HW # 3

MAE 200B

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Problem 5.3.2

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5.3.2 $$\rho \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} + \alpha u + \beta \frac{\partial u}{\partial t}$$

a) Give brief interpretation. what sign must α, β have to be physicall?

Considering u to be displacement. this is a wave equation. then αu is a functn that depends on position. this is similar to restoring force in spring. so $\boxed{\alpha < 0}$

$\beta \frac{\partial u}{\partial t}$. here we have force proportional to speed. this is like damping force. so $\boxed{\beta < 0}$ since restoring or opposing force as well.

b) Allow $\rho, \alpha \beta$ to be functn of x. show that separation of variables work only if $\beta = c\rho$, c constant.

Assume $U = XT$. substitute in PDE

$$\rho(x) T'' X = T_0 \, TX'' + \alpha(x) TX + \beta(x) T'X$$

divide by XT we get

$$\rho(x) \frac{T''}{T} = T_0 \frac{X''}{X} + \alpha(x) + \beta(x) \frac{T'}{T}$$

so

$$\rho(x) \frac{T''}{T} - \beta(x) \frac{T'}{T} = T_0 \frac{X''}{X} + \alpha(x)$$

Since ρ, α, β functn of x, need to be in x side. so divide by ρ we set

$$\frac{T''}{T} - \frac{\beta(x)}{\rho(x)} \frac{T'}{T} = \frac{T_0}{\rho(x)} \frac{X''}{X} + \frac{\alpha(x)}{\rho(x)} \longrightarrow$$

so we see right away that for LHS to depend on time only, we must have $\frac{\beta}{\rho}$ to be Not a function of x. hence this is possible only if $\beta = c\rho$ where c is constant because then we can write

$$\frac{T''}{T} - \frac{c\rho(x)}{\rho(x)}\frac{T'}{T} = \frac{T_0}{\rho}\frac{X''}{X} + \frac{\alpha}{\rho}$$

$$\frac{T''}{T} - c\frac{T'}{T} = \frac{T_0}{\rho(x)}\frac{X''}{X} + \frac{\alpha(x)}{\rho(x)} \qquad (1)$$

and now we can do separation of variables since LHS depends on t only and RHS depends on x only.

QED.

part c

if $\beta = c\rho$ show that the spatial eq. is S.L. and solve the time equation.

from (1) we set

$$\frac{T''}{T} - c\frac{T'}{T} = \lambda$$

and $$\frac{T_0}{\rho(x)}\frac{X''}{X} + \frac{\alpha(x)}{\rho(x)} = \lambda$$

notice: I wrote λ here because I don't yet know the sign. most of the time this is $-w^2$ for $w > 0$.

so spatial equation is $$\boxed{\frac{T_0}{\rho(x)}X'' + \frac{\alpha(x)}{\rho(x)}X - \lambda X = 0}$$

divide by $\frac{T_0}{\rho(x)}$ we set

$$X'' + \frac{\alpha(x)}{T_0}X - \frac{\rho(x)}{T_0}\lambda X = 0$$

Compare to S.L.

$$\frac{d}{dx}(p(x)\phi') + q(x)\phi + \sigma(x)\lambda\phi = 0 \rightarrow$$

we see that this is a S.L. problem with

$$p(x) = 1, \quad q(x) = \frac{\alpha(x)}{T_0}, \quad \sigma(x) = -\frac{\rho(x)}{T_0}$$

notice since T_0 is constant, I can also write

$$p(x) = T_0, \quad q(x) = \alpha(x), \quad \sigma(x) = -\rho(x)$$

now solve the time equation:

$$\frac{T''}{T} - c\frac{T'}{T} = \lambda$$

$$T'' - cT' = \lambda T$$

i.e. $T'' - cT' - \lambda T = 0$ let $T = Ae^{Bt}$

char eq $B^2 - cB - \lambda = 0$

so $B = \frac{c \pm \sqrt{c^2+4\lambda}}{2} = -\frac{c}{2} \pm \sqrt{c^2+4\lambda}$

so $T_1 = Ae^{-\frac{c}{2} + \sqrt{c^2+4\lambda}\, t}$ or $T_2 = Ae^{-\frac{c}{2} - \sqrt{c^2+4\lambda}\, t}$.

or $T_1 = Ae^{-\frac{c}{2}t} e^{\sqrt{c^2+4\lambda}\, t}$ or $T_2 = Ae^{-\frac{c}{2}t} e^{-\sqrt{c^2+4\lambda}\, t}$

3 cases see online solution -1

since both are solutions, then the scaled sum is a solution.

so $T(t) = A_1 e^{-\frac{c}{2}t} e^{\sqrt{c^2+4\lambda}\, t} + A_2 e^{-\frac{c}{2}t} e^{-\sqrt{c^2+4\lambda}\, t}$

$$T(t) = e^{-\frac{c}{2}t}\left[A_1 e^{\sqrt{c^2+4\lambda}\, t} + A_2 e^{-\sqrt{c^2+4\lambda}\, t}\right]$$

where A_1 and A_2 can be found from initial conditions which are not given here. so I stop here.

Problem 5.3.9

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5.3.9

Consider $x^2 \frac{d^2\phi}{dx^2} + x \frac{d\phi}{dx} + \lambda\phi = 0$ ——— (1)

with $\phi(1) = 0$, $\phi(b) = 0$

a) Show that multiplying by $\frac{1}{x}$ puts this eq. in the Sturm-Liouville form.

multiply (1) by $\frac{1}{x} \Rightarrow$ $x \frac{d^2\phi}{dx^2} + \frac{d\phi}{dx} + \lambda \frac{1}{x}\phi = 0$ ——— (2)

Compare to eq. 5.3.2, page 162 (Sturm-Liouville)

$$\frac{d}{dx}\left(p(x) \frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0 \qquad a < x < b$$

$$p'(x) \frac{d\phi}{dx} + p(x) \frac{d^2\phi}{dx^2} + q(x)\phi + \lambda\sigma(x)\phi = 0 \quad \text{——— (3)}$$

Compare (3) to (2) we see that when $\boxed{p(x) = x,\ q(x) = 0,}$ $\boxed{\sigma(x) = \frac{1}{x}}$ then (3) becomes

$$x \frac{d^2\phi}{dx^2} + \frac{d\phi}{dx} + \lambda \frac{1}{x}\phi = 0$$

which is eq (2). QED.

b) Show that $\lambda \geq 0$.

Since we showed in part (a) that this ODE is a Sturm-Liouville Form, then we can use its properties, one of them is property #6 on page 163 which is that any eigenvalue is given by (Rayleigh quotient)

$$\lambda = \frac{\left[-p\phi \frac{d\phi}{dx}\right]_a^b + \int_a^b \left[p\left(\frac{d\phi}{dx}\right)^2 - q\phi^2\right] dx}{\int_a^b \phi^2 \sigma \, dx} \quad \rightarrow$$

The denominator $\int_a^b \phi^2 \sigma \, dx > 0$ since $\phi \neq 0$ everywhere and $\sigma = \frac{1}{x}$ which is positive since the limit is from 1 to b.

So looking at Numerator $\underbrace{\left[-p\phi \frac{d\phi}{dx}\right]_a^b}_{\Downarrow} + \int_a^b p\left(\frac{d\phi}{dx}\right)^2 - q\phi^2 \, dx$

$\Downarrow$ 0 since $\phi(b) = \phi(a) = 0$

$q\phi^2$: $=0$ given.

So Numerator is $\int_a^b p\left(\frac{d\phi}{dx}\right)^2 dx = \int_a^b x \underbrace{\left(\frac{d\phi}{dx}\right)^2}_{\geq 0 \text{ since squared.}} dx$

So $\int_a^b x\left(\frac{d\phi}{dx}\right)^2 dx \geq 0$.

hence $\lambda \geq 0$ QED.

part c

$$x^2 \frac{d^2\phi}{dx^2} + x\frac{d\phi}{dx} + \lambda\phi = 0$$

an equidimensional, or Cauchy-Euler equation is of the form

$$x^2\phi'' + bx\phi' + c\phi = 0$$

where b, c are constants.

hence our ODE is an equidemensional with $\boxed{b=1,\ c=\lambda}$

this Cauchy-Euler eq. can be assumed to have solution

$$\boxed{\phi = x^a}$$

Sub into the ODE we set

$$x^2 a(a-1)(x^{a-2}) + x a x^{a-1} + \lambda x^a = 0$$

$$a(a-1)x^a + xa x^{a-1} + \lambda x^a = 0$$

$$a(a-1)x^a + a x^a + \lambda x^a = 0$$

$$[a(a-1) + a + \lambda]\, x^a = 0 \longrightarrow$$

bnt $x \neq 0 \Rightarrow \quad a(a-1) + a + \lambda = 0$

or $\quad a^2 - a + a + \lambda = 0$

so $\boxed{a^2 = -\lambda}$

i.e $\boxed{a = \pm i\sqrt{\lambda}}$

So solution $\phi(x) = x^{\pm i\sqrt{\lambda}}$

to convert to exponential (to use sin/cos), do

$$\phi(x) = x^{\pm i\sqrt{\lambda}} = e^{\ln(x^{\pm i\sqrt{\lambda}})}$$

since $e^{\ln k} = k$

so $\phi(x) = e^{\ln(x^{\pm i\sqrt{\lambda}})} = e^{\pm i\sqrt{\lambda}\ln x}$

$$\boxed{\phi(x) = A\cos\sqrt{\lambda}\ln x + B\sin\sqrt{\lambda}\ln x}$$

from Boundary conditions $\phi(1) = 0$

$$\phi(1) = 0 = A\cos\sqrt{\lambda}\ln 1 + B\sin\sqrt{\lambda}\ln 1$$

$$0 = A\cos 0 + B\sin 0$$

$$\boxed{0 = A}$$

so $\phi(x) = B\sin\sqrt{\lambda}\ln x$

from B.C. $\phi(b) = 0$ we get

$$0 = B\sin\sqrt{\lambda}\ln b \Rightarrow \sin\sqrt{\lambda}\ln b = 0 \text{ for nontrivial solution } B \neq 0$$

i.e $\sqrt{\lambda}\ln b = n\pi \qquad n = 1, 2, \ldots$

so $\sqrt{\lambda_n} = \frac{n\pi}{\ln b} \Rightarrow \boxed{\lambda_n = \left(\frac{n\pi}{\ln b}\right)^2}$

so $n = 1 \Rightarrow$ first $\lambda \neq 0$
so $\lambda = 0$ is NOT an eigenvalue
so we have ∞ number of λ with smallest $\left(\frac{\pi}{\ln b}\right)^2$ bnt No largest
QED.

part d

from page 163, property #5

Eigenfunctions are orthogonal relative to weight $\sigma(x)$

$$\int_a^b \phi_n(x)\,\phi_m(x)\,\sigma(x)\,dx = 0 \qquad \text{if } n \neq m$$

but in our problem $\boxed{\sigma(x) = \frac{1}{x}}$ and we found

$$\phi_n(x) = \sin\sqrt{\lambda_n}\ln x$$

so

$$I = \int_1^b \sin(\sqrt{\lambda_n}\ln x)\ \sin(\sqrt{\lambda_m}\ln x)\ \frac{1}{x}\,dx$$

to verify this is zero if $n \neq m$ carry out the integral

use identity that $\sin A \sin B = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$

so

$$I = \int_1^b \frac{1}{x}\left[\frac{1}{2}\cos(\sqrt{\lambda_n}\ln x - \sqrt{\lambda_m}\ln x) - \frac{1}{2}\cos(\sqrt{\lambda_n}\ln x + \sqrt{\lambda_m}\ln x)\right]dx$$

$$I = \int_1^b \frac{1}{2x}\left[\cos(\ln x(\sqrt{\lambda_n} - \sqrt{\lambda_m})) - \cos(\ln x(\sqrt{\lambda_n} + \sqrt{\lambda_m}))\right]dx$$

where $\sqrt{\lambda_n} = \frac{n\pi}{\ln b}$, $\sqrt{\lambda_m} = \frac{m\pi}{\ln b}$.

so

$$I = \frac{1}{2}\int_1^b \frac{1}{x}\left(\cos\left[\frac{\pi}{\ln b}(n-m)\ln x\right] - \cos\left[\frac{\pi}{\ln b}(n+m)\ln x\right]\right)dx$$

since $n \neq m$, then $n-m$ and $n+m$ are integer values

if n even, m even $\Rightarrow$ $n+m$ even, and $n-m$ even
and hence integrand $= 0$ and $I = 0$ (since cosine function is even and periodic)

if n odd, m even $\Rightarrow$ $n+m$ odd, and $n-m$ odd
and Integrand $= 0$ as well $\Rightarrow I = 0$

if n odd, m odd $\Rightarrow$ $n+m$ = even, and $n-m$ = even same case as
so if $n \neq m \Rightarrow I = 0$.

but if $n=m$, then we set

$$\frac{1}{2}\int_1^b \frac{1}{x}\left(\cos\left(\frac{\pi}{\ln b}(n-m)\ln x\right)-\cos\left(\frac{\pi}{\ln b}(n+m)\ln x\right)\right)dx$$

$$=\frac{1}{2}\int_1^b \frac{1}{x}\left(1-\cos\left(\frac{\pi}{\ln b}2n\ln x\right)\right)dx \neq 0$$

So we showed that if $n\neq m$, orthogonality is verified, and integral $\to 0$. and if $n=m$ integral $\neq 0$.

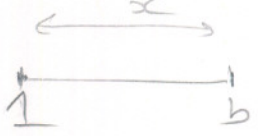
QED.

part e

show that n^{th} eigenfunction has $n-1$ zeros.

the n^{th} eigenfunction is

$$\phi_n(x)=\sin\left(\frac{n\pi}{\ln b}\ln x\right)$$



the first mode is $n=1$

$\phi_1(1)=0$ i.e $\sin\left(\frac{\pi}{\ln b}\ln(1)\right)=0$

$\phi_1(b)=0$ i.e $\sin\left(\frac{\pi}{\ln b}\ln b\right)=0$

So for first mode, i.e $n=1$ we have No zeros in between left and right ends.

So number of zeros is one less n

i.e $\underline{n-1}$

QED

Problem #3

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3. $X'' = -\omega^2 X$, $-L < x < L$

$x(-L) = X(L)$, $X'(-L) = X'(L)$

a show by direct substitution that

$$\phi_n^- = e^{-i\omega_n x}$$

and $\phi_n^+ = e^{i\omega_n x}$

are two eigenfunctions for eigenvalues $\omega_n = \frac{n\pi}{L}$

Since ϕ_n^- and ϕ_n^+ are eigenfunctions, then each is a solution to the ODE, and also their sum is a solution. (and their difference as well).

i.e we can say that $\boxed{\phi_n(x) = \phi_n^-(x) + \phi_n^+(x)}$

so $\phi' = \phi_n^{-\prime} + \phi_n^{+\prime} = -i\omega_n e^{-i\omega_n x} + i\omega_n e^{i\omega_n x}$

and $\phi'' = i^2\omega_n^2 e^{-i\omega_n x} + i^2\omega_n^2 e^{i\omega_n x}$

$= -\omega_n^2 e^{-i\omega_n x} - \omega_n^2 e^{i\omega_n x} = -\omega_n^2\left[e^{i\omega_n x} + e^{-i\omega_n x}\right]$

$= -\omega_n^2\left[2\cos\omega_n x\right] = \boxed{-2\omega_n^2\cos\omega_n x}$ ——— (1)

but $\phi = e^{-i\omega_n x} + e^{i\omega_n x} = (\cos\omega_n x - i\sin\omega_n x) + (\cos\omega_n x + i\sin\omega_n x)$

$= \boxed{2\cos\omega_n x}$. ——— (2)

Substitute (1) and (2) into ODE to see if we get identity.

$-2\omega_n^2\cos\omega_n x = -\omega_n^2\left[2\cos\omega_n x\right]$

$\cos\omega_n x = \cos\omega_n x$

$1 = 1$ ⟶ ok. QED

part b

show that the above two eigenfunctions, and eigenfunctions for different eigenvalues are orthogonal with the following definition of inner product

$$\langle f, g \rangle = \int_{-L}^{L} f(x)\, \overline{g(x)}\, dx.$$

the eigenfunctions are orthogonal if $\langle f, g \rangle = 0$ when $n \neq m$, and $\langle f, g \rangle \neq 0$ if $n = m$, where n and m represent different modes or eigenvalues.

by direct integration we need to show that

$$I = \int_{-L}^{L} e^{-i\omega_n x}\, e^{i\omega_m x}\, dx = 0 \quad \text{if } n \neq m$$

$$I = \int_{-L}^{L} [\cos\omega_n x - i \sin\omega_n x][\cos\omega_m x + i \sin\omega_m x]\, dx$$

$$= \int_{-L}^{L} \cos\omega_n x \cos\omega_m x + i \cos\omega_n x \sin\omega_m x - i \sin\omega_n x \cos\omega_m x + \sin\omega_n x \sin\omega_m x\, dx$$

use identities: $\cos A \cos B = \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right)$

$\cos A \sin B = \frac{1}{2}\left(\sin(A+B) + \sin(B-A)\right)$

$\sin A \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$

→

so using these identities we set

$$\cos w_n x \cos w_m x = \frac{1}{2}\left[\cos(w_n+w_m)x + \cos(w_n-w_m)x\right]$$

$$\cos w_n x \sin w_m x = \frac{1}{2}\left[\sin(w_n+w_m)x + \sin(w_n-w_m)x\right]$$

$$\sin w_n x \cos w_m x = \frac{1}{2}\left[\sin(w_n+w_m)x + \sin(w_m-w_n)x\right]$$

$$\sin w_n x \sin w_m x = \frac{1}{2}\left[\cos(w_n-w_m)x - \cos(w_n+w_m)x\right]$$

so looking at integral again

$$I = \int_{-L}^{L} \cos w_n x \cos w_m x + \sin w_n x \sin w_m x \, dx$$

$$+ i \int_{-L}^{L} \cos w_n x \sin w_m x - \sin w_n x \cos w_m x \, dx$$

so we set

$$I = \int_{-L}^{L} \frac{1}{2}\left[\cos(w_n+w_m)x + \cos(w_n-w_m)x\right] + \frac{1}{2}\left[\cos(w_n-w_m)x - \cos(w_n+w_m)x\right] dx$$

$$+ i \int_{-L}^{L} \frac{1}{2}\left[\sin(w_n+w_m)x + \sin(w_n-w_m)x\right] - \frac{1}{2}\left[\sin(w_n+w_m)x + \sin(w_m-w_n)x\right] dx$$

$$I = \frac{1}{2}\int_{-L}^{L} \cos(w_n-w_m)x + \cos(w_n-w_m)x \, dx$$

$$+ \frac{1}{2} i \int_{-L}^{L} \sin(w_n-w_m)x - \sin(w_m-w_n)x \, dx$$

rewrite as $-\sin(w_n-w_m)$

$$I = \int_{-L}^{L} \cos(w_n-w_m)x \, dx + i \int_{-L}^{L} \sin(w_n-w_m) \, dx$$

→

$$I = \int_{-L}^{L} \cos(w_n - w_m)x \, dx + i \int_{-L}^{L} \sin(w_n - w_m) \, dx$$

if $n = m$ we get

$$I = \int_{-L}^{L} dx = [x]_{-L}^{L} = 2L$$

if $n \neq m$ we get

$$I = \frac{1}{w_n - w_m} \left[\sin(w_n - w_m)x \right]_{-L}^{L} + i \frac{1}{w_n - w_m} \left[-\cos(w_n - w_m)x \right]_{-L}^{L}$$

$$= \frac{1}{w_n - w_m} \left[\sin(w_n - w_m)L - \sin(w_n - w_m)(-L) \right] - \frac{i}{w_n - w_m} \left[\cos(w_n - w_m)L - \cos(w_n - w_m)(-L) \right]$$

$$= \frac{1}{w_n - w_m} \left[\sin(w_n - w_m)L + \sin(w_n - w_m)L \right] - \frac{i}{w_n - w_m} \left[0 \right]$$

$$= \frac{1}{(w_n - w_m)} 2 \sin(w_n - w_m)L$$

$$= \frac{2}{\frac{n\pi}{L} - \frac{m\pi}{L}} \sin\left(\frac{n\pi}{L} - \frac{m\pi}{L} \right) L = \frac{2L}{(n-m)\pi} \sin\left((n-m)\pi \right)$$

but $n - m$ is an integer. $\Rightarrow \sin(n-m)\pi = 0$

i.e. $\boxed{I = 0}$

hence showed that inner product $= 0$ if $n \neq m$

and $\neq 0$ if $n = m$ $\Longrightarrow$ valid definition of inner product.

QED

part c

given function $f(x)$, expand it in terms of above basis functions

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{i\frac{n\pi}{L}x}$$

determine the formula for C_n.

consider eigenfunction $\phi_n = e^{i\omega_n x}$.

so $C_n = \dfrac{\langle f(x), \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} = \dfrac{\int_{-L}^{L} f(x)\, \overline{e^{i\omega_n x}}\, dx}{\int_{-L}^{L} e^{i\omega_n x}\, \overline{e^{i\omega_n x}}\, dx} = \dfrac{\int_{-L}^{L} f(x) e^{-i\omega_n x}\, dx}{2L}$

note $\langle \phi_n, \phi_n \rangle = \int_{-L}^{L} \phi_n\, \overline{\phi_n}$

so $\boxed{C_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i\omega_n x}\, dx}$

Can be simplified more using Euler

$$C_n = \frac{1}{2L} \int_{-L}^{L} f(x) \left[\cos \omega_n x - i \sin \omega_n x\right] dx \qquad (1)$$

part d show that $C_{-n} = \overline{C_n}$

from (1) $C_{-n} = \frac{1}{2L} \int_{-L}^{L} f(x) \left[\cos\left(-\frac{n\pi}{L}x\right) - i \sin\left(\frac{-n\pi}{L}\right)x\right] dx$

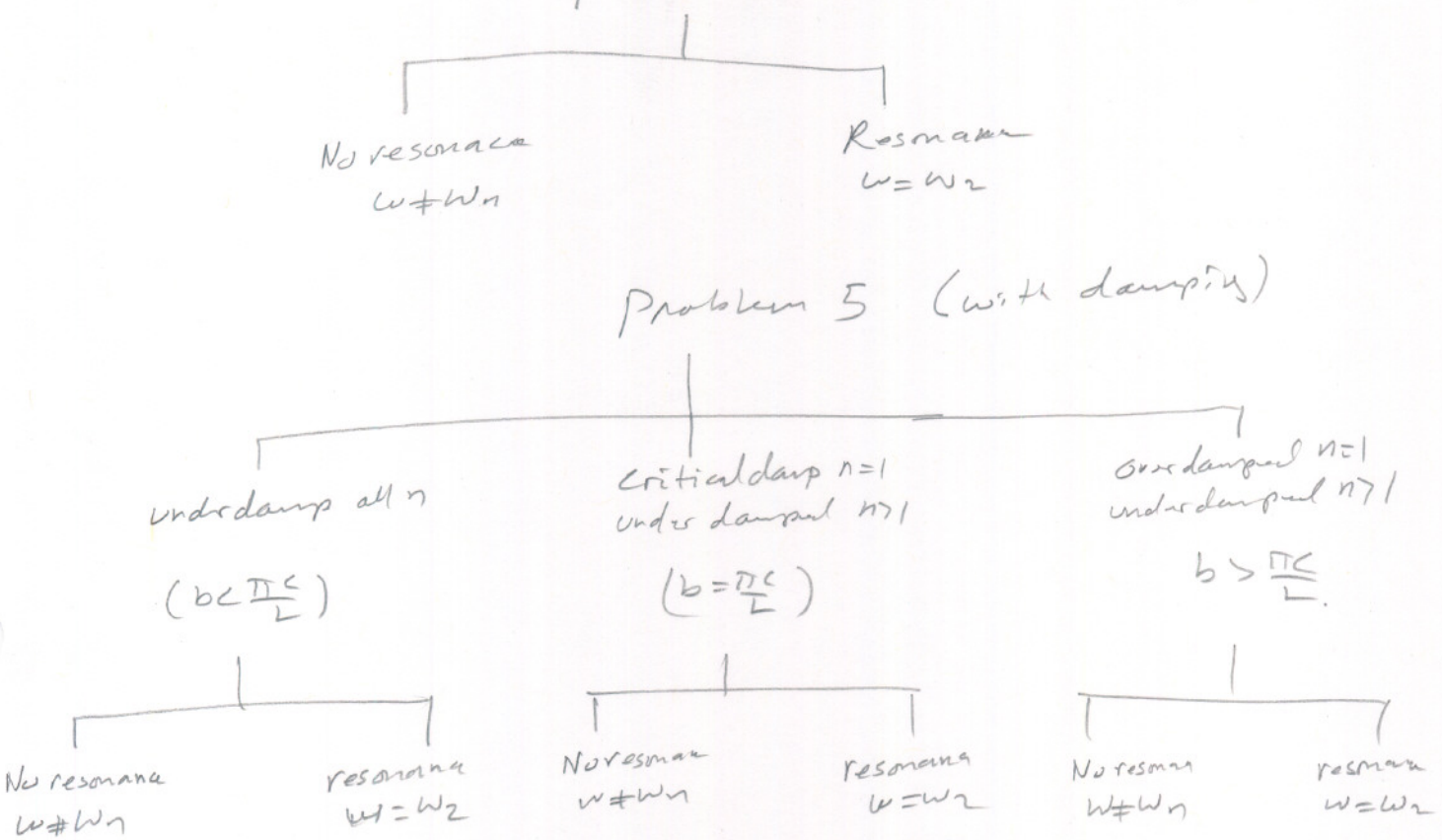
$= \frac{1}{2L} \int_{-L}^{L} f(x) \left[\cos \frac{n\pi}{L}x + i \sin\left(\frac{n\pi}{L}\right)x\right] dx$

$= \boxed{\frac{1}{2L} \int_{-L}^{L} f(x)\, e^{i\omega_n x}\, dx}$

but $\overline{C_n} = \frac{1}{2L} \int_{-L}^{L} f(x)\, \overline{e^{-i\omega_n x}}\, dx$ since $f(x)$ is real.

$= \frac{1}{2L} \int_{-L}^{L} f(x)\, e^{+i\omega_n x}\, dx$ which is C_{-n} QED

Over view of problems #4, and #5



difference between problem 4 and 5, is that in problem 4 when we have resonance we have ∞ displacment occure. but in problem 5, due to damping, the effect of damping cause the resonance displacment to be much smaller since No division by zero occure in this case due to the addition of the damping term in the denomator to the $(\omega_n - \omega)$ term that cause the trouble.

problem #4

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Forced vibration of a finite string.

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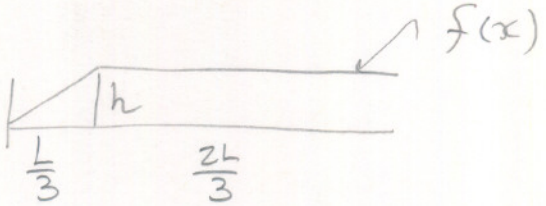
Initial conditions
$t=0$



so the problem is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + A \sin \omega t \qquad 0 < x < L,\ t > 0$$

$$\left.\begin{aligned} &\frac{\partial u(L,t)}{\partial x} = 0 \\ &u(0,t) = 0 \end{aligned}\right\} \text{B.C.} \qquad t>0$$

$$\left.\begin{aligned} &u(x,0) = f(x) \\ &\frac{\partial u(x,0)}{\partial t} = 0 \end{aligned}\right\} \text{I.C.} \qquad 0<x<L$$

notice here we have an inhomogeneous wave equation with a forcing function that is time depedent.
Chapter 8, page 364 for reference. Haberman book.

Solve the homogenous PDE... $u = TX$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \Rightarrow u = XT \Rightarrow \frac{T''}{c^2 T} = \frac{X''}{X} = \lambda = -\omega^2$$

by separation of variables, we obtain the eigenvalue problem

$$X'' = -\omega^2 X \qquad \text{where } \omega > 0$$

$$\left.\begin{aligned} &X(0) = 0 \\ &X'(L) = 0 \end{aligned}\right\} \text{B.C.}$$

solution is $X(x) = A \cos \omega x + B \sin \omega x$

apply first B.C. $\Rightarrow 0 = A \Rightarrow X(x) = B \sin \omega x$

apply second B.C. $\Rightarrow 0 = B\omega \cos \omega L \Rightarrow \omega L = \frac{(2n-1)\pi}{2} \Rightarrow \boxed{\omega_n = \frac{(2n-1)}{2L}\pi}$ $n = 1, 2, \ldots$

→

so $\boxed{X_n(x) = \sin \omega_n x}$ (B will be absorbed into C_n later).

now expand in Fourier the Forcing function.

$$A \sin \omega t = \sum_{n=1}^{\infty} C_n(t) X_n(x)$$

$$\text{so } C_n(t) = \frac{\langle A \sin \omega t, X_n(x) \rangle}{\langle X_n(x), X_n(x) \rangle} = \frac{\int_0^L A \sin \omega t \sin \omega_n x \, dx}{\int_0^L \sin^2 \omega_n x \, dx}$$

$$= A \sin \omega t \, \frac{\int_0^L \sin \omega_n x \, dx}{\int_0^L \sin^2 \omega_n x \, dx}$$

$\int_0^L \sin \omega_n x \, dx \longrightarrow -\frac{1}{\omega_n} \cos \omega_n x$

$\int_0^L \sin^2 \omega_n x \, dx \longrightarrow \frac{L}{2}$

$$\text{so } C_n(t) = A \sin \omega t \, \frac{\left[-\frac{1}{\omega_n} \cos \omega_n x \right]_0^L}{\frac{L}{2}} = -\frac{2A}{L\omega_n} \sin \omega t \left[\cos \omega_n L - \cos \omega_n 0 \right]$$

$$= -\frac{2A}{L\omega_n} \sin \omega t \left[\cos \frac{(2n-1)\pi}{2L} L - 1 \right]$$

$\cos \frac{(2n-1)\pi}{2L} L = 0$

$$\boxed{C_n(t) = \frac{2A}{L\omega_n} \sin \omega t} \Rightarrow \boxed{A \sin \omega t \equiv \sum_{n=1}^{\infty} \frac{2A}{L\omega_n} \sin \omega t \sin \omega_n x}$$

decomposition of nonhomogeneous term by eigenfunction expansion

so sub into PDE we set

$$\sum T_n''(t) X_n(x) = c^2 \sum T_n(t) X_n''(x) + \sum \frac{2A}{L\omega_n} \sin \omega t \, X_n(x)$$

but $X_n''(x) = -\omega_n^2 X_n(x)$.

$$\text{so } \sum T_n''(t) X_n(x) - \sum c^2 T_n(t) \left(-\omega_n^2 X_n(x) \right) = \sum \frac{2A}{L\omega_n} \sin \omega t \, X_n(x)$$

$$\text{or } \sum \left[T_n''(t) + \omega_n^2 c^2 T_n(t) \right] X_n(x) = \sum \left[\frac{2A}{L\omega_n} \sin \omega t \right] X_n(x)$$

$$\text{so } \boxed{T_n''(t) + \omega_n^2 c^2 T_n(t) = \frac{2A}{L\omega_n} \sin \omega t} \longrightarrow$$

need now to solve $T'' + \omega_n^2 c^2 T = \frac{2A}{L\omega_n} \sin\omega t$.

let $B_n = \frac{2A}{L\omega_n}$ $\Rightarrow$ $\boxed{T_n'' + \omega_n^2 c^2 T_n = B_n \sin\omega t}$

but $e^{i\omega t} = \cos\omega t + i\sin\omega t$

so solve $Y'' + \omega_n^2 c^2 Y = B_n e^{i\omega t}$ and take the particular solution for Y and take the imaginary part of that. this will be the particular solution to

assume $Y_p = Me^{i\omega t}$, $Y_p' = Mi\omega e^{i\omega t}$, $Y_p'' = Mi^2\omega^2 e^{i\omega t} = -M\omega^2 e^{i\omega t}$.

so that $-M\omega^2 e^{i\omega t} + \omega_n^2 c^2 M e^{i\omega t} = B_n e^{i\omega t}$

so $-M\omega^2 + \omega_n^2 c^2 M = B_n$ or $M = \left(\frac{B_n}{\omega_n^2 c^2 - \omega^2}\right)$

so $Y_p = \frac{B_n}{\omega_n^2 c^2 - \omega^2} e^{i\omega t}$

take imaginary part of Y_p $\Rightarrow$ $\boxed{\frac{B_n}{\omega_n^2 c^2 - \omega^2} \sin\omega t}$

this is the particular solution to $T'' + \omega_n^2 c^2 T = B_n \sin\omega t$.

so now find homogeneous solution to

$$T'' + \omega_n^2 c^2 T = 0$$

let $T = R_1 e^{R_2 t} \rightarrow T' = R_1 R_2 e^{R_2 t} \rightarrow T'' = R_1 R_2^2 e^{R_2 t}$

$\Rightarrow$ characteristic equation is $R_2^2 + \omega_n^2 c^2 = 0 \Rightarrow R_2^2 = -\omega_n^2 c^2$

$\boxed{R_2 = \pm i\omega_n c}$

$\Rightarrow T_n(\text{homogeneous}) = K_1 \cos\omega_n ct + K_2 \sin\omega_n ct$

so Total solution is

$$\boxed{T_n = \overbrace{K_1\cos\omega_n ct + K_2 \sin\omega_n ct}^{\text{homogeneous}} + \overbrace{\frac{B_n}{\omega_n^2 c^2 - \omega^2} \sin\omega t}^{\text{particular}}}$$

h) now apply I.C. to find K_1, K_2 $\longrightarrow$

$$T_n' = -K_1 w_n c \sin w_n c t + K_2 w_n c \cos w_n c t + \frac{\omega B_n}{w_n^2 c^2 - \omega^2} \cos \omega t$$

but $T_n' = 0$ at $t = 0$

so $0 = K_2 w_n c + \frac{\omega B_n}{w_n^2 c^2 - \omega^2}$

$$\Rightarrow K_2 = \frac{-\omega B_n (w_n c)}{w_n^2 c^2 - \omega^2} = \frac{-\omega w_n B_n c}{w_n^2 c^2 - \omega^2}$$

to find K_1 apply Fourier expansion

so $u(x,t) = \sum_{n=1}^{\infty} T_n X_n$

$$u(x,t) = \sum_{n=1}^{\infty} \left(K_1 \cos w_n c t + K_2 \sin w_n c t + \frac{B_n}{w_n^2 c^2 - \omega^2} \sin \omega t \right) \sin w_n x$$

but $\frac{\partial u(x,0)}{\partial t} = 0 = \sum_{n=1}^{\infty} \left(-K_1 w_n c \sin w_n c t + K_2 w_n c \cos w_n c t + \frac{B_n \omega \cos \omega t}{w_n^2 c^2 - \omega^2} \right) \sin w_n x$

at $t=0 \Rightarrow 0 = \sum_{n=1}^{\infty} \left(K_2 w_n c + \frac{B_n \omega}{w_n^2 c^2 - \omega^2} \right) \sin w_n x$

$$\Rightarrow K_2 w_n c + \frac{B_n \omega}{w_n^2 c^2 - \omega^2} = 0 \Rightarrow K_2 = \frac{\left(\frac{-B_n \omega}{w_n^2 c^2 - \omega^2} \right)}{w_n c} = \boxed{\frac{-B_n \omega}{(w_n^2 c^2 - \omega^2) w_n c}}$$

so $u(x,t) = \sum_{n=1}^{\infty} \left(K_1 \cos w_n c t - \frac{B_n \omega}{w_n c (w_n^2 c^2 - \omega^2)} \sin w_n c t + \frac{B_n}{w_n^2 c^2 - \omega^2} \sin \omega t \right) \sin w_n x$

apply second I.C. at $t = 0 \quad u(x,0) = f(x)$.

so $\boxed{f(x) = \sum_{n=1}^{\infty} K_1 \sin w_n x}$

$$\Rightarrow K_1 = \frac{\langle f(x), \sin w_n x \rangle}{\langle \sin w_n x, \sin w_n x \rangle} \Rightarrow K_1 = \frac{\int_0^L f(x) \sin w_n x \, dx}{\int_0^L \sin^2 w_n x \, dx}$$

$$K_1 = \frac{2}{L} \left[\int_0^{\frac{L}{3}} \frac{3h}{L} x \sin w_n x + \int_{\frac{L}{3}}^{L} h \sin w_n x \right]$$

$$K_1 = \frac{2}{L}\left[\frac{3h}{L}\int_0^{L/3} x \sin w_n x\, dx + h\int_{L/3}^{L} \sin w_n x\, dx\right]$$

$$\int_0^{L/3} x \sin w_n x\, dx \Rightarrow \int u\,dv = uv - \int v\,du$$

$$\Rightarrow \left[-x\frac{1}{w_n}\cos w_n x\right]_0^{L/3} - \int_0^{L/3}\left(\frac{-1}{w_n}\cos w_n x\right) dx \Rightarrow$$

$$\Rightarrow \left[-\frac{L}{3}\frac{1}{w_n}\cos w_n\frac{L}{3}\right] + \frac{1}{w_n}\left[\frac{1}{w_n}\sin w_n x\right]_0^{L/3} = -\frac{L}{3w_n}\cos\frac{Lw_n}{3} + \frac{1}{w_n^2}\left[\sin w_n\frac{L}{3}\right]$$

$$= \frac{1}{w_n^2}\sin\frac{Lw_n}{3} - \frac{L}{3w_n}\cos\frac{Lw_n}{3}$$

$$\therefore K_1 = \frac{2}{L}\left[\frac{3h}{L}\left(\frac{1}{w_n^2}\sin\frac{Lw_n}{3} - \frac{L}{3w_n}\cos\frac{Lw_n}{3}\right) + h\left[\frac{1}{w_n}[-\cos w_n x]\right]_{L/3}^{L}\right]$$

$$= \frac{2}{L}\left[\frac{3h}{L}\left(\frac{1}{w_n^2}\sin\frac{Lw_n}{3} - \frac{L}{3w_n}\cos\frac{Lw_n}{3}\right) - \frac{h}{w_n}\left(\cos w_n L - \cos w_n\frac{L}{3}\right)\right]$$

$$K_1 = \left[\frac{6h}{L^2}\left(\frac{1}{w_n^2}\sin\frac{Lw_n}{3} - \frac{L}{3w_n}\cos\frac{Lw_n}{3}\right) - \frac{2h}{Lw_n}\left(\cos w_n L - \cos w_n\frac{L}{3}\right)\right]$$

$\cos w_n L = 0$ since $w_n = \frac{(2n-1)\pi}{2}$

$$\therefore K_1 = \frac{6h}{L^2 w_n^2}\sin\frac{Lw_n}{3} \quad \text{or} \quad \boxed{K_1 = \frac{24h}{(2n-1)^2\pi^2}\sin\frac{(2n-1)\pi}{6}}$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty}\left(K_1\cos w_n ct + K_2\sin w_n ct + \frac{B_n}{w_n^2c^2 - w^2}\sin wt\right)\sin w_n x}$$

where K_1 is as shown above, $K_2 = -\frac{B_n w}{(w_n^2c^2 - w^2)w_n c}$ and $B_n = \frac{2A}{Lw_n}$

now can solve parts (a) and (b).

→

Let me rewrite $u(x,t)$ to make it more clear.

the forcing freqweng is ω

the wave freqweng is given by $\omega_n c$

when $\boxed{\omega = \omega_n c}$ resonance will occure. this can be seen by rewriting $u(x,t)$ as

$$u(x,t) = \sum_{n=1}^{\infty} \left(k_1 \cos \omega_n c t - \frac{B_n \omega}{(\omega_n^2 c^2 - \omega^2)\omega_n c} \sin \omega_n c t + \frac{B_n}{\omega_n^2 c^2 - \omega^2} \sin \omega t \right) \sin \omega_n x$$

$$= \sum_{n=1}^{\infty} \left(k_1 \cos \omega_n c t - \frac{B_n}{(\omega_n^2 c^2 - \omega^2)} \left[\frac{\omega}{\omega_n c} \sin \omega_n c t + \sin \omega t \right] \right) \sin \omega_n x$$

this term goes to zero when $\omega_n c = \omega$ which causes large value in term $\rightarrow \infty$. / part 15 in solution

$$\boxed{= \sum_{n=1}^{\infty} \left(k_1 \cos \omega_n c t - \frac{2A}{L\omega_n} \frac{1}{(\omega_n^2 c^2 - \omega^2)} \left[\frac{\omega}{\omega_n c} \sin \omega_n c t + \sin \omega t \right] \right) \sin \omega_n x}$$

to plot this, I get division-by-zero error at resonance. So I check for this case explicitly, and replace the whole term by some large negative value, say (-100).

---

to find T, the period of fundamental mode:

$$2\pi f = \omega_n c \rightarrow 2\pi f = \frac{(2n-1)}{2L}\pi c$$

$c = 4, L = 1 \Rightarrow 2\pi f = \frac{(2n-1)}{2} 4\pi \Rightarrow f = 2n-1$ in Hz.

so for $n=1$ $f = 1$ Hz. i.e $\boxed{T = 1 \text{ sec}}$

Notice that freqweng are given by $\omega_n c$ term and not ω_n.

for <u>part a</u> use $\omega \neq \omega_n$, so I pick $\omega = \frac{1}{2}\omega_1 c = \frac{1}{2}\frac{\pi}{2} 4 = \boxed{\pi}$

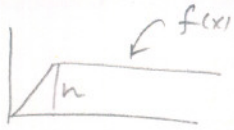
for <u>part b</u> use $\omega = \omega_2 c \Rightarrow \omega = \frac{3}{2}\pi 4 = \boxed{6\pi}$

for both parts I use $A = 10$ since not specified in problem.

Problem # 5

PS. in this solution I did not show lots of details, but the main results to make it easy to follow.

Problem #5



$$\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} + A\sin\omega t \quad , \quad \left.\begin{aligned}\frac{\partial u(x,0)}{\partial t} &= 0\\ u(x,0) &= f(x)\end{aligned}\right\} \text{I.C.}$$

$$\left.\begin{aligned}\frac{\partial u(L,t)}{\partial x} &= 0\\ u(0,t) &= 0\end{aligned}\right\} \text{B.C.}$$

Solve the homogeneous PDE

$$\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$$

Separation of Variables $\Rightarrow$ $X'' = -k^2 X$ $\quad k>0$

solution is $\boxed{X_n = \sin k_n x}$ $\qquad \boxed{k_n = \frac{2n-1}{2L}\pi}$ $\quad n=1,2,\cdots$

now expand the forcing function by eigenfunction expansion

$$A\sin\omega t = \sum_{n=1}^{\infty} g_n(t)\, X_n(x)$$

we obtain

$$\boxed{g_n(t) = \frac{2A}{LK_n}\sin\omega t}$$

so PDE is now

$$\sum T_n'' X_n + \sum b T_n' X_n = \sum c^2 X_n'' T_n + \sum g_n(t) X_n(t)$$

so we get $\qquad \boxed{T_n'' + bT_n' + c^2 k_n^2 T_n = g_n(t)}$

The homogeneous ODE $T_n'' + bT_n' + c^2 k_n^2 T_n = 0$ is written as

underdamp critical

$$T_n'' + 2\xi\omega_n T_n' + \omega_n^2 T_n = 0 \qquad \boxed{\omega_n = ck_n} \qquad \boxed{\xi = \frac{b}{2\omega_n}}$$

3 cases, $\xi<1$, $\xi=1$, $\xi>1 \rightarrow$ overdamped.

$b = \frac{\pi c}{L}$

Case 1 $\quad b < \frac{\pi c}{L} \Rightarrow T_n = e^{-\frac{b}{2}t}\left[C_n\cos(\beta_n t) + D_n\sin(\beta_n t)\right]$, $\boxed{\beta_n = \omega_n\sqrt{1-\xi^2}}$

Case 2 $\quad b = \frac{\pi c}{L} \Rightarrow$ $n=1$ critically damped $T_1 = C_1 e^{-\frac{b}{2}t} + D_1 t e^{-\frac{b}{2}t}$

$n>1$ underdamped. as case 1.

Case 3 $\quad b > \frac{2\pi c}{L} \Rightarrow$ $n=1$ overdamped. $T_1 = C_1 e^{\left(-\frac{b}{2}+\omega_1\sqrt{\xi^2-1}\right)t} + D_1 e^{\left(-\frac{b}{2}-\omega_1\sqrt{\xi^2-1}\right)t}$

$n>1$ underdamped. as case 1 solution

Case 1

$T_n = T_n(\text{homogeneous}) + T_n(\text{particular})$ — as HW#2

$$T_n(\text{particular}) = \frac{2A}{LK_n} \frac{\left[(\omega_n^2-\omega^2)\sin\omega t - 2\zeta\omega_n\omega\cos\omega t\right]}{(\omega_n^2-\omega^2)^2 + (2\zeta\omega_n\omega)^2}$$

$$\text{so } T_n = e^{-\frac{b}{2}t}\left[C_n\cos(\beta_n t) + D_n\sin(\beta_n t)\right] + \frac{2A}{LK_n}\frac{\left[(\omega_n^2-\omega^2)\sin\omega t - 2\zeta\omega_n\omega\cos\omega t\right]}{(\omega_n^2-\omega^2)^2 + (2\zeta\omega_n\omega)^2}$$

apply IC to find C_n, D_n.

we set $C_n = \frac{6h}{L^2K_n^2}\sin\left(\frac{LK_n}{3}\right) + \frac{4A\zeta\omega_n\omega}{LK_n\left[(\omega_n^2-\omega^2)^2 + (2\zeta\omega_n\omega)^2\right]}$

$$D_n = \frac{-2A\left(\omega(\omega_n^2-\omega^2)\right)}{LK_n\beta_n\left[(\omega_n^2-\omega^2)^2 + (2\zeta\omega_n\omega)^2\right]} + \frac{bC_n}{2\beta_n}$$

Case 2

for $n=1$ $T_1 = T_1(\text{homogeneous}) + T_1(\text{particular})$ — as in HW#2; as case #1, $n=1$

$$\text{so } T_1 = \left(C_1 e^{-\frac{b}{2}t} + D_1 t e^{-\frac{b}{2}t}\right) + \frac{4A}{\pi}\frac{\left[(\omega_1^2-\omega^2)\sin\omega t - 2\zeta\omega_1\omega\cos\omega t\right]}{(\omega_1^2-\omega^2)^2 + (2\zeta\omega_1\omega)^2}$$

$n>1$

$$T_n = \left(C_n e^{-\frac{b}{2}t} + D_n t e^{-\frac{b}{2}t}\right) + \frac{2A}{LK_n}\frac{\left[(\omega_n^2-\omega)\sin\omega t - 2\zeta\omega_n\omega\cos\omega t\right]}{(\omega_n^2-\omega^2)^2 + (2\zeta\omega_n\omega)^2}$$

apply I.C. we set $D_1 = -\frac{4A}{\pi}\frac{\omega(\omega_1^2-\omega^2)}{(\omega_1^2-\omega^2)^2 + (2\zeta\omega_1\omega^2)} + \frac{b}{2}C_1$

and for $n=2,3\ldots$ $D_n = \frac{-2A}{LK_n}\frac{\omega(\omega_n^2-\omega^2)}{\beta_n\left[(\omega_n^2-\omega^2)^2 + (2\zeta\omega_n\omega)^2\right]} + \frac{bC_n}{2\beta_n}$

and $C_n = \frac{6h}{L^2K_n^2}\sin\frac{LK_n}{3} + \frac{4A\zeta\omega_n\omega}{LK_n}\frac{1}{\left[(\omega_n^2-\omega^2)^2 + (2\zeta\omega_n\omega)^2\right]}$

→

So find solution for case 2 is

$$u(x,t) = C_1\left(e^{-\frac{b}{2}t} + D_1 e^{-\frac{b}{2}t} + R_1\frac{[(w_1^2-w^2)\sin wt - 2\zeta w_1 w\cos wt)}{(w_1^2-w^2)^2+(2\zeta w_1 w)^2}\right)\sin K_1 x + \sum_{n=2}^{\infty}\left(\left[C_n\cos\beta_n t + D_n\sin\beta_n t\right]e^{-\frac{b}{2}t} + \frac{2A}{LK_n}\frac{[(w_n^2-w^2)\sin wt - 2\zeta w_n w\cos wt]}{(w_n^2-w^2)^2+(2\zeta w_n w)^2}\right)\sin K_n x$$

i.e $u(x,t)$ = transient response + steady state

where $D_1 = \frac{-4A}{\pi}\frac{w(w_1^2-w^2)}{(w_1^2-w^2)^2+(2\zeta w_1 w)^2} + \frac{b}{2}C_1$

$n = 1, 2, \ldots$

and C_n = as in last page

and $D_n = \frac{-2A\,(w(w_n^2-w^2))}{LK_n\beta_n\left[(w_n^2-w^2)^2+(2\zeta w_n w)^2\right]} + \frac{bC_n}{2\beta_n}$

for $n = 2, 3, \ldots$

Case 3 overdamped.

$\zeta > 1$. $b = \frac{2\pi c}{L} = 4w_1$

$\zeta = \frac{b}{2w_n} = \frac{2\pi c}{L}\times\frac{1}{2w_n} = \frac{2\pi c}{L}\times\frac{1}{2cK_n} = \frac{2\pi c}{L}\frac{\chi L}{\chi c(2n-1)\pi} = \frac{1}{(2n-1)}$

so for $n=1$, $\zeta = 2$ i.e critically damped!

for $n=2$, $\zeta = \frac{2}{3} < 1$ i.e underdamped.

so for $n = 2, 3, \ldots$ underdamped. same as case 2.

so just need to solve for $n=1$.

$u_1(x,t) = \left(C_1 e^{\left(-\frac{b}{2}+w_1\sqrt{\zeta^2-1}\right)t} + D_1 e^{\left(-\frac{b}{2}-w_1\sqrt{\zeta^2-1}\right)t}\right)\sin K_1 x$

this is for the homogeneous part. particular solution

as before $R_1\frac{[(w_1^2-w^2)\sin wt - 2\zeta w_1 w\cos wt}{(w_1^2-w^2)^2+(2\zeta w_1 w)^2}$ →

So $\frac{\partial u_1(x,t)}{\partial t} = 0$ at $t = 0$

$$\Rightarrow \left(C_1\left[-\frac{b}{2}+\omega_1\sqrt{\zeta^2-1}\right] + D_1\left[-\frac{b}{2}-\omega_1\sqrt{\zeta^2-1}\right]\right)\sin K_1 x$$

$$+ \frac{R_1}{(\omega_1^2-\omega^2)^2+(2\zeta\omega_1\omega)^2}\left(\omega(\omega_1^2-\omega^2)\right)\sin K_1 x = 0$$

i.e
$$D_1 = \frac{-R_1\left(\omega(\omega_1^2-\omega^2)\right)}{(\omega_1^2-\omega^2)^2+(2\zeta\omega_1\omega)^2} - C_1\left[-\frac{b}{2}+\omega_1\sqrt{\zeta^2-1}\right]$$
$$\overline{\qquad\qquad -\frac{b}{2}-\omega_1\sqrt{\zeta^2-1}}$$

So $u(x,t) = \underbrace{T_1 X_1}_{u_1(x,t)} + \sum_{n=2}^{\infty} T_n X_n$

and the $C_n, D_{n=2,3,\ldots}$ is as was found in Case 2.

where $C_1 = \frac{-R_1\left(\omega(\omega_1^2-\omega^2)\right)}{(\omega_1^2-\omega^2)^2+(2\zeta\omega_1\omega)^2} \quad \frac{-1}{1-\frac{-\frac{b}{2}+\omega_1\sqrt{\zeta^2-1}}{-\frac{b}{2}-\omega_1\sqrt{\zeta^2-1}}} \quad \frac{6h}{L^2K_1^2}\sin\left(\frac{LK_1}{3}\right)$

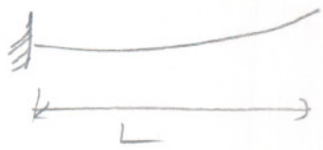
and $C_{2,3,\ldots} =$ as Case 2

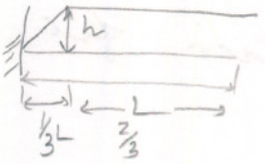
10/10

problem # 5

done with more details
Can be ignored as I
included a short version of
this problem. but I keep this
here for reference

problem #5





initial position $t=0$

$$\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} + A\sin\omega t.$$

assume $u = TX$ and substitute in $\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ (i.e. homogeneous PDE).

$$T''X + bT'X = c^2X''T$$

divide by XT we obtain

$$\frac{T''}{T} + b\frac{T'}{T} = c^2\frac{X''}{X} = \lambda$$

we obtain 2 ODE's:

$$\frac{1}{c^2}\left(\frac{T''}{T} + b\frac{T'}{T}\right) = \lambda \quad \text{and} \quad \frac{X''}{X} = \lambda$$

or $$T'' + bT' - \lambda c^2 T = 0 \quad \text{and} \quad X'' = \lambda X$$

as before, we can determine that only for $\lambda < 0$ we obtain non trivial solutions for the spatial equation. Hence $\lambda = -K^2$ for $K > 0$

hence we write

$$T'' + bT' + K^2c^2T = 0 \quad \text{and} \quad \boxed{X'' = -K^2X}$$

the spatial ODE has solution $X(x) = A\cos Kx + B\sin Kx$

as before we find that $\boxed{X_n(x) = \sin K_n x}$ where $\boxed{K_n = \frac{2n-1}{2L}\pi}$ $n=1,2\ldots$

now expand the nonhomogeneous term by eigenfunction expansion

$$A\sin\omega t = \sum_{n=1}^{\infty} g_n(t)\, X_n(x)$$

as before we find that $\boxed{g_n(t) = \frac{2A\sin\omega t}{LK_n}}$

now plug this into the PDE we obtain

$$\boxed{\sum T_n''X_n + b\sum T_n'X_n = \sum c^2X_n''T_n + \sum C_n(t)X_n}$$

but $X_n'' = -\omega_n^2 X_n$ so above becomes

$$\sum T_n''X_n + \sum bT_n'X_n = \sum -c^2K_n^2X_nT_n + \sum C_n(t)X_n \quad \rightarrow$$

hence we obtain the second order ODE for T_n

$$\boxed{T_n'' + b T_n' + c^2 k_n^2 T_n = g_n(t)}$$

Solve the homogeneous ODE

$$T_n'' + b T_n' + c^2 k_n^2 T_n = 0$$

as in HW # 2 this is a standard 2nd order system.

write it as $\boxed{T_n'' + 2\xi\omega_n T_n' + \omega_n^2 T_n = 0}$

where $\boxed{\omega_n = c k_n}$ and $\boxed{\xi = \frac{b}{2\omega_n}}$

the roots of the characteristic equation is

$$r_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

$$\boxed{r_{1,2} = -\frac{b}{2} \pm \omega_n\sqrt{\xi^2 - 1}}$$

So we get 3 cases, depending if $\xi < 1$, $\xi = 1$, $\xi > 1$

for each case we obtain a homogeneous solution for T_n, then for each such case we have to find a particular solution. then add both solutions to obtain the general solution for T_n. then we use initial conditions to solve for the 2 unknown integration constants. Now $\omega_1 = ck_1 = \frac{c(\pi)}{2L}$, $\omega_2 = ck_2 = c\frac{(3\pi)}{2L} = 1.5\frac{c\pi}{L}$

$\omega_3 = \frac{c(5\pi)}{2L} = 2.5\frac{c\pi}{L}$ etc..

Case 1 $b < \frac{\pi c}{L}$ i.e $\xi < 1$ $\Rightarrow$ all modes are under damped.

homogeneous solution is $T_n = e^{-\frac{b}{2}t}\left[C_n \cos(\beta_n t) + D_n \sin(\beta_n t)\right]$

where $\beta_n = \omega_n\sqrt{1-\xi^2}$ and $\omega_n = ck_n$

to find a particular solution for $T_n'' + 2\xi\omega_n T_n' + \omega_n^2 T_n = g_n(t)\sin k_n x$

solve for $Y'' + 2\xi\omega_n Y' + \omega_n^2 Y = \frac{2A}{Lk_n} e^{i\omega t}$ and take the particular solution of this, then take the imaginary part $\rightarrow$

let $y_p = M e^{i\omega t}$, $y_p' = Mi\omega e^{i\omega t}$, $y_p'' = -M\omega^2 e^{i\omega t}$

so $-M\omega^2 e^{i\omega t} + 2\xi\omega_n M i\omega e^{i\omega t} + \omega_n^2 M e^{i\omega t} = \frac{2A}{Lk_n} e^{i\omega t}$

so $-M\omega^2 + 2\xi\omega_n M i\omega + \omega_n^2 M = \frac{2A}{Lk_n}$

i.e. $M(-\omega^2 + 2\xi\omega_n i\omega + \omega_n^2) = \frac{2A}{Lk_n} \Rightarrow M = \frac{2A}{Lk_n} \frac{1}{\omega_n^2 + 2\xi\omega_n i\omega - \omega^2}$

so $y_p = \frac{2A}{Lk_n} \frac{1}{\omega_n^2 + 2\xi\omega_n i\omega - \omega^2} e^{i\omega t}$

take the imaginary part of this and that will be the particular solution we need.

let $R_n = \frac{2A}{Lk_n}$ and write $y_p = R_n \frac{\cos\omega t + i\sin\omega t}{(\omega_n^2 - \omega^2) + i2\xi\omega_n\omega}$

$\Rightarrow y_p = R_n \frac{(\cos\omega t + i\sin\omega t)((\omega_n^2 - \omega^2) - i2\xi\omega_n\omega)}{((\omega_n^2 - \omega^2) + i2\xi\omega_n\omega)((\omega_n^2 - \omega^2) - i2\xi\omega_n\omega)}$ (multiply by complex conjugate of denominator)

$= R_n \frac{[(\omega_n^2 - \omega^2)\cos\omega t - i2\xi\omega_n\omega\cos\omega t + i(\omega_n^2 - \omega^2)\sin\omega t + 2\xi\omega_n\omega\sin\omega t]}{(\omega_n^2 - \omega^2)^2 + (2\xi\omega_n\omega)^2}$

$= R_n \frac{(\omega_n^2 - \omega^2)\cos\omega t + 2\xi\omega_n\omega\sin\omega t}{(\omega_n^2 - \omega^2)^2 + (2\xi\omega_n\omega)^2} + i R_n \frac{[(\omega_n^2 - \omega^2)\sin\omega t - 2\xi\omega_n\omega\cos\omega t]}{(\omega_n^2 - \omega^2)^2 + (2\xi\omega_n\omega)^2}$

So imaging part gives particular solution

So T_n particular solution is $R_n \frac{[(\omega_n^2 - \omega^2)\sin\omega t - 2\xi\omega_n\omega\cos\omega t]}{(\omega_n^2 - \omega^2)^2 + (2\xi\omega_n\omega)^2}$

hence for <u>Case 1</u>, the full solution for T_n is

$= e^{-\frac{b}{2}t}[C_n\cos(\beta_n t) + D_n\sin(\beta_n t)] + R_n \frac{[(\omega_n^2 - \omega^2)\sin\omega t - 2\xi\omega_n\omega\cos\omega t]}{(\omega_n^2 - \omega^2)^2 + (2\xi\omega_n\omega)^2}$

now app'l I.C. to find C_n and D_n

→

$$u(x,t) = \sum_{n=1}^{\infty} T_n X_n$$

$$= \sum_{n=1}^{\infty} \left(e^{-\frac{b}{2}t}\left[C_n \cos(\beta_n t) + D_n \sin(\beta_n t)\right] + R_n \left[\frac{(\omega_n^2-\omega^2)\sin\omega t - 2\xi\omega_n\omega\cos\omega t}{(\omega_n^2-\omega^2)^2 + (2\xi\omega_n\omega)^2}\right]\right) \sin k_n x$$

$\frac{\partial u}{\partial t} = 0$ at $t=0 \Rightarrow$

$$0 = \sum_{n=1}^{\infty} \left(-\frac{b}{2} e^{-\frac{b}{2}t}\left[C_n\cos(\beta_n t) + D_n \sin(\beta_n t)\right] + e^{-\frac{b}{2}t}\left[-C_n\beta_n \sin(\beta_n t) + D_n\beta_n \cos\beta_n t\right] + \right.$$

$$\left. \frac{R_n}{(\omega_n^2-\omega^2)^2 + (2\xi\omega_n\omega)^2}\left(\omega(\omega_n^2-\omega^2)\cos\omega t + 2\xi\omega_n\omega^2 \sin\omega t\right)\right) \sin k_n x$$

$$0 = \sum_{n=1}^{\infty} \left[-\frac{b}{2}C_n + D_n\beta_n + \frac{R_n}{(\omega_n^2-\omega^2)^2 + (2\xi\omega_n\omega)^2}\left(\omega(\omega_n^2-\omega^2)\right)\right] \sin k_n x$$

because of uniqueness of Fourier series we obtain that

$$\boxed{-\frac{b}{2}C_n + D_n\beta_n + \frac{R_n\left(\omega(\omega_n^2-\omega^2)\right)}{(\omega_n^2-\omega^2)^2 + (2\xi\omega_n\omega)^2} = 0} \quad \text{——— (1)}$$

now apply second IC.

$u(x,t) = f(x)$ at $t=0$.

so $$\boxed{f(x) = \sum \left(C_n + \frac{R_n(-2\xi\omega_n\omega)}{(\omega_n^2-\omega^2)^2 + (2\xi\omega_n\omega)^2}\right) \sin k_n x} \quad \text{——— (2)}$$

from (2) we find C_n and use that to find D_n from (1).

to make it easier to write, call the term $\frac{R_n(-2\xi\omega_n\omega)}{(\omega_n^2-\omega^2)^2 + (2\xi\omega_n\omega)^2} = H_n$.

so $$f(x) = \sum (C_n + H_n) \sin k_n x.$$

so $$C_n + H_n = \frac{\langle f(x), \sin k_n x\rangle}{\langle \sin k_n x, \sin k_n x\rangle} = \frac{\int_0^L f(x)\sin k_n x}{\int_0^L \sin^2 k_n x}$$

$$\boxed{C_n + H_n = \frac{6h}{L^2 k_n^2}\sin\frac{Lk_n}{3}} \text{ as we did before.} \rightarrow$$

$$\text{so } C_n = \frac{6h}{L^2 k_n^2}\sin\frac{Lk_n}{3} - \frac{2A}{Lk_n}\frac{(-2\xi\omega_n w)}{(\omega_n^2-w^2)^2+(2\xi\omega_n w)^2}$$

$$\boxed{C_n = \frac{6h}{L^2k_n^2}\sin\left(\frac{Lk_n}{3}\right) + \frac{4A\,\xi\omega_n w}{Lk_n\left[(\omega_n^2-\omega^2)^2 + (2\xi\omega_n w)^2\right]}}$$

From (1) we now find D_n

$$D_n = \frac{-\frac{R_n\left(w(\omega_n^2-w^2)\right)}{(\omega_n^2-w^2)^2+(2\xi\omega_n w)^2} + \frac{b}{2}C_n}{\beta_n} = \boxed{\frac{-2A\left(w(\omega_n^2-w^2)\right)}{Lk_n\beta_n\left[(\omega_n^2-w^2)^2+(2\xi\omega_n w)^2\right]} + \frac{bC_n}{2\beta_n}}$$

So final solution for Case 1 is

$$u(x,t) = \sum T_n X_n$$

$$= \sum_{n=1}^{\infty}\left(e^{-\frac{b}{2}t}\left[C_n\cos(\beta_n t) + D_n\sin(\beta_n t)\right] + \frac{2A}{Lk_n}\frac{\left[(\omega_n^2-w^2)\sin wt - 2\xi\omega_n w\cos wt\right]}{(\omega_n^2-w^2)^2+(2\xi\omega_n w)^2}\right)\sin k_n x$$

where C_n and D_n is as given above. $\boxed{k_n = \frac{2n-1}{2L}\pi,\ \beta_n = \omega_n\sqrt{1-\xi^2}}$

$$\boxed{\xi = \frac{b}{2\omega_n}}$$

Plots follow for $b = \frac{0.5\pi c}{L}$, using $L=1$, $h=.1$, $c=4$

and animation.

first find T_1. $\quad 2\pi f t = \beta_n t = \omega_n\sqrt{1-\xi^2}\,t$

$$= ck_n\sqrt{1-\left(\frac{b}{2\omega_n}\right)^2}\,t = c\,\frac{(2n-1)}{2L}\pi\sqrt{1-\left(\frac{b}{2ck_n}\right)^2}\,t$$

$$\text{so } 2f = \frac{c(2n-1)}{2L}\sqrt{1-\left(\frac{b}{2c\frac{(2n-1)}{2L}\pi}\right)^2} \Rightarrow n=1 \text{ we set}$$

$$2f = \frac{c}{2L}\sqrt{1-\left(\frac{bL}{c\pi}\right)^2} = \frac{c}{2L}\sqrt{1-\frac{b^2L^2}{c^2\pi^2}} \quad . c=4, L=1 \text{ we set}$$

$$2f = \frac{4}{2}\sqrt{1-\frac{b^2}{16\pi^2}} \Rightarrow f = \sqrt{1-\frac{b^2}{16\pi^2}} = \sqrt{1-\frac{(0.5\pi\times4)^2}{16\pi^2}} = \sqrt{0.75} = 0.866\text{Hz} \Rightarrow \boxed{T_1 = 1.155\ \text{sec}}$$

Case 2 $b = \frac{\pi c}{L} \Rightarrow \xi = \frac{b}{2\omega_n} = \frac{\pi c}{L}\frac{1}{2\omega_n} = \frac{\pi c}{L}\frac{1}{2ck_n} = \frac{\pi c}{L}\frac{1}{2c}\frac{2L}{(2n-1)\pi}$

i.e. $\xi = \frac{1}{2n-1}$ so for $n=1 \Rightarrow \xi_1 = 1$, for $n=2$, $\xi_2 = \frac{1}{3} < 1$

i.e. for $n=1$ only we have critically damped, and for $n=2,3,\ldots$ underdamped.

for $n=1$

in this case $T_1 = C_1 e^{-\frac{b}{2}t} + D_1 t e^{-\frac{b}{2}t}$ is the homogeneous solution

and T_1 particular solution is as before.

so $T_1 = \left(C_1 e^{-\frac{b}{2}t} + D_1 t e^{-\frac{b}{2}t}\right) + R_1 \frac{\left[(\omega_1^2-\omega^2)\sin\omega t - 2\xi\omega_1\omega\cos\omega t\right]}{(\omega_1^2-\omega^2)^2 + (2\xi\omega_1\omega)^2}$

but $\omega_1 = ck_1 = \frac{1}{2}\frac{c\pi}{L}$, $R_1 = \frac{2A}{Lk_1} = \frac{2A2L}{L\pi} = \frac{4A}{\pi}$

so $T_1 = \left(C_1 e^{-\frac{b}{2}t} + D_1 t e^{-\frac{b}{2}t}\right) + \frac{4A}{\pi}\left[\frac{(\omega_1^2-\omega^2)\sin\omega t - 2\xi\omega_1\omega\cos\omega t}{(\omega_1^2-\omega^2)^2 + (2\xi\omega_1\omega)^2}\right]$

and T_n for $n=2,3,\ldots$ is as before

$$T_n = e^{-\frac{b}{2}t}\left[C_n\cos(\beta_n t) + D_n\sin(\beta_n t)\right] + R_n\left[\frac{(\omega_n^2-\omega^2)\sin\omega t - 2\xi\omega_n\omega\cos\omega t}{(\omega_n^2-\omega^2)^2 + (2\xi\omega_n\omega)^2}\right]$$

so $u(x,t) = \sum_{n=1}^{\infty} T_n X_n$

$= T_1 X_1 + \sum_{n=2}^{\infty} T_n X_n =$

Apply first I.C. to $n=1$

$\frac{\partial u}{\partial t} = 0 \Rightarrow \left(-\frac{b}{2}C_1 e^{-\frac{b}{2}t} + D_1 e^{-\frac{b}{2}t} + D_1 t\left(-\frac{b}{2}\right)e^{-\frac{b}{2}t}\right)\sin(k_1 x)$

$$+ \frac{R_1}{(\omega_1^2-\omega^2)^2 + (2\xi\omega_1\omega)^2}\left[\omega(\omega_1^2-\omega^2)\cos\omega t + 2\xi\omega_1\omega^2\sin\omega t\right]\sin k_1 x$$

so $0 = \left[-\frac{b}{2}C_1 + D_1\right]\sin k_1 x + \frac{R_1\sin k_1 x}{(\omega_1^2-\omega^2)^2 + (2\xi\omega_1\omega^2)}\left[\omega(\omega_1^2-\omega^2)\right]$

$$\Rightarrow \quad -\frac{b}{2}C_1 + D_1 + \frac{R_1\,\omega(\omega_1^2-\omega^2)}{(\omega_1^2-\omega^2)^2 + (2\xi\omega_1\omega^2)} = 0$$

$\longrightarrow$

so $\boxed{D_1 = -\frac{4A}{\pi}\frac{\omega(\omega_1^2-\omega^2)}{(\omega_1^2-\omega^2)^2+(2\zeta\omega_1\omega)^2} + \frac{b}{2}C_1}$

so $u_1(x,t) = \left[C_1 e^{-\frac{b}{2}t} + [\ ]\, t e^{-\frac{b}{2}t} + R_1\frac{[(\omega_1^2-\omega^2)\sin\omega t - 2\zeta\omega_1\omega\cos\omega t]}{(\omega_1^2-\omega^2)^2+(2\zeta\omega_1\omega)^2}\right]\sin K_1 x$

$u_{2,3,\ldots}(x,t) = \sum_{n=2}^{\infty} T_n X_n$

apply $\frac{\partial u}{\partial t} = 0$ at $t=0$. as in case 1, we get

$$D_n = \frac{-2A(\omega(\omega_n^2-\omega^2))}{L K_n \beta_n[(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2} + \frac{b C_n}{2\beta_n}$$

$n = 2, 3, \ldots$

so $u_{2,3\ldots}(x,t) = \sum_{n=2}^{\infty} C_n e^{-\frac{b}{2}t}\left(\cos(\beta_n t) + D_n \sin(\beta_n t)\right)\sin K_n x$

$$+ R_n\frac{[(\omega_n^2-\omega^2)\sin\omega t - 2\zeta\omega_n\omega\cos\omega t]}{(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2}\sin K_n x$$

So $u(x,t) = u_1(x,t) + u_{2,3\ldots}(x,t)$

at $t=0$, $u_0(x,t) = f(x)$.

so $f(x) = \left[C_1 + \frac{R_1(-2\zeta\omega_1\omega)}{(\omega_1^2-\omega^2)^2+(2\zeta\omega_1\omega)^2}\right]\sin(K_1 x)$

$$+ \sum_{2}^{\infty} C_n(1)\sin K_n x + \frac{R_n(-2\zeta\omega_n\omega)}{(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2}\sin K_n x$$

i.e. $f(x) = \sum_{n=1}^{\infty}\left[C_n - \frac{2R_n\zeta\omega_n\omega}{(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2}\right]\sin K_n x$

as in case 1, we set

$$C_n = \frac{6h}{L^2K_n^2}\sin\frac{LK_n}{3} + \frac{4A\zeta\omega_n\omega}{LK_n[(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2]}$$