HW 1, MAE 170.
Problem B 6-1, Modern Control Engineering, 4th edition by Ogata
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## Question

Plot the root loci for $G(s)=\frac{k}{s(s+1)\left(s^{2}+4 s+5\right)}$

## Solution

The closed loop transfer function is

$$
G_{c l}(s)=\frac{G}{1+G}=\frac{\frac{k}{s(s+1)\left(s^{2}+4 s+5\right)}}{1+\frac{k}{s(s+1)\left(s^{2}+4 s+5\right)}}=\frac{k}{s(s+1)\left(s^{2}+4 s+5\right)+k}
$$

Hence the system is


## step 1

Plot the open loop poles and zeros:
Poles are at $s=0, s=-1$, and roots of $\left(s^{2}+4 s+5\right)$ which is $s=\frac{-4 \pm \sqrt{16-4 \times 5}}{2}=\frac{-4 \pm 2 i}{2}=-2 \pm i$ There are no finite zeros (there are however 4 zeros at $\infty$ )


## step 2

Apply angle condition to find initial segments on real axis that could be part of the root loci.
We see that on the RHP, it is not possible to have loci, since then the sum of angles to a text point will not add to $180 \pm n 360$. For segments between p1 and p2, we will have a loci. for the segment to the right of p2, it is not possible to have a loci. Hence we now get this plot: (I use bold line to show where loci is)


## step 3

Since loci starts at a pole and ends at a zero of the open loop poles/zeros, then we know loci must start at p1 as well it must start at p2. Hence there must be a break away point between p1 and p2. Now we find this break away point. For this we use the condition that $\frac{d k}{d s}=0$ The characteristic equation for the close loop is $s(s+1)\left(s^{2}+4 s+5\right)+k=0$ hence $\frac{d k}{d s}=\frac{d}{d s}\left(s(s+1)\left(s^{2}+4 s+5\right)\right)=\frac{d}{d s}\left(5 s+9 s^{2}+5 s^{3}+s^{4}\right)=5+18 s+15 s^{2}+4 s^{3}=0$, Solution is: $[s=-1.6785-0.60278 i],[s=-1.6785+0.60278 i],[s=-0.39299]$
Since we are looking for a solution on the real axis, and one that is between p 1 and p 2 , hence only possible solution is $s=-0.39299$, I add this point to the diagram, now it looks as follows


## step 4

Now I need to find where the asymptotes lines cross at the real axis, and need to find angles that asymptotes leave the real axis at.
to find where asymptotes meet at the real axis:
$\sigma_{a}=\frac{\sum_{m} z_{i}-\sum_{n} p_{i}}{n-m}=\frac{0-(0+1+(2-i)+(2+i))}{4}=\frac{-(1+4)}{4}=-1.25$
To find angles, put a test point s very far away, and consider the sum of angles from the finite poles and zeros to that test point. we have here only 4 finite poles and no finite zeros. We know that sums of these angles must be 180 degrees, so we write

$$
\begin{aligned}
4 \theta & =180 \pm n 360 \\
\theta & =45 \pm n 90
\end{aligned}
$$

Hence the angles are $45,45+90,45+180,45+270$ or $45,135,225,315$ so now I get this diagram, where I just added the asymptotes lines

step 5
Now I need to find angles of departures of loci from p 3 and p 4 . To do this, put a test point s very close to p3 and solve for the angle conditions. a little bit of geometry is needed here. We get


Hence, for a test point 's' near p3, we get from the angle condition, the following

$$
\theta_{1}+\theta_{2}+\theta_{3}+\alpha=180 \pm n 360
$$

but
$\theta_{3}=90^{\circ}$
$\theta_{2}=90^{0}+\tan ^{-1}(1 / 1)=135^{0}$
$\theta_{1}=90^{0}+\tan ^{-1}(2 / 1)=90^{0}+63.4^{0}=153.4^{0}$
hence

$$
\begin{aligned}
\alpha+90+135+153.4 & =180^{0} \pm n 360 \\
\alpha+378.4 & =180^{0} \pm n 360 \\
\alpha & =180^{0} \pm n 360-378.4 \\
& =-198.4 \pm n 360
\end{aligned}
$$

Since angles are positive anticlockwise, then this angle of -198.4 means this:


Hence, this is the angle of departures of p3. By symmetry we know the angle of departure of p4. Hence the plot now looks like this:


Note: at the break away point, -0.39 , the loci is at $90^{\circ}$ from the real axis.

## step 6

Find where loci cross the j axis.
Looking at the charaterstic equation for the close loop $s(s+1)\left(s^{2}+4 s+5\right)+k$, set $s=j \omega$ and solve

$$
s(s+1)\left(s^{2}+4 s+5\right)+k=0
$$

Hence

$$
\begin{array}{r}
\left(s^{2}+s\right)\left(s^{2}+4 s+5\right)+k=0 \\
5 s+9 s^{2}+5 s^{3}+s^{4}+k=0
\end{array}
$$

Let $s=j \omega$

$$
5 j \omega-9 \omega^{2}-5 j \omega^{3}+\omega^{4}+k=0
$$

Hence equating real parts and imaginary parts, we get

$$
\begin{aligned}
5 \omega-5 \omega^{3} & =0 \\
-9 \omega^{2}+\omega^{4}+k & =0
\end{aligned}
$$

From first equation, we get $1-\omega^{2}=0$ or $\omega= \pm 1$
Hence the loci crosses the imaginary axis at $\pm i$
To find the gain $k$ at these point, using the second equation, we get

$$
\begin{aligned}
-9 i^{2}+i^{4}+k & =0 \\
+9+1 & =-k
\end{aligned}
$$

hence gain is 10 (negative gain) where it cross the point $(0, i)$ by symmetry, the gain will be 10 (positive gain) where it cross at point $(0,-i)$

