MAE 295. Solid mechanics

HW1, UCI Winter 2006. by Nasser Abbasi

Question

Assigned by Professor Atluri on Thursday 1/12/2006

Explain why the gradient of a vector function field is a second order tensor

Answer

First I describe what is a second order tensor, then show that the gradient of a vector field gives a second order tensor.

A second order tensor is a set whose elements are each uniquely identified by 2 components. As an example, to uniquely identify a stress component at a point in a solid, one must specify not only the direction of the stress (the direction of the force) but also on which plane the stress acts on. Hence we need 2 different components to uniquely identify each stress element.

From a pure data structure point view with no physical interpretation given, a 2D matrix can be viewed as a second order tensor because to identify each element of the matrix we need to specify 2 things, the row the element is on, and the column it is on. This is the reason a vector is called a first order tensor. Each element of a vector requires only one index or component to identify it, which is the row number the element is located at.

Now that we know what a second order tensor is, we will try to see if the gradient of a vector function field will result in such an object.

In a vector function field, each point in space is identified by 3 elements (the coordinates of the tip of the vector). i.e. $\mathbf{V}(x_1, x_2, x_3) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = x_i\mathbf{e}_i$ The gradient of this vector field means to find the rate of the vector as it moves from one point to a near by point. But each vector has 3 components, and each one of these components can change differently in each of the 3 directions, hence to uniquely identify each element of a the gradient we must specify which vector components we are considering and on which direction are calculating its rate of change. Hence we need 2 components to uniquely identify each element of a gradient field. Using mathematics, the gradient operator is $\nabla = \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}\right)$, hence the gradient of a vector field is

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\begin{aligned}
\nabla\mathbf{V} &= \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}\right)(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3) \\
&= \left(\frac{\partial x_1}{\partial x_1} + \frac{\partial x_1}{\partial x_2} + \frac{\partial x_1}{\partial x_3}\right)\mathbf{e}_1 \\
&\quad + \left(\frac{\partial x_2}{\partial x_1} + \frac{\partial x_2}{\partial x_2} + \frac{\partial x_2}{\partial x_3}\right)\mathbf{e}_2 \\
&\quad + \left(\frac{\partial x_3}{\partial x_1} + \frac{\partial x_3}{\partial x_2} + \frac{\partial x_3}{\partial x_3}\right)\mathbf{e}_3
\end{aligned}
$$

In matrix format this can be written as $\begin{bmatrix} \frac{\partial x_1}{\partial x_1} & \frac{\partial x_1}{\partial x_2} & \frac{\partial x_1}{\partial x_3} \\ \frac{\partial x_2}{\partial x_1} & \frac{\partial x_2}{\partial x_2} & \frac{\partial x_2}{\partial x_3} \\ \frac{\partial x_3}{\partial x_1} & \frac{\partial x_3}{\partial x_2} & \frac{\partial x_3}{\partial x_3} \end{bmatrix}$