PROBLEM 13.61

A thin circular rod is supported in a vertical plane by a bracket at A. Attached to the bracket and loosely wound around the rod is a spring of constant $k = 40 \text{ N/m}$ and undeformed length equal to the arc of circle $AB$. A 200-g collar C, not attached to the spring, can slide without friction along the rod. Knowing that the collar is released from rest when $\theta = 30^\circ$, determine (a) the maximum height above point B reached by the collar, (b) the maximum velocity of the collar.

SOLUTION

(a) Maximum height

Above B is reached when the velocity at E is zero

\[
\begin{align*}
T_C &= 0 \\
T_E &= 0 \\
V &= V_e + V_g
\end{align*}
\]

Point C

\[
\Delta L_{BC} = (0.3 \text{ m})\left(\frac{\pi}{6} \text{ rad}\right)
\]

\[
\Delta L_{BC} = \frac{\pi}{20} \text{ m}
\]

\[
(V_C)_e = \frac{1}{2} k (\Delta L_{BC})^2 = \frac{1}{2} (40 \text{ N/m})\left(\frac{\pi}{20} \text{ m}\right)^2 = 0.4935 \text{ J}
\]

\[
(V_C)_g = WR(1 - \cos \theta) = (0.2 \text{ kg} \times 9.81 \text{ m/s}^2)(0.3 \text{ m})(1 - \cos 30^\circ)
\]

\[
(V_C)_g = 0.07886 \text{ J}
\]

\[
(V_E)_e = 0 \quad \text{(spring is unattached)}
\]

\[
(V_E)_g = WH = (0.2 \times 9.81)(H) = 1.962H \text{ (J)}
\]

\[
T_C + V_C = T_E + V_E
\]

\[
0 + 0.4935 + 0.07886 = 0 + 0 + 1.962H
\]

\[
H = 0.292 \text{ m} \uparrow
\]
PROBLEM 13.61 CONTINUED

(b) The maximum velocity is at $B$ where the potential energy is zero, $v_B = v_{\text{max}}$

\[ T_C = 0 \quad V_C = 0.4935 + 0.07886 = 0.5724 \text{ J} \]

\[ T_B = \frac{1}{2}mv_B^2 = \frac{1}{2}(0.2 \text{ kg})v_{\text{max}}^2 \]

\[ T_B = 0.1v_{\text{max}}^2 \]

\[ V_B = 0 \]

\[ T_C + V_C = T_B + V_B \quad 0 + 0.5724 = (0.1)v_{\text{max}}^2 \]

\[ v_{\text{max}}^2 = 5.72 \text{ m}^2/\text{s}^2 \]

\[ v_{\text{max}} = 2.39 \text{ m/s} \]
A thin circular rod is supported in a vertical plane by a bracket at A. Attached to the bracket and loosely wound around the rod is a spring of constant \( k = 40 \) N/m and undeformed length equal to the arc of circle AB. A 200-g collar C, not attached to the spring, can slide without friction along the rod. Knowing that the collar is released from rest at an angle \( \theta \) with respect to the vertical, determine (a) the smallest value of \( \theta \) for which the collar will pass through D and reach point A, (b) the velocity of the collar as it reaches point A.

**SOLUTION**

(a) Smallest angle \( \theta \) occurs when the velocity at D is close to zero

\[
\begin{align*}
V_C &= 0 \\
V_D &= 0
\end{align*}
\]

\[
\begin{align*}
T_C &= 0 \\
T_D &= 0
\end{align*}
\]

\[
V = V_e + V_g
\]

Point C

\[
\Delta L_{BC} = (0.3 \text{ m}) \theta = 0.3 \theta \text{ m}
\]

\[
(V_C)_e = \frac{1}{2}k(\Delta L_{BC})^2
\]

\[
(V_C)_e = 1.8 \theta^2
\]

\[
(V_C)_g = WR(1 - \cos \theta)
\]

\[
(V_C)_g = (1.962 \text{ N})(0.3 \text{ m})(1 - \cos \theta)
\]

\[
V_C = (V_C)_e + (V_C)_g = 1.8 \theta^2 + 0.5886(1 - \cos \theta)
\]

Point D

\[
(V_D)_e = 0 \quad \text{(spring is unattached)}
\]

\[
(V_D)_g = W(2R) = (2)(1.962 \text{ N})(0.3 \text{ m}) = 1.1772 \text{ J}
\]

\[
T_C + V_C = T_D + V_D, \quad 0 + 1.8 \theta^2 + 0.5886(1 - \cos \theta) = 1.1772 \text{ J}
\]

\[
(1.8) \theta^2 - (0.5886) \cos \theta = 0.5886
\]

By trial \( \theta = 0.7522 \text{ rad} \)

\[ \theta = 43.1^\circ \]

\[ \theta = 43.1^\circ \]
PROBLEM 13.62 CONTINUED

(b) Velocity at A

Point D

\[ V_D = 0 \quad T_D = 0 \quad V_D = 1.1772 \text{ J (see Part (a))} \]

Point A

\[ T_A = \frac{1}{2} mv_A^2 = \frac{1}{2} (0.2 \text{ kg}) v_A^2 \]

\[ T_A = 0.1v_A^2 \]

\[ V_A = (V_A)_g = W(R) = (1.962 \text{ N})(0.3 \text{ m}) = 0.5886 \text{ J} \]

\[ T_A + V_A = T_D + V_D \]

\[ 0.1v_A^2 + 0.5886 = 0 + 1.1772 \]

\[ v_A^2 = 5.886 \text{ m}^2/\text{s}^2 \]

\[ v_A = 2.43 \text{ m/s} \]
PROBLEM 13.63

A 6-lb collar can slide without friction on a vertical rod and is resting in equilibrium on a spring. It is pushed down, compressing the spring 6 in., and released. Knowing that the spring constant is \( k = 15 \text{ lb/in.} \), determine (a) the maximum height \( h \) reached by the collar above its equilibrium position, (b) the maximum velocity of the collar.

SOLUTION

(a) Maximum height when \( v_2 = 0 \)

\[
T_2 = T_2 = 0
\]

\[ V = V_g + V_e \]

Position \( \Phi \)

\[ \left( V_g \right)_1 = 0 \]

\[
\left( V_e \right)_1 = \frac{1}{2} k x_1^2 = \frac{1}{2} (15 \text{ lb/in.})(6.4 \text{ in.})^2
\]

\[ = 307.2 \text{ lb-in.} = 25.6 \text{ lb-ft} \]

Position \( \Theta \)

\[ \left( V_g \right)_2 = mg \left( \frac{6}{12} + h \right) = 6(0.5 + h) \]

\[ \left( V_e \right)_2 = 0 \]

\[ T_1 + V_1 = T_2 + V_2 : \left( V_g \right)_1 + \left( V_e \right)_1 = \left( V_g \right)_2 + \left( V_e \right)_2 \]

\[ 25.6 = 6(0.5 + h) \]

\[ h = 3.767 \text{ ft} \]

\[ h = 45.2 \text{ in.} \]

(b) Maximum velocity occurs when acceleration is 0, equilibrium position

\[ T_3 = \frac{1}{2} m v_3^2 = \frac{1}{2} \left( \frac{6}{32.2} \right) v_3^2 = 0.093167 v_3^2 \]

\[ V_3 = \left( V_g \right)_3 + (V_e)_3 = 6(6) + \frac{1}{2} k (x_1 - 6)^2 = 36 + 7.5(6.4 - 6)^2 \]

\[ = 37.2 \text{ lb-in.} = 3.1 \text{ lb-ft} \]

\[ T_1 + V_1 = T_3 + V_3 : 25.6 = 0.093167 v_3^2 + 3.1 \]

\[ v_{max} = 15.54 \text{ ft/s} \]
(b) Force of rod on collar $AC$

$F_z = 0$ (no friction)

$F = F_x i + F_y j$

$\theta = \tan^{-1} \frac{75}{300} = 14.04^\circ$

$F_e = (k\Delta L_{AC})(\cos \theta i + \sin \theta k)$

$F_e = (320)(0.10923)(\cos 14.04^\circ i + \sin 14.04^\circ k)$

$F_e = 33.909 i + 8.4797 k$ (N)

$\Sigma F = (F_x + 33.909)i + (F_y - 4.905)j + 8.4797k = \frac{mv^2}{r} j + mgk$

$F_x + 33.909$ N = 0

$F_y = 4.905$ N + (0.5) \left( \frac{8.5212 \text{ m}^2/\text{s}^2}{0.15 \text{ m}} \right)

$F_x = -33.909$ N

$F_y = 33.309$ N

$F = -33.9 \text{ N} i + 33.3 \text{ N} j$
PROBLEM 13.70

A thin circular rod is supported in a vertical plane by a bracket at A. Attached to the bracket and loosely wound around the rod is a spring of constant $k = 40 \text{ N/m}$ and undeformed length equal to the arc of circle $AB$. A 200-g collar C is unattached to the spring and can slide without friction along the rod. Knowing that the collar is released from rest when $\theta = 30^\circ$, determine (a) the velocity of the collar as it passes through point B, (b) the force exerted by the rod on the collar as it passes through B.

(a) $v_C = 0, \quad T_C = 0$

$T_B = \frac{1}{2}mv_B^2$

$T_B = \frac{1}{2}(0.2 \text{ kg})v_B^2$

$T_B = 0.1v_B^2$

$V_C = (V_C)_e + (V_C)_g$

arc $BC = \Delta L_{BC} = R\theta$

$\Delta L_{BC} = (0.3 \text{ m})(30^\circ)\left(\frac{\pi}{180^\circ}\right)$

$\Delta L_{BC} = 0.15708 \text{ m}$

$(V_C)_e = \frac{1}{2}k(\Delta L_{BC})^2 = \frac{1}{2}(40 \text{ N/m})(0.15708 \text{ m})^2 = 0.49348 \text{ J}$

$(V_C)_g = WR(1 - \cos\theta) = (0.2 \text{ kg})(9.81 \text{ m/s}^2)(0.3 \text{ m})(1 - \cos 30^\circ)$

$(V_C)_g = 0.078857 \text{ J}$

$V_C = (V_C)_e + (V_C)_g = 0.49348 \text{ J} + 0.078857 \text{ J} = 0.57234 \text{ J}$

$V_B = (V_B)_e + (V_B)_g = 0 + 0 = 0$

$T_C + V_C = T_B + V_B; \quad 0 + 0.57234 = 0.1v_B^2$

$v_B^2 = 5.7234 \text{ m}^2/\text{s}^2 \quad v_B = 2.39 \text{ m/s}$

(b) $+\sum F = FR - W = \frac{mv_B^2}{R}$

$FR = 1.962 \text{ N} + (0.2 \text{ kg})\left(\frac{5.7234 \text{ m}^2/\text{s}^2}{0.3 \text{ m}}\right)$

$FR = 1.962 \text{ N} + 3.8156 \text{ N} = 5.7776 \text{ N} \quad FR = 5.78 \text{ N}$
An 8-oz package is projected upward with a velocity \( v_0 \) by a spring at \( A \); it moves around a frictionless loop and is deposited at \( C \). For each of the two loops shown, determine (a) the smallest velocity \( v_0 \) for which the package will reach \( C \), (b) the corresponding force exerted by the package on the loop just before the package leaves the loop at \( C \).

**Loop 1**

(a) The smallest velocity at \( B \) will occur when the force exerted by the tube on the package is zero.

\[
\sum F = 0 + mg = \frac{mv_B^2}{r}
\]

\[
v_B^2 = rg = 1.5 \text{ ft} (32.2 \text{ ft/s}^2)
\]

\[
v_B^2 = 48.30
\]

At \( B \)

\[
T_A = \frac{1}{2} m v_0^2
\]

\[
V_A = 0 \quad \left(8 \text{ oz} = 0.5 \text{ lb} \Rightarrow v = \frac{0.5}{32.2} = 0.01553 \right)
\]

At \( B \)

\[
T_B = \frac{1}{2} m v_B^2 = \frac{1}{2} m(48.30) = 24.15 \text{ m}
\]

\[
V_B = mg(7.5 + 1.5) = 9mg = 9(0.5) = 4.5 \text{ lb} \cdot \text{ft}
\]

\[
T_A + V_A = T_B + V_B: \quad \frac{1}{2}(0.01553)v_0^2 = 24.15(0.01553) + 4.5
\]

\[
v_0^2 = 627.82
\]

\[
V_0 = 25.056
\]

\[
v_0 = 25.1 \text{ ft/s}
\]

\[
T_C = \frac{1}{2} m v_C^2 = 0.007765 v_C^2
\]

\[
V_C = 7.5mg = 7.5(0.5) = 3.75
\]

\[
T_A + V_A = T_C + V_C: \quad 0.007765v_0^2 = 0.007765v_C^2 + 3.75
\]

\[
0.007765(25.056)^2 - 3.75 = 0.007765v_C^2
\]

\[
v_C^2 = 144.87
\]
PROBLEM 13.75 CONTINUED

(b) \[ N_c = \frac{\gamma v_c^2}{\gamma} \]

\[ \sum F = ma: \quad N = 0.01553 \frac{(144.87)}{1.5} \]

\[ N = 1.49989 \]

{Package in tube} \( N_c = 1.500 \text{ lb} \)

(a) At B, tube supports the package so,

\[ v_B \approx 0 \]

\[ v_B = 0, \quad T_B = 0 \]

\[ V_B = mg(7.5 + 1.5) \]

\[ = 4.5 \text{ lb-ft} \]

\[ T_A + V_A = T_B + V_B \]

\[ \frac{1}{2} (0.01553)v_A^2 = 4.5 \Rightarrow v_A = 24.073 \]

\[ v_A = 24.1 \text{ ft/s} \]

(b) At C

\[ T_C = 0.007765v_C^2, \quad V_C = 7.5mg = 3.75 \]

\[ T_A + V_A = T_C + V_C: \quad 0.007765(24.073)^2 = 0.007765v_C^2 + 3.75 \]

\[ v_C^2 = 96.573 \]

\[ N_c = \frac{\gamma v_c^2}{\gamma} \]

\[ \gamma \]

\[ 0.5 \]

\[ N_c = 0.01553 \frac{96.573}{1.5} = 0.99985 \]

{Package on tube} \( N_c = 1.000 \text{ lb} \)
PROBLEM 13.83

Knowing that the velocity of an experimental space probe fired from the earth has a magnitude $v_A = 32.5$ Mm/h at point $A$, determine the velocity of the probe as it passes through point $B$.

SOLUTION

At $A$,

$$v_A = 32.5 \text{ Mm/h} = 9028 \text{ m/s}$$

$$T_A = \frac{1}{2} m (9028 \text{ m/s})^2 = 40.752 \times 10^6 \text{ m}$$

$$V_A = -\frac{GMm}{r_A} = -\frac{gR^2m}{r_A}$$

$$r_A = 10.67 \text{ Mm} = 10.67 \times 10^6 \text{ m}$$

$$R = 6370 \text{ km} = 6.37 \times 10^6 \text{ m}$$

$$V_A = \left(\frac{9.81 \text{ m/s}^2}{(6.37 \times 10^6 \text{ m})^2} \cdot \frac{(6.37 \times 10^6 \text{ m})^2}{(10.67 \times 10^6 \text{ m})}\right) \cdot m = -37.306 \times 10^6 \text{ m}$$

At $B$

$$r_B = 19.07 \text{ Mm} = 19.07 \times 10^6 \text{ m}$$

$$T_B = \frac{1}{2} m v_B^2; \quad V_B = -\frac{GMm}{r_B} = -\frac{gR^2m}{r_B}$$

$$V_B = \left(\frac{9.81 \text{ m/s}^2}{(6.37 \times 10^6 \text{ m})^2} \cdot \frac{(6.37 \times 10^6 \text{ m})^2}{(19.07 \times 10^6 \text{ m})}\right) \cdot m = -20.874 \times 10^6 \text{ m}$$

$$T_A + V_A = T_B + V_B; \quad 40.752 \times 10^6 \text{ m} - 37.306 \times 10^6 \text{ m} = \frac{1}{2} m v_B^2 = 20.874 \times 10^6 \text{ m}$$

$$v_B^2 = 2 \left[40.752 \times 10^6 - 37.306 \times 10^6 + 20.874 \times 10^6\right]$$

$$v_B^2 = 48.64 \times 10^6 \text{ m}^2/\text{s}^2$$

$$v_B = 6.9742 \times 10^3 \text{ m/s} = 25.107 \text{ Mm/h}$$

$$v_B = 25.1 \text{ Mm/h}$$
**PROBLEM 13.116**

A spacecraft of mass \( m \) describes a circular orbit of radius \( r_1 \) around the earth. (a) Show that the additional energy \( \Delta E \) which must be imparted to the spacecraft to transfer it to a circular orbit of larger radius \( r_2 \) is

\[
\Delta E = \frac{GMm(r_2 - r_1)}{2r_1r_2}
\]

where \( M \) is the mass of the earth. (b) Further show that if the transfer from one circular orbit to the other circular orbit is executed by placing the spacecraft on a transitional semielliptic path \( AB \), the amounts of energy \( \Delta E_A \) and \( \Delta E_B \) which must be imparted at \( A \) and \( B \) are, respectively, proportional to \( r_1 \) and \( r_2 \):

\[
\Delta E_A = \frac{r_2}{r_1 + r_2} \Delta E \quad \Delta E_B = \frac{r_1}{r_1 + r_2} \Delta E
\]

---

**SOLUTION**

(a) For a circular orbit of radius \( r \)

\[
F = ma \quad \Rightarrow \quad \frac{GMm}{r^2} = \frac{mv^2}{r}
\]

\[
v^2 = \frac{GM}{r}
\]

\[
E = T + V = \frac{1}{2}mv^2 = \frac{1}{2} m \left( \frac{GM}{r} \right) = -\frac{1}{2} \frac{GMm}{r}
\]

Thus \( \Delta E \) required to pass from circular orbit of radius \( r_1 \) to circular orbit of radius \( r_2 \) is

\[
\Delta E = E_1 - E_2 = \frac{1}{2} \frac{GMm}{r_1} + \frac{1}{2} \frac{GMm}{r_2}
\]

\[
\Delta E = \frac{GMm(r_2 - r_1)}{2r_1r_2} \quad \text{(Q.E.D.)}
\]

(b) For an elliptic orbit we recall Equation (3) derived in Problem 13.113 (with \( v_p = v_1 \))

\[
v_1^2 = \frac{2Gm}{r_1} \left( \frac{r_2}{(r_1 + r_2)} \right)
\]

At point \( A \): Initially spacecraft is in a circular orbit of radius \( r_1 \)

\[
v_{circ}^2 = \frac{GM}{r_1}
\]

\[
T_{circ} = \frac{1}{2}mv_{circ}^2 = \frac{1}{2} m \left( \frac{GM}{r_1} \right)
\]
PROBLEM 13.116 CONTINUED

After the spacecraft engines are fired and it is placed on a semi-elliptic path $AB$, we recall

$$ v_1^2 = \frac{2GM}{r_1} \cdot \frac{r_2}{r_1} $$

And

$$ T_1 = \frac{1}{2} m v_1^2 = \frac{1}{2} m \frac{2GMr_2}{r_1(r_1 + r_2)} $$

At point $A$, the increase in energy is

$$ \Delta E_A = T_1 - T_{circ} = \frac{1}{2} m \frac{2GMr_2}{r_1(r_1 + r_2)} - \frac{1}{2} m \frac{GM}{r_1} $$

$$ \Delta E_A = \frac{GMM (2r_2 - r_1 - r_2)}{2r_1(r_1 + r_2)} = \frac{GMM (r_2 - r_1)}{2r_1(r_1 + r_2)} $$

Recall Equation (2): \[ \Delta E_A = \frac{r_2}{\eta + r_2} \Delta E \] (Q.E.D)

A similar derivation at point $B$ yields,

$$ \Delta E_B = \frac{r_1}{\eta + r_2} \Delta E \] (Q.E.D)