

ω plane



$$(-i)^i = \left(e^{\ln(-i)}\right)^i$$

$$= \left(e^{\ln\left(1 \cdot e^{i\left(-\frac{\pi}{2} \pm 2n\pi\right)}\right)}\right)^i$$

$$= \left(e^{\operatorname{Ln}(1) + \ln\left(e^{i\left(-\frac{\pi}{2} \pm 2n\pi\right)}\right)}\right)^i$$

$$= \left(e^{\operatorname{Ln}(1)} \; e^{\ln\left(e^{i\left(-\frac{\pi}{2} \pm 2n\pi\right)}\right)}\right)^i$$

$$= \left((e^0)\left(e^{i\left(-\frac{\pi}{2} \pm 2n\pi\right)}\right)\right)^i$$

$$= e^{-\left(-\frac{\pi}{2} \pm 2n\pi\right)} = e^{\frac{\pi}{2} \mp 2n\pi} \qquad n = 0, 1, 2, \ldots$$

So $(-i)^i = e^{\pi/2}$ or $e^{\pi/2 - 2\pi}$ or $e^{\pi/2 + 2\pi}$ or $e^{\pi/2 - 4\pi}$ or ...

$= e^{\pi/2}$ or $e^{-\frac{3}{2}\pi}$ or $e^{5/2\,\pi}$ or ...