

HW # 6

MAE 170

winter 2005

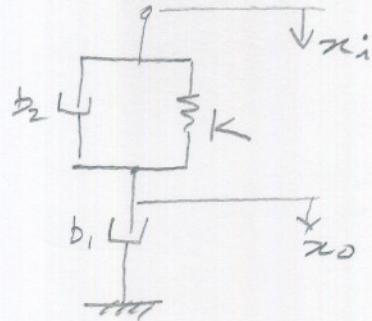
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HW # 6

Problem B-7-1

Consider mechanical system below. Obtain transfer function of system. x_i is input, x_o is output. Is this a Lead or Lag network?



Solution

Write the dynamic equations of system:

$$b_2(\ddot{x}_i - \ddot{x}_o) + K(x_i - x_o) = b_1(\dot{x}_o) \quad \checkmark$$

Take Laplace transform

$$b_2 s(X_i - X_o) + K(X_i - X_o) = b_1 sX_o$$

$$X_i [b_2 s + K] = X_o [b_1 s + b_2 s + K]$$

so

$$\frac{X_o}{X_i} = \frac{b_2 s + K}{s(b_1 + b_2) + K}$$

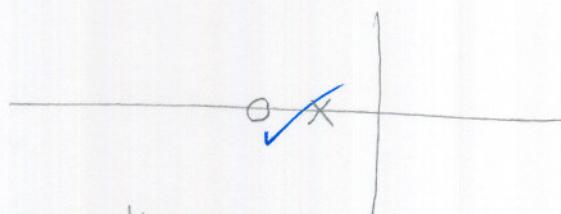
i.e.

$$X_i \rightarrow \frac{b_2 s + K}{(b_1 + b_2)s + K} \rightarrow X_o$$

So poles/zeros of the transfer function are.

$$\text{Zero: } s = -\frac{K}{b_2}$$

$$\text{Pole: } s = -\frac{K}{(b_1 + b_2)}$$



Since $b_1 + b_2 > b_2$, then $\frac{K}{b_1 + b_2} < \frac{K}{b_2}$

This means The zero is to the left of the pole \Rightarrow

Lag Network

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Problem B-7-3

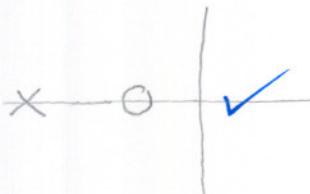
is the following $G_c(s)$ a lead or lag?

$$G_c(s) = \frac{3.5s + 1.4}{s + 2}$$

Answer

$$G_c(s) = \frac{s + \frac{1.4}{3.5}}{s + 2} = \frac{s + 0.4}{s + 2}$$

since zero of $G_c(s)$ is at $s = -0.4$
and pole of $G_c(s)$ is at $s = -2$



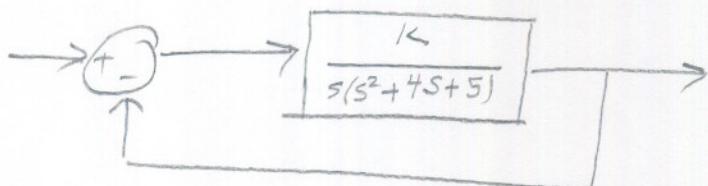
then this is a ~~lead~~ component
(pole to left of zero).

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Problem B-7-6

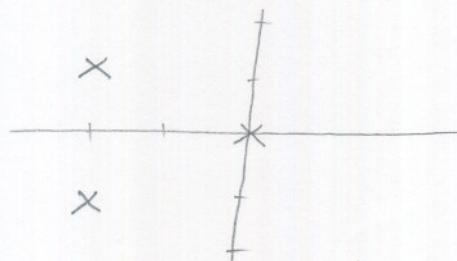
Consider system shown, Plot loci for system.

Determine value for K such that 3 of dominant pole for closed loop is 0.5. Then determine all closed loop poles. Plot unit step using Matlab.



Answer

Step 1 Put poles/zeros of open loop

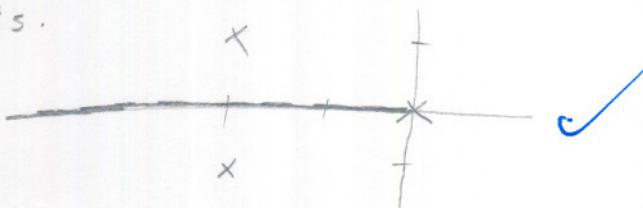


$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-4 \pm \sqrt{16 - 4 \times 5}}{2}$$
$$= -2 \pm i$$

Step 2 determine loci on real axis. (apply angle condition).

No loci on positive real axis.

Loci on all negative real axis.

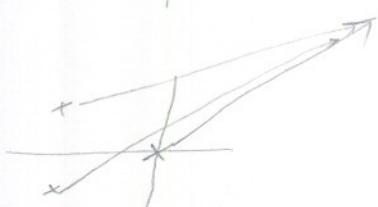


Step 3 determine asymptotes.

put a test point very far from origin

$$\text{then } 3\theta = 180 \pm n360$$

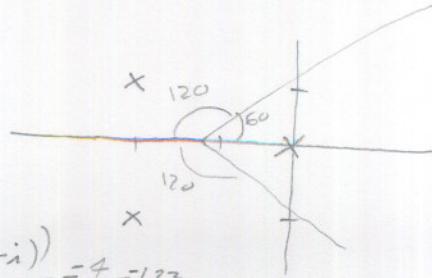
$$\text{so } \theta = 60^\circ \pm n120^\circ$$



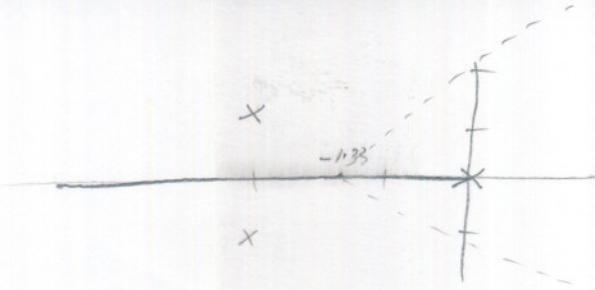
so

now find where asymptotes meet in σ_a .

$$\sigma_a = \frac{\sum P - \sum Z}{n-m} = \frac{(0 + (-2+i) + (-2-i))}{3} = \frac{4}{3} = 1.33$$



so we have this:



Step 4 find s_b (break away) use $\frac{dk}{ds} = 0$.

The char. equation for closed loop is

$$K + s(s^2 + 4s + 5) = 0$$

∴

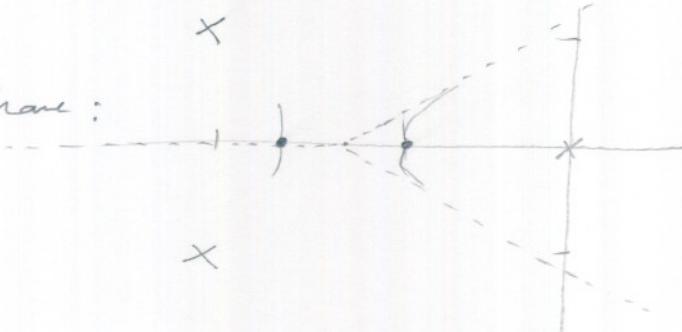
$$\begin{aligned}\frac{dk}{ds} &= \frac{d}{ds}(s^3 + 4s^2 + 5s) \\ &= 3s^2 + 8s + 5 = 0\end{aligned}$$

$$\text{so } s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-8 \pm \sqrt{64 - 4 \times 3 \times 5}}{6} = \frac{-8 \pm \sqrt{4}}{6}$$

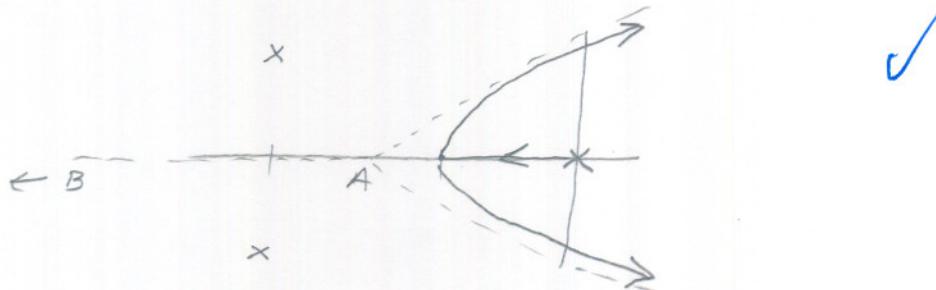
$$= \frac{-8 \pm 2}{6} = \frac{-8+2}{6} \text{ or } \frac{-8-2}{6}$$

i.e. -1 or -1.6

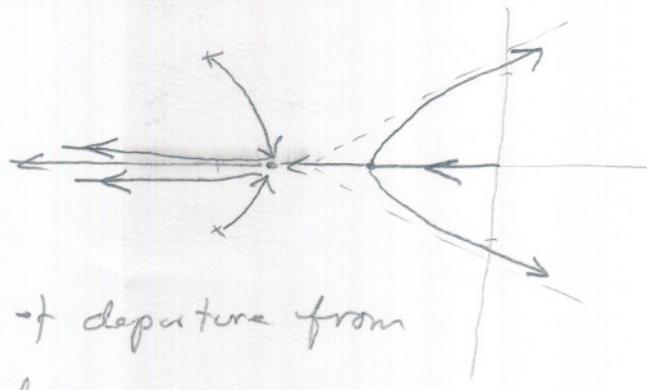
i.e we now have:



now since a loci starts at a pole and at a zero, then loci must start at pole at origin. take point -1 as breakaway. we set

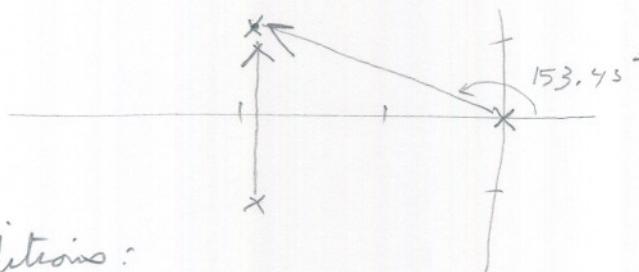


and since line AB is an asymptote, then it mean point -1.6 is a break in for branches coming down from conjugate poles i.e. →



Step 5 to find angle of departure from the conjugate poles:

Put a small test point close to $(-2+i)$.

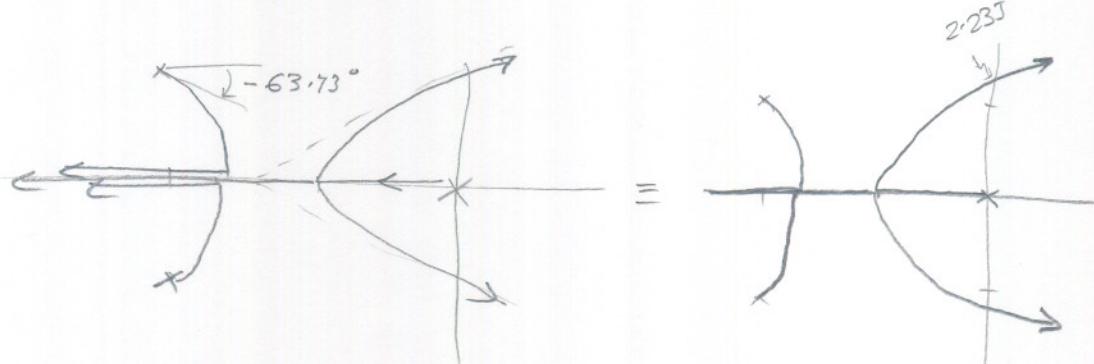


apply angle conditions:

$$153.43^\circ + 90^\circ + \alpha = 180 \pm n360$$

$$\therefore \alpha = 180 - 243.43^\circ = -63.43^\circ \pm n360 \quad \text{in} \quad +\sqrt{-63.43^\circ}$$

\therefore departure angle is -63.43°



Step 6 find intersection at imaginary axis.

Characteristic equation is $K + S(S^2 + 4S + 5) = 0$

$$\therefore K + S^3 + 4S^2 + 5S = 0$$

$$\therefore K + -j\omega^3 - 4\omega^2 + 5j\omega = 0$$

$$\therefore K - 4\omega^2 + j(-\omega^3 + 5\omega) = 0 \Rightarrow \begin{cases} K - 4\omega^2 = 0 \\ -\omega^3 + 5\omega = 0 \end{cases} \Rightarrow \begin{cases} K = 4\omega^2 \\ \omega^3 = 5\omega \end{cases} \Rightarrow \omega^2 = 5$$

$$\therefore \omega = \sqrt{5} \Rightarrow \text{intersection points } \boxed{\pm 2.236j}$$

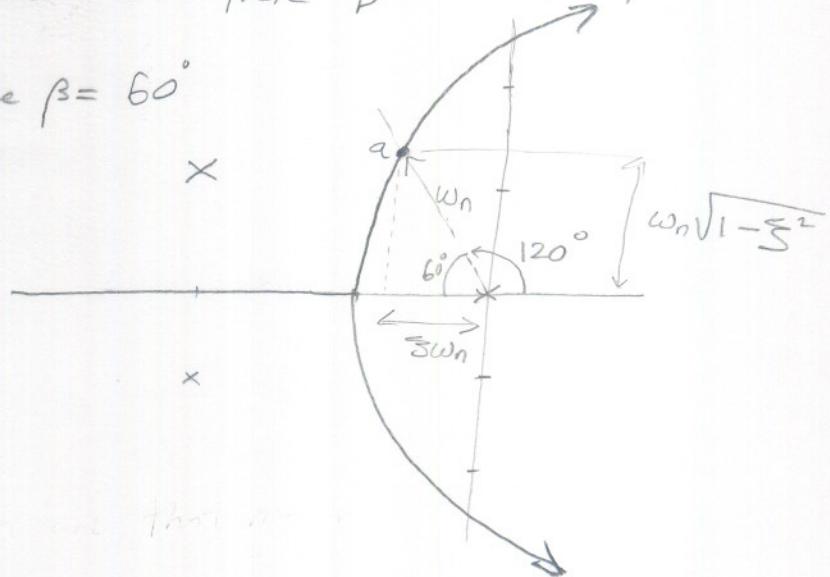
and K at that point is $\boxed{20}$



now that loci is plotted, complete solution of problem.

find K such that ξ of dominant pole for closed loop is 0.5.

$$\xi = 0.5 \Rightarrow \text{Re}\beta = \xi \Rightarrow \text{i.e. } \beta = 60^\circ$$



so need to find point 'a' on the locus

$$\text{closed loop } \left[G(s) = \frac{K}{K + s(s^2 + 4s + 5)} \right]$$

The dominant closed loop pole by definition is on loci. assume it is at point 'a'. this point makes angle 60° as shown.

roots of $K + s(s^2 + 4s + 5) = 0$ are the closed loop poles.

let point 'a' be $(\sigma + i\beta)$

$$K + (\sigma + i\beta)((\sigma + i\beta)^2 + 4(\sigma + i\beta) + 5) = 0$$

$$\therefore (\sigma + i\beta)[\sigma^2 + 2i\sigma\beta - \beta^2 + 4\sigma + 4i\beta + 5] = -K$$

$$\sigma^3 + 2i\sigma^2\beta - \sigma\beta^2 + 4\sigma^2 + 4i\sigma\beta + 5\sigma$$

$$+ i\sigma\beta - 2\sigma\beta^2 - i\beta^3 + 4\sigma i\beta - 4\beta^2 + 5i\beta = K.$$

$$\text{i.e. } (\sigma^3 - \sigma\beta^2 + 4\sigma^2 + 5\sigma - 2\sigma\beta^2 - 4\beta^2) + i(2\sigma^2\beta + 4\sigma\beta + \sigma\beta - \beta^3 + 4\sigma\beta + 5\beta) = -K.$$

equate real parts and imaginary parts.

so

$$\left[\begin{array}{l} \sigma^3 - 3\sigma\beta^2 - 4\beta^2 + 4\sigma^2 + 5\sigma = -K \\ 2\sigma^2\beta + 9\sigma\beta - \beta^3 + 5\beta = 0 \end{array} \right]$$

This is 2 eqn, but 3 unknowns. now using $\tan 60^\circ = \frac{\beta}{\sigma}$ gives 3rd eq. \rightarrow

So we have

$$\left. \begin{array}{l} \sigma^3 - 3\sigma\beta^2 - 4\beta^2 + 4\sigma + 5\sigma = -K \\ 2\sigma^2\beta + 9\sigma\beta - \beta^3 + 5\beta = 0 \\ 1.732 = \frac{\beta}{\sigma} \end{array} \right\} \begin{array}{l} \text{--- (1)} \\ \text{--- (2)} \\ \text{--- (3)} \end{array}$$

3 equations, 3 unknowns. now we can find K .

$$\beta = 1.732\sigma$$

sub in eq (2)

$$2\sigma^2(1.732\sigma) + 9\sigma(1.732\sigma) - (1.732\sigma)^3 + 5(1.732\sigma) = 0$$

$$\text{so } 3.464\sigma^3 + 15.588\sigma^2 - 5.195\sigma^3 + 8.66\sigma = 0$$

$$\text{or } -1.731\sigma^3 + 15.5886\sigma^2 + 8.66\sigma = 0$$

$$\text{or } -1.731\sigma^2 + 15.5886\sigma + 8.66 = 0$$

$$\text{so } \sigma = \frac{-15.588 \pm \sqrt{15.588^2 - 4 \times (-1.731) \times 8.66}}{2 \times (-1.731)} = \frac{-15.588 \pm 17.4}{-3.462}$$
$$= \frac{-15.588 + 17.4}{-3.462} \quad \text{or} \quad \frac{-15.588 - 17.4}{-3.462}$$

$$= -0.523 \quad \text{or} \quad 9.5 \quad \Rightarrow \text{since pole is on negative plan} \Rightarrow \boxed{\sigma = -0.523}$$

$$\text{so } \beta = (1.732)(-0.523) = \boxed{-0.9058}$$

so $\beta = -0.9058$

from (1) we find K :

$$(-0.523)^3 - 3(-0.523)(-0.9058)^2 - 4(-0.9058)^2 + 4(-0.523) + 5(-0.523) = -K$$
$$-0.143 + 9.058^2 - 1.093 + 6.06 - 5 = -K$$
$$\Rightarrow K = 3.67$$

now to find all closed loop poles. i.e. solutions of

char eq

$$\boxed{3.67 + s(s^2 + 4s + 5) = 0}$$



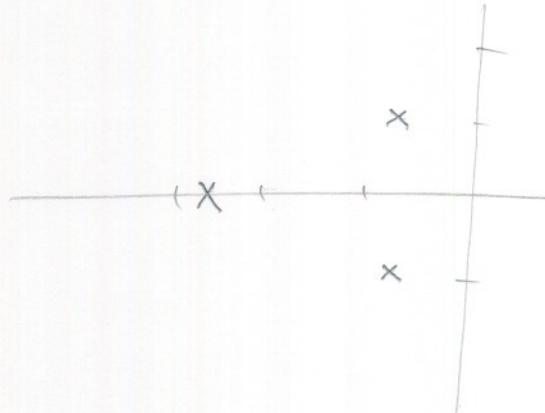
now to find Poles of Closed Loop system,
solved char eq.

$$3.67 + s(s^2 + 4s + 5) = 0$$

solutions are

$$\boxed{\begin{aligned} s &= -2.629 \\ s &= -0.685 - 0.962i \checkmark \\ s &= -0.685 + 0.962i \checkmark \end{aligned}}$$

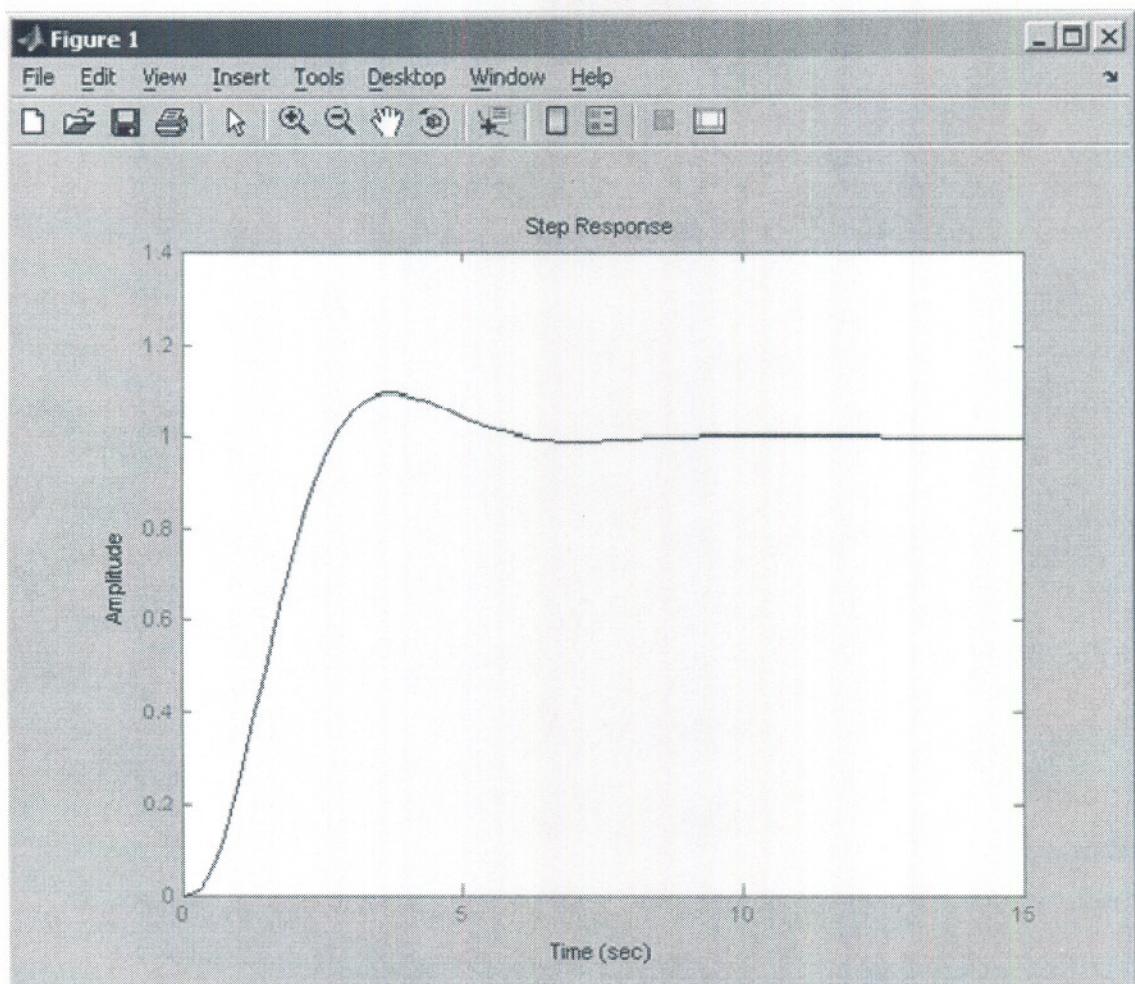
these are the closed loop poles



now I plot using matlab the step response:

✓

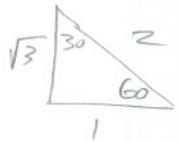
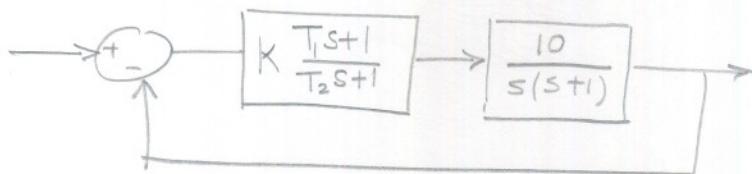
```
s=tf('s');
k=3.67
G=k/ (s*(s^2+4*s+5));
sys=G/(1+G);
step(sys)
```



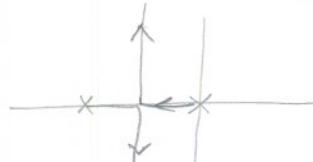
HW#6

Problem B-7-7

Determine the values of K, T_1, T_2 of system shown so that the dominant closed-loop poles have $\xi = 0.5$ and the undamped $\omega_n = 3 \text{ rad/sec}$



Loci for uncompensated system.



Solution

Poles/Zeros for uncompensated system is

$$\zeta_b = \frac{-1}{2} = -0.5$$

we see that for uncompensated system, closed pole will not be on loci, we want closed loop pole to be at:

so need the compensator.

if we put the zero of the compensator on top of the pole at -1, we get

now apply angle conditions:

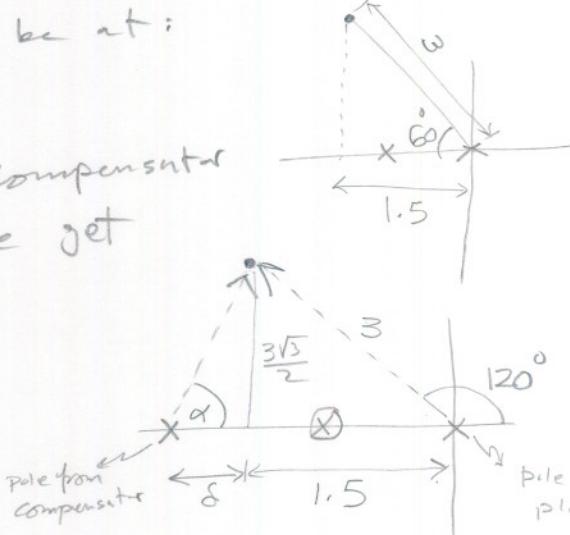
$$120^\circ + \alpha = 180^\circ \pm n360^\circ$$

$$\Rightarrow \alpha = 60^\circ$$

$$\text{so } \tan 60^\circ = \frac{3\sqrt{3}/2}{s} \Rightarrow s = \frac{3\sqrt{3}/2}{\sqrt{3}} = \frac{3}{2} = 1.5$$

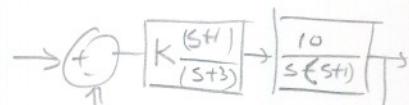
so pole from compensator at -3 .

$$\text{so } G_c(s) = K \frac{(s+1)}{(s+3)}$$



Now apply magnitude conditions. need char. equation

$$\text{so } \left| \frac{10K(s+1)}{s(s+3)(s+1)} \right| \Rightarrow \left| K \frac{(s+1)}{(s+3)} \frac{10}{s(s+1)} \right| = 1$$



let $s = \left(-1.5 + \frac{3\sqrt{3}}{2}j\right)$. The location of one dominant closed loop pole. now we find K

$$\left| \frac{10K \left(-1.5 + j\frac{3\sqrt{3}}{2} + 1 \right)}{\left(-1.5 + j\frac{3\sqrt{3}}{2} + 3 \right) \left(-1.5 + j\frac{3\sqrt{3}}{2} \right) \left(-1.5 + j\frac{3\sqrt{3}}{2} + 1 \right)} \right| = 1$$

$$\left| \frac{10K \left(-0.5 + j\frac{3\sqrt{3}}{2} \right)}{\left(1.5 + j\frac{3\sqrt{3}}{2} \right) \left(-1.5 + j\frac{3\sqrt{3}}{2} \right) \left(-0.5 + j\frac{3\sqrt{3}}{2} \right)} \right| = 1$$

$$\left| \frac{10K}{-1.5 - \frac{9(3)}{4}} \right| = 1$$

$$\left| \frac{10K}{2.25 - 6.75} \right| = 1 \Rightarrow \left| \frac{10K}{-4.5} \right| = 1$$

so $\frac{10K}{4.5} = 1 \Rightarrow K = \frac{4.5}{10} = 0.45$

so $G_c(s) = 0.45 \frac{(s+1)}{(s+3)}$ compare to

$$K \frac{T_1 s + 1}{T_2 s + 1}$$

so $\frac{0.45s + 0.45}{s+3} = \frac{K T_1 s + K}{T_2 s + 1}$

multiply, and equate coefficients of s^2, s, K , we set

and
$$\begin{cases} 0.45 T_2 = T_1 K \\ 0.45 + T_2 \cdot 0.45 = 3 T_1 K + K \\ 0.45 = 3 K \end{cases}$$

so $K = 0.15 \Rightarrow \begin{cases} 0.45 + 0.45 T_2 = 0.45 T_1 + 0.15 \\ 0.45 T_2 = 0.15 T_1 \end{cases}$

so $T_1 = 3T_2 \Rightarrow 0.45 + 0.45 T_2 = 0.45 (3T_2) + 0.15 \Rightarrow 0.45 + 0.45 T_2 - 1.35 T_2 - 0.15 = 0$

so $T_2 (0.45 - 1.35) = 0.15 - 0.45 \Rightarrow T_2 = \frac{-0.3}{-0.9} = \frac{1}{3}$

so $T_1 = 3(\frac{1}{3}) = 1 \Rightarrow G_c(s) = 0.15 \frac{(1s+1)}{(\frac{1}{3}s+1)} \Rightarrow \begin{cases} K = 0.15 \\ T_1 = 1 \\ T_2 = \frac{1}{3} \end{cases}$