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HW # 5

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MAE 170

HW 1, MAE 170.

Problem B 6-1, Modern Control Engineering, 4th edition by Ogata
by Nasser Abbasi
UCI, Winter 2005.

Question

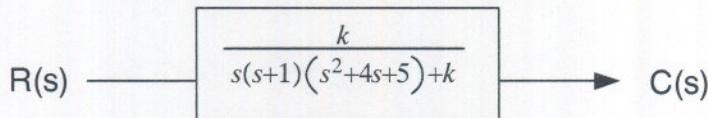
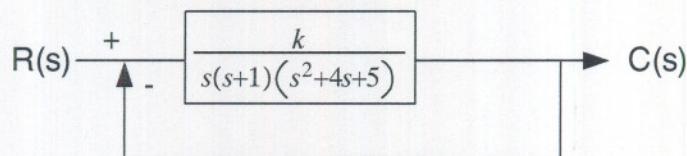
Plot the root loci for $G(s) = \frac{k}{s(s+1)(s^2+4s+5)}$

Solution

The closed loop transfer function is

$$G_{cl}(s) = \frac{G}{1+G} = \frac{\frac{k}{s(s+1)(s^2+4s+5)}}{1 + \frac{k}{s(s+1)(s^2+4s+5)}} = \frac{k}{s(s+1)(s^2+4s+5) + k}$$

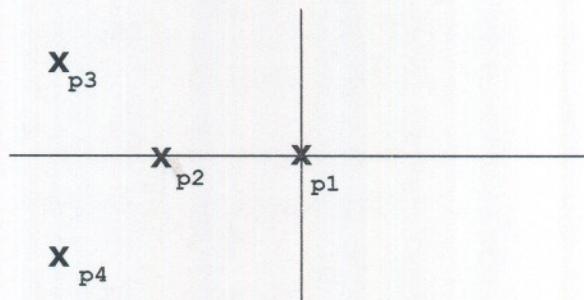
Hence the system is



step 1

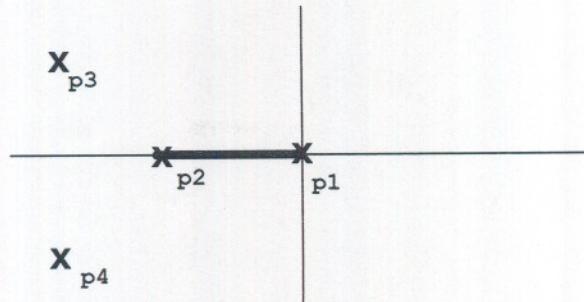
Plot the open loop poles and zeros:

Poles are at $s = 0$, $s = -1$, and roots of $(s^2 + 4s + 5)$ which is $s = \frac{-4 \pm \sqrt{16 - 4 \times 5}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$
There are no finite zeros (there are however 4 zeros at ∞)



step 2

Apply angle condition to find initial segments on real axis that could be part of the root loci.
We see that on the RHP, it is not possible to have loci, since then the sum of angles to a test point will not add to $180 \pm n360$. For segments between p1 and p2, we will have a loci. for the segment to the right of p2, it is not possible to have a loci. Hence we now get this plot: (I use bold line to show where loci is)



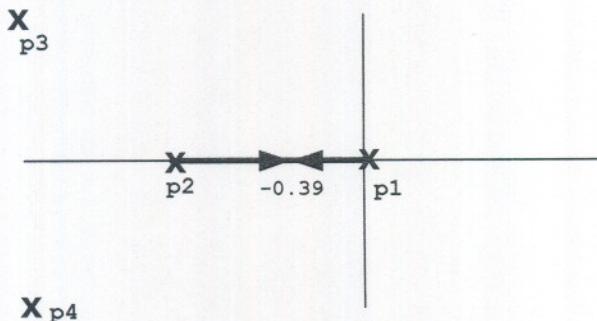
step 3

Since loci starts at a pole and ends at a zero of the open loop poles/zeros, then we know loci must start at p1 as well it must start at p2. Hence there must be a break away point between p1 and p2. Now we find this break away point. For this we use the condition that $\frac{dk}{ds} = 0$

The characteristic equation for the close loop is $s(s+1)(s^2 + 4s + 5) + k = 0$ hence

$$\frac{dk}{ds} = \frac{d}{ds}(s(s+1)(s^2 + 4s + 5)) = \frac{d}{ds}(5s + 9s^2 + 5s^3 + s^4) = 5 + 18s + 15s^2 + 4s^3 = 0, \text{ Solution is: } [s = -1.6785 - 0.60278i], [s = -1.6785 + 0.60278i], [s = -0.39299]$$

Since we are looking for a solution on the real axis, and one that is between p1 and p2, hence only possible solution is $s = -0.39299$, I add this point to the diagram, now it looks as follows



step 4

Now I need to find where the asymptotes lines cross at the real axis, and need to find angles that asymptotes leave the real axis at.

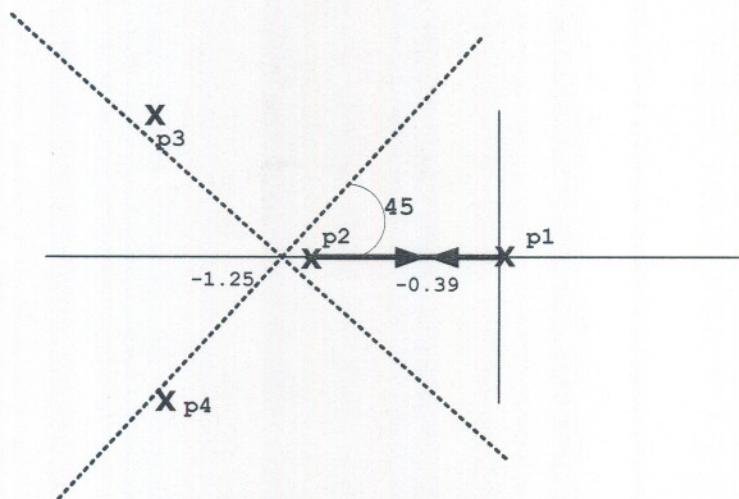
to find where asymptotes meet at the real axis:

$$\sigma_a = \frac{\sum_m z_i - \sum_n p_i}{n-m} = \frac{0 - (0+1+(2-i)+(2+i))}{4} = \frac{-(1+4)}{4} = -1.25$$

To find angles, put a test point s very far away, and consider the sum of angles from the finite poles and zeros to that test point. we have here only 4 finite poles and no finite zeros. We know that sums of these angles must be 180 degrees, so we write

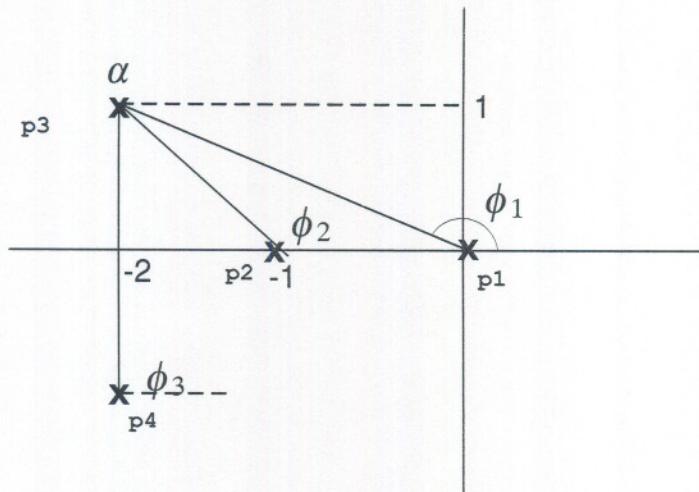
$$\begin{aligned} 4\theta &= 180 \pm n360 \\ \theta &= 45 \pm n90 \end{aligned}$$

Hence the angles are 45, 45 + 90, 45 + 180, 45 + 270 or 45, 135, 225, 315 so now I get this diagram, where I just added the asymptotes lines



step 5

Now I need to find angles of departures of loci from p3 and p4. To do this, put a test point s very close to p3 and solve for the angle conditions. a little bit of geometry is needed here. We get



Hence, for a test point 's' near p3, we get from the angle condition, the following

$$\theta_1 + \theta_2 + \theta_3 + \alpha = 180^\circ \pm n360^\circ$$

but

$$\theta_3 = 90^\circ$$

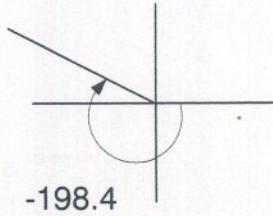
$$\theta_2 = 90^\circ + \tan^{-1}(1/1) = 135^\circ$$

$$\theta_1 = 90^\circ + \tan^{-1}(2/1) = 90^\circ + 63.4^\circ = 153.4^\circ$$

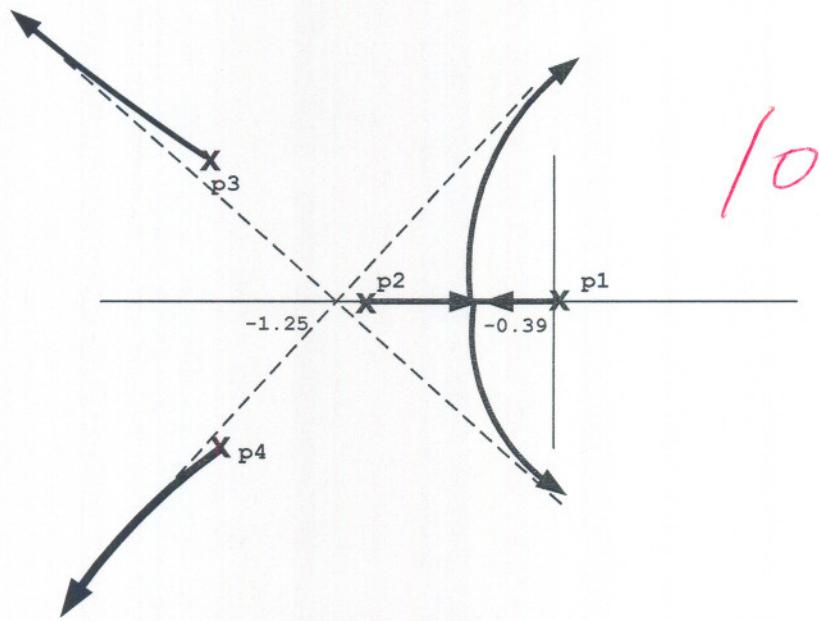
hence

$$\begin{aligned}
 \alpha + 90 + 135 + 153.4 &= 180^\circ \pm n360^\circ \\
 \alpha + 378.4 &= 180^\circ \pm n360^\circ \\
 \alpha &= 180^\circ \pm n360^\circ - 378.4 \\
 &= -198.4 \pm n360^\circ
 \end{aligned}$$

Since angles are positive anticlockwise, then this angle of -198.4 means this:



Hence, this is the angle of departures of p₃. By symmetry we know the angle of departure of p₄. Hence the plot now looks like this:



Note: at the break away point, -0.39, the loci is at 90° from the real axis.

step 6

Find where loci cross the j axis.

Looking at the characteristic equation for the close loop $s(s+1)(s^2 + 4s + 5) + k$, set $s = j\omega$ and solve

$$s(s+1)(s^2 + 4s + 5) + k = 0$$

Hence

$$\begin{aligned} (s^2 + s)(s^2 + 4s + 5) + k &= 0 \\ 5s + 9s^2 + 5s^3 + s^4 + k &= 0 \end{aligned}$$

Let $s = j\omega$

$$5j\omega - 9\omega^2 - 5j\omega^3 + \omega^4 + k = 0$$

Hence equating real parts and imaginary parts, we get

$$\begin{aligned} 5\omega - 5\omega^3 &= 0 \\ -9\omega^2 + \omega^4 + k &= 0 \end{aligned}$$

From first equation, we get $1 - \omega^2 = 0$ or $\omega = \pm 1$

Hence the loci crosses the imaginary axis at $\pm i$

To find the gain k at these point, using the second equation, we get

$$\begin{aligned}-9i^2 + i^4 + k &= 0 \\ +9 + 1 &= -k\end{aligned}$$

hence gain is 10 (negative gain) where it cross the point $(0, i)$ by symmetry, the gain will be 10 (positive gain) where it cross at point $(0, -i)$

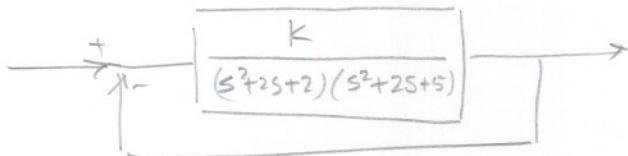
HW#5

Problem B-6-5

Plot Root Loci For

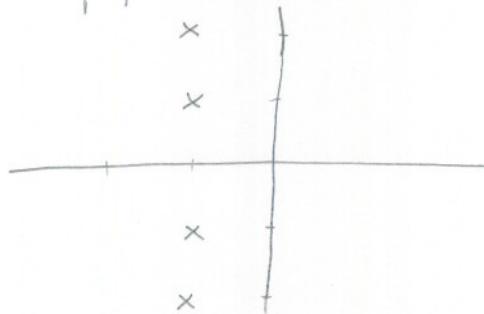
$$G(s) = \frac{K}{(s^2 + 2s + 2)(s^2 + 2s + 5)}$$

Solution



Step 1

Plot open Loop Poles and Zeros.



For $s^2 + 2s + 2 = 0 \rightarrow -1 \pm i$
For $s^2 + 2s + 5 = 0 \rightarrow -1 \pm 2i$

Step 2 Find loci on real axis. we see that using angle conditions, there will be no loci on real axis. it is not possible to put have test point on real axis for loci.

Step 3 since loci starts at poles and ends at zeros, then since there are no finite zeros, then all loci will end at zeros at ∞ . since there are no break away or break in points here, nothing to do in this step.

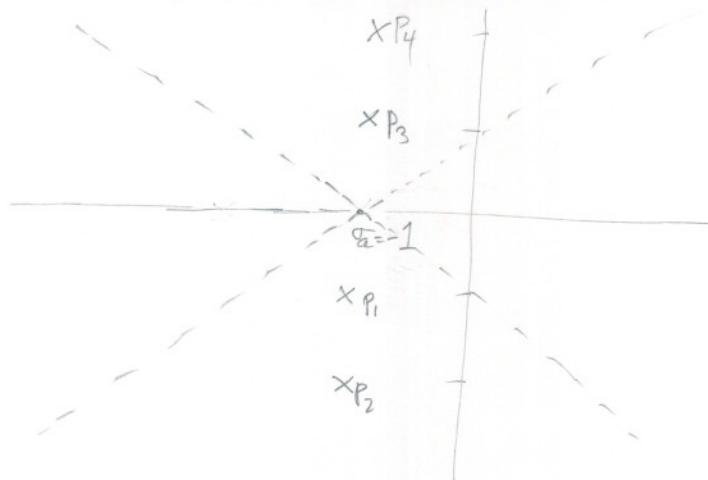
Step 4 determine number of asymptote lines and σ_a .

$$\sigma_a = \frac{\sum z_i - \sum p_i}{n-m} = \frac{0 - (1+i + 1-i + 1-2i + 1+2i)}{4} = -\left(\frac{4}{4}\right) = -1$$

now find angles of asymptotes.

Put a test point very far. we see that $4\theta = 180 \pm n360$

or $\theta = 45^\circ \pm n90^\circ$, i.e 4 asymptotes \rightarrow



Step 5 Find angles of departures for P_3, P_4 (by symmetry we)

Find angle of departures for P_1, P_2 .

For P_3 Put test point 's' very close to P_3 , we set

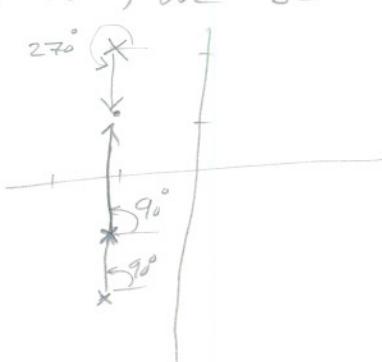
$$\text{so } \alpha + 270 + 90 + 90 = 180 \pm n360$$

$$\alpha + 450 = 180 \pm n360$$

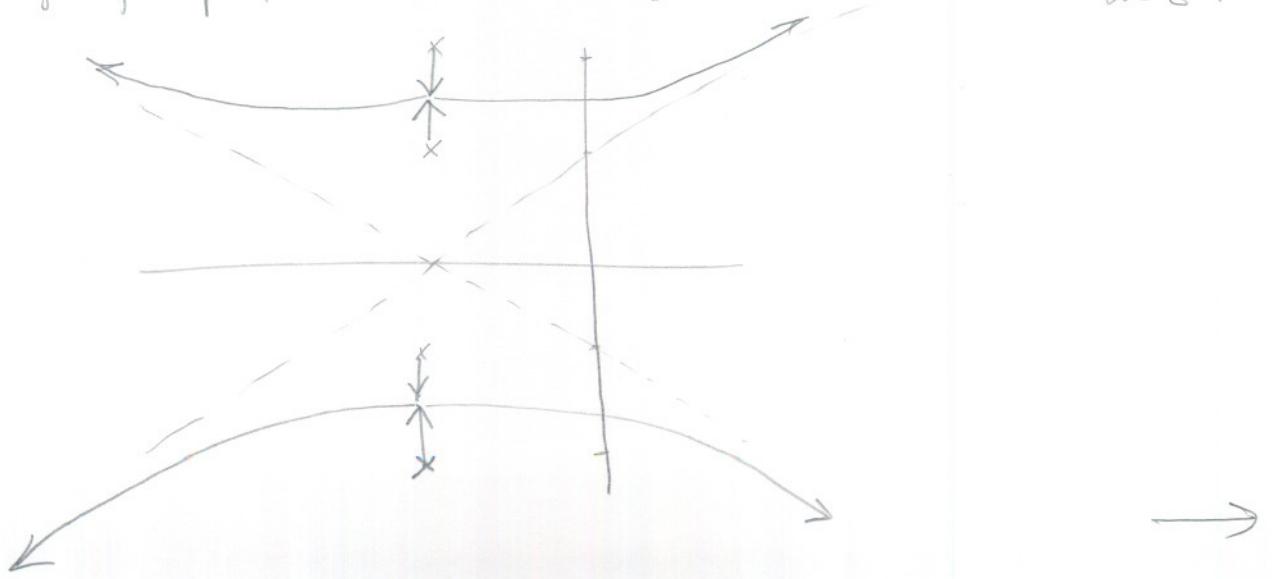
$$\alpha = (180 - 450) \pm n360$$

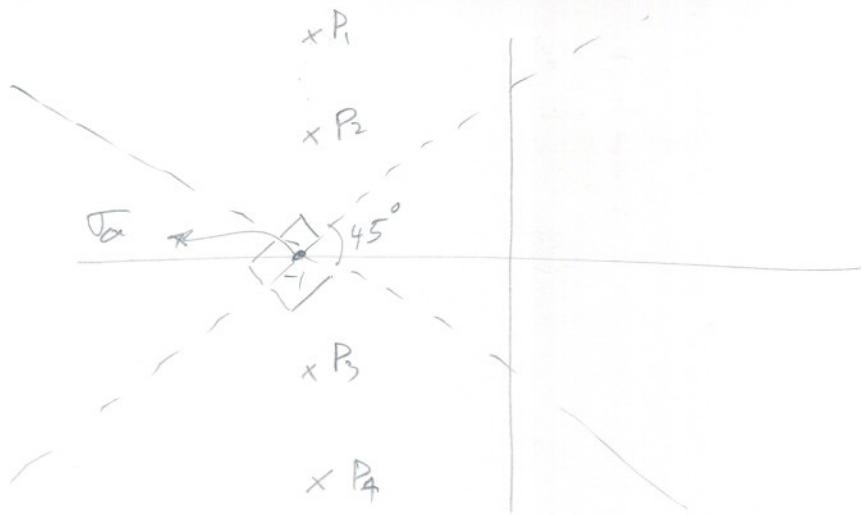
$$\alpha = -270 \pm n360$$

hence α is



hence angle of departures are -270° (or 90° from horizontal), by symmetry we set:

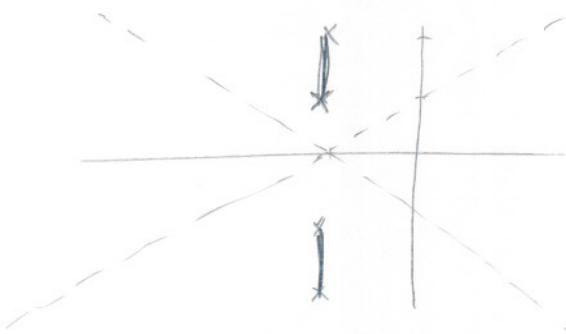
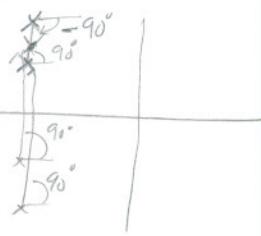




since we have 4 poles, we must have 4 branches.
Put a test point 's' between P_1, P_2 , then angle condition is
 $-90^\circ + 90^\circ + 90^\circ + 90^\circ = 180^\circ$.

hence Path between P_1, P_2 is an loci.

Similarly path between P_3, P_4 is loci



Step 6 Find where loci cross JW axis.

From char. equation of closed loop

$$(s^2+2s+2)(s^2+2s+5) + k = 0$$

$$s^4 + 2s^3 + 5s^2 + 2s^3 + 4s^2 + 10s + 2s + 4s + 10 + k = 0$$

$$s^4 + 4s^3 + 11s^2 + 14s + 10 + k = 0 \quad \text{let } s = j\omega$$

$$\omega^4 - 4j\omega^3 - 11\omega^2 + 14j\omega + 10 + k = 0$$

equate real and imaginary parts

$$\omega^4 - 11\omega^2 + 10 + k = 0$$

$$-4\omega^3 + 14\omega = 0 \implies 14 = 4\omega^2 \rightarrow \omega = \pm\sqrt{7} = \pm 1.87$$

so loci crosses imaginary axis at $\boxed{\pm 1.87i}$

HW#5

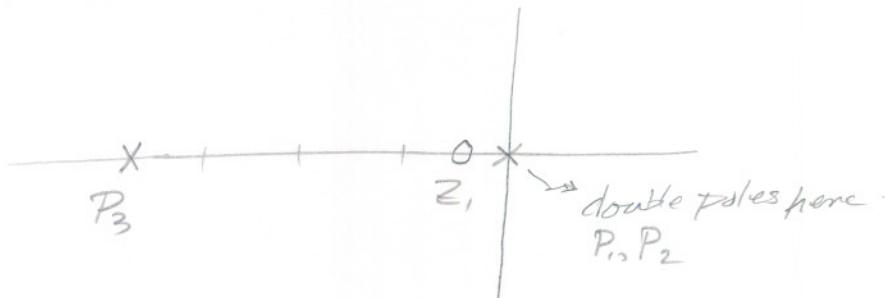
Problem B6-7

Plot loci for $G(s) = \frac{K(s+0.2)}{s^2(s+3.6)}$ $H(s) = 1$

Solution

Step 1 Plot Poles and zeros of open loop

$$P_1 = 0, P_2 = 0, P_3 = -3.6, Z_1 = -0.2$$



Step 2 Determine loci on real axis:

between P_1 and Z_1 , angle conditions is (count vectors from poles only to test point).

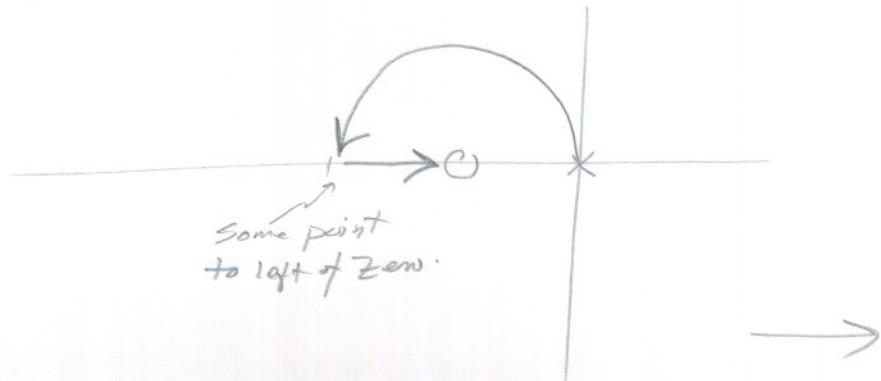
$$180^\circ + 180^\circ + 0 \neq 180^\circ$$

hence line between P_1, Z_1 is not on loci.

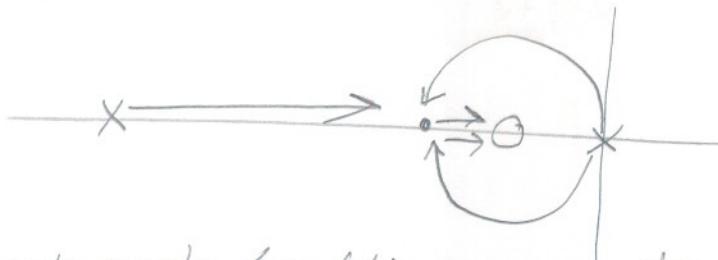
Line on the real axis also not on loci. since angle

$$\text{conditions there is } 0 + 0 + 0 = 0.$$

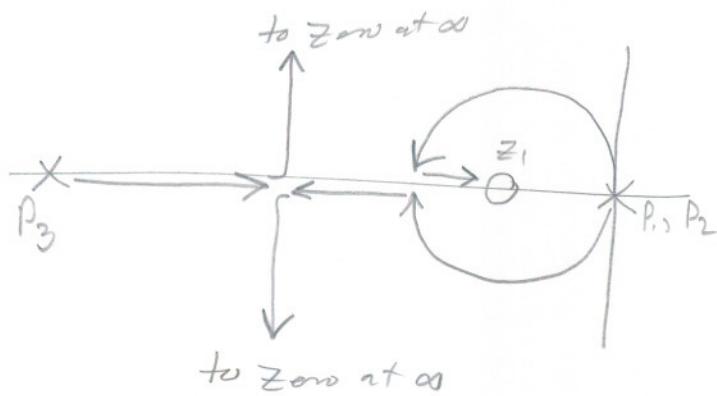
we know that loci must start at P_1 and end at a zero Z_1 . but since line between P_1, Z_1 is not on loci, how will occur then? it must be that loci jump over the imaginary plan and falls back to real axis and go back to the zero Z_1 as follows:



to verify, put a test point to left of z_0 as follows



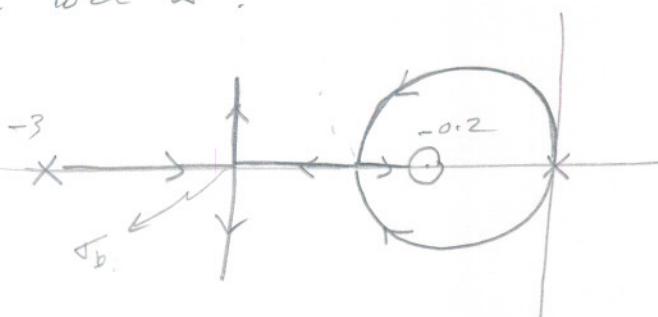
Now apply angle conditions, we get $0 + 0 + 0 \neq 180^\circ$.
Then it must be that branch from P_2 will not go back to Z_1 , but will go to P_3 and have a break point in between:



now try again with a test point to left of Z_1 , we set

$$0 + 180^\circ + 0 = 180^\circ - \text{OK}$$

hence loci is :



Step 3

now I need to find point δ_b .

$\frac{dK}{ds} = 0$ at this points.

char eq. for closed Loop is $s^2(s+3 \cdot 6) + K(s+0 \cdot 2) = 0$

$$\text{so } K = \frac{-s^2(s+3 \cdot 6)}{s+0 \cdot 2} \Rightarrow \frac{dK}{ds} = \frac{s^2(s+3 \cdot 6)}{-(s+0 \cdot 2)^2} + \frac{1}{(s+0 \cdot 2)^2} [s^2 + 2s(s+3 \cdot 6)]$$

$$\frac{dK}{ds} = \frac{s^3 + 3 \cdot 6s^2}{-(s+0 \cdot 2)^2} + \frac{s^2 + 2s^2 + 7 \cdot 2s}{s+0 \cdot 2} = \frac{s^3 + 3 \cdot 6s^2 - (s+0 \cdot 2)(s^2 + 2s^2 + 7 \cdot 2s)}{-(s+0 \cdot 2)^2} \rightarrow$$

∴ set $\frac{dk}{ds} = 0$

Then $s^3 + 3.6s^2 - [s^3 + 2s^3 + 7.2s^2 + 0.2s^2 + 0.4s^2 + 1.44s] = 0$

i.e. $s^3 + 3.6s^2 - s^3 - 2s^3 - 7.4s^2 - 0.4s^2 - 1.44s = 0$

i.e. $-2s^3 - 4.2s^2 - 1.44s = 0$

i.e. $-2s^2 - 4.2s - 1.44 = 0$

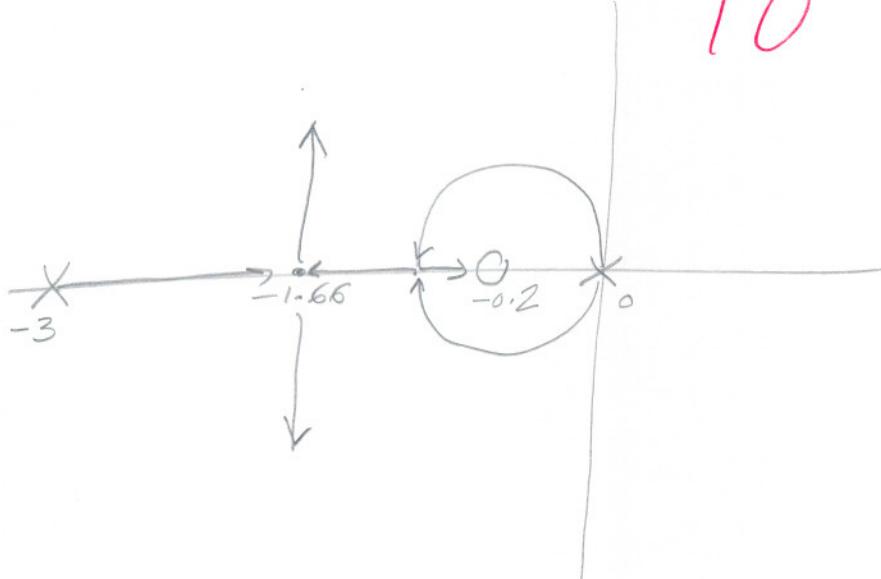
∴ $s^2 + 2.1s + .72 = 0$

i.e. $s = \frac{-2.1 \pm \sqrt{2.1^2 - 4 \times .72}}{2} = \frac{-2.1 \pm 1.2369}{2} = -1.668 \approx 0.018$

∴ $s_b = -1.668$

hence Final loci is

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HW#5

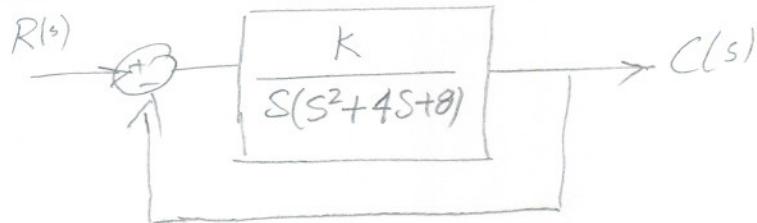
Problem B 6-11

Consider Unity feedback control systems with the following feedforward transfer function.

$$G(s) = \frac{K}{s(s^2 + 4s + 8)}$$

Plot the root loci for the system. if value of gain K is set equal to 2, where are the closed loop poles located?

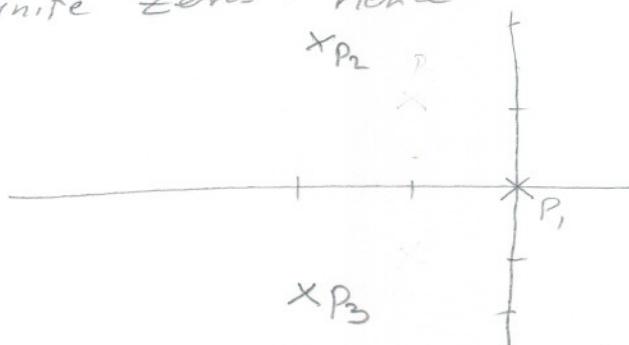
Answer



Step 1 Plot the open loop poles and zeros.

$$P_1 = 0, P_2 = \text{roots } s^2 + 4s + 8 = 0 \Rightarrow \frac{-4 \pm \sqrt{16 - 4 \times 8}}{2} = -2 \pm 2i$$

No finite zeros. Hence

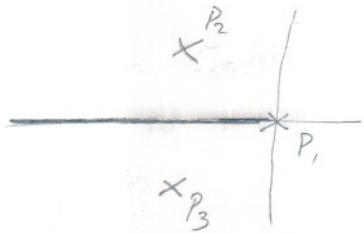


Step 2 apply angle conditions to find loci on real axis.

No loci to right of P_1 .

Put test point to left of P_1 . $\angle \text{lco} = 180^\circ$. ok. so loci on all of Left real axis.





Step 3 locate σ_b (break away points).
since no other pole on real axis. nothing to do.

Step 4 locate asymptotes:

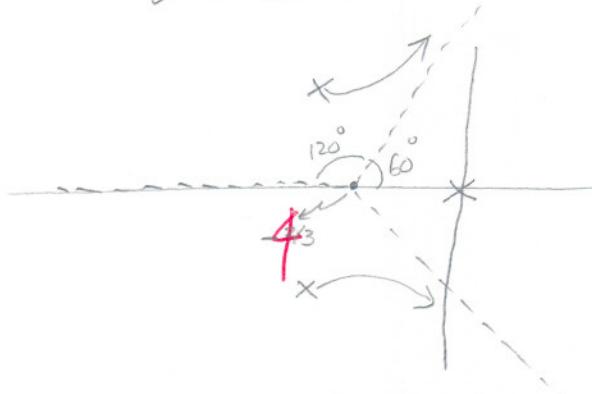
$$\sigma_a = \frac{\sum z_i - \sum p_i}{n-m} = \frac{0 - (0 + (2+1) + (2+1))}{3} = -\frac{(2)}{3} = -2/3$$

Put test point very far.

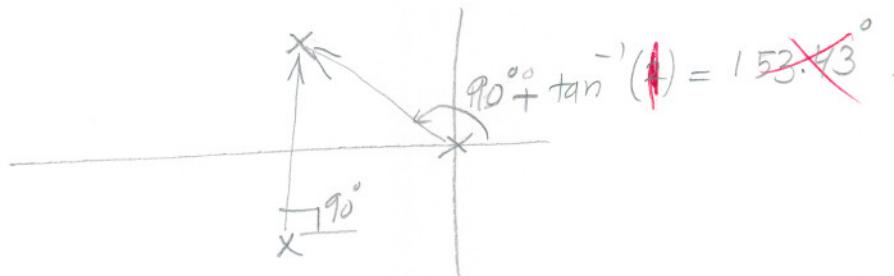
Then we have $3\theta = 180^\circ \pm n360^\circ$

$$\theta = 60^\circ \pm 120^\circ n$$

so

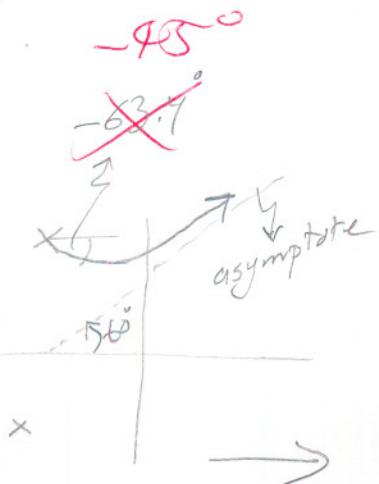


Step 5 Find angles of departures of P_2, P_3 to asymptotes.
Put test point very close to P_2 . we get

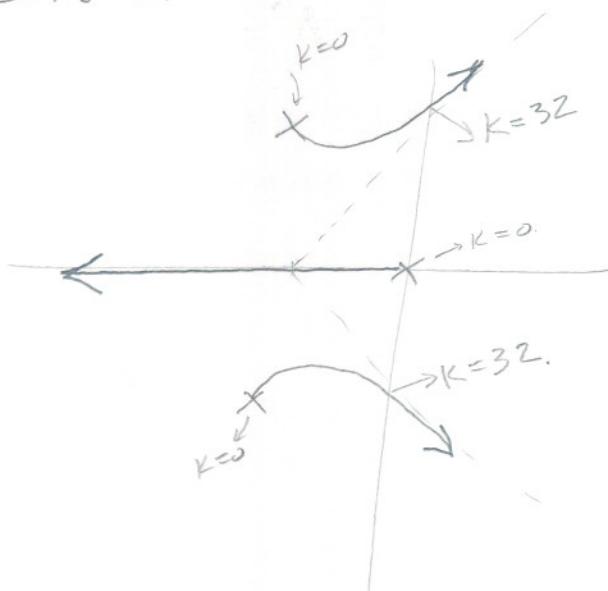


$$\text{so } 153.4^\circ + 90^\circ + \alpha = 180^\circ \pm n360^\circ$$

$\alpha = -63.4^\circ \pm n360^\circ$, hence angle of departure is



by symmetry, angle of departure from P_2 is found.
so loci so far looks like



Step 6) Find where loci cross imaginary axis.

char equation for closed Loop is

$$5(s^2 + 4s + 8) + K = 0$$

$$s^3 + 4s^2 + 8s + K = 0$$

let $s = j\omega$.

$$\text{so } -j\omega^3 - 4\omega^2 + 8j\omega + K = 0$$

$$\text{so } -\omega^3 + 8\omega = 0 \rightarrow \omega^2 = 8 \rightarrow \omega = \pm 2.828$$

so loci cross imaginary axis at $\boxed{\pm 2.828i}$

$$\text{and } -4\omega^2 + K = 0$$

$$\text{so } K = 4(2.828)^2 = -32$$

now need to answer final part

Put $K=2$. So need to find where on loci is $K=2$.

From loci diagram we see that system is stable

for $\boxed{0 < K < 32}$



when $K=2$

$$\text{Closed Loop TF} = \frac{K}{s(s^2 + 4s + 8) + K}$$

so char eq when $K=2$ is

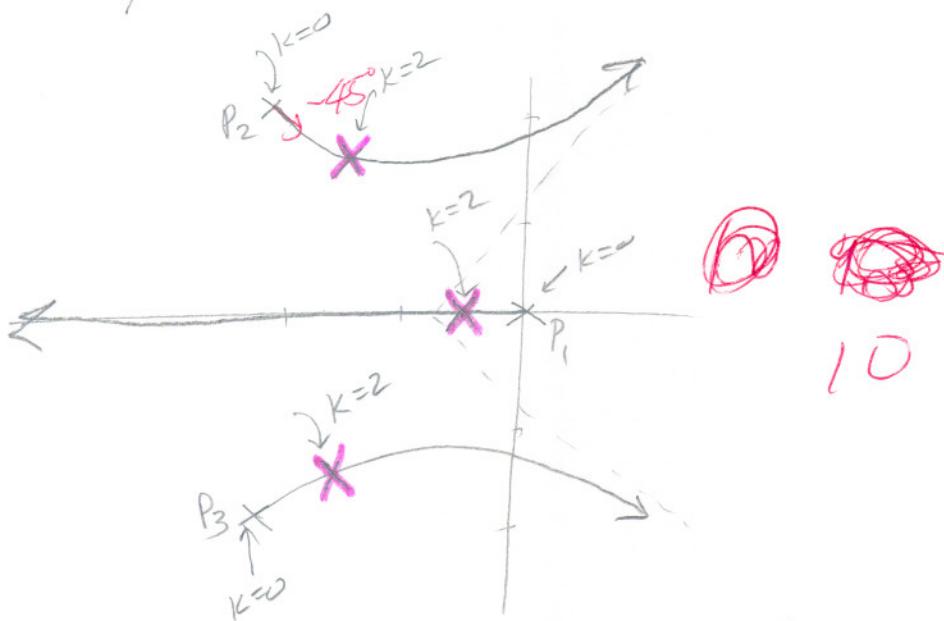
$$s(s^2 + 4s + 8) + 2 = 0$$

roots are $s = -0.288$

$$s = -1.855 + 1.866i$$

$$s = -1.855 - 1.866i$$

These poles are on loci as shown:

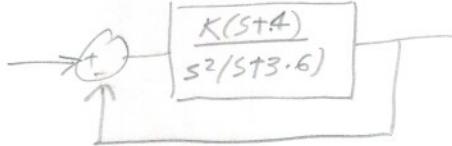


so when gain = 2, we see poles have moved to the right to new location, and pole P_1 have moved to left.

HW #6

problem A-6-6

sketch root loci for

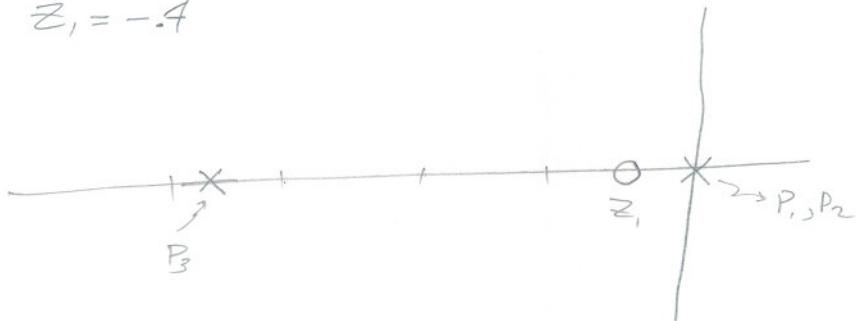


Solution

Step 1 Plot open loop poles and zeros.

$$P_1 = 0, P_2 = 0, P_3 = -3.6$$

$$Z_1 = -4$$



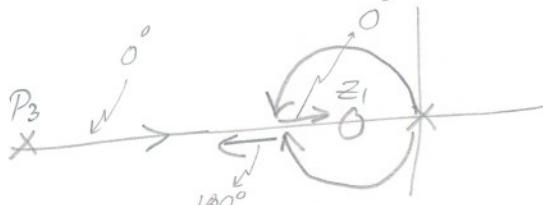
Step 2 Use angle conditions to find loci on real axis

$$\text{RHP: } 0 + 0 + 0 \neq 180^\circ$$

$$\text{footpoint between } P_1, Z_1: 180 + 180 + 0 \neq 180^\circ + n360^\circ$$

so line between P_1 and Z_1 not part of Loci

footpoint between Z_1, P_3 : since line $P_1 Z_1$ not on loci, how will loci reach Line $Z_1 P_3$ then? it must go up around Z_1 as follows:

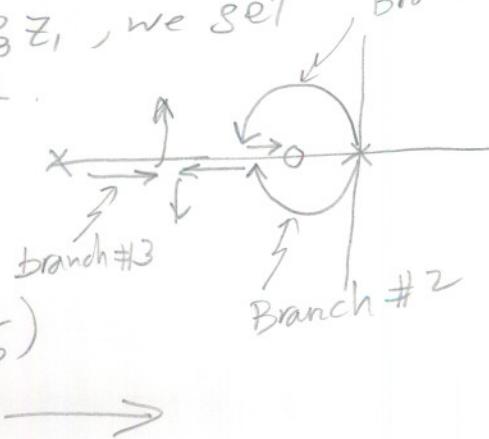


so need for a test point on line $P_3 Z_1$, we set Branch #1

$$\theta + 180^\circ + 0^\circ = 180^\circ \quad \text{OK}$$

so we have this loci so far

so now need to find breakaway point for branches 2,3 (σ_b)



Step 3 Find σ_b ; break away point for branches # 2,3.

Q. $\frac{ds}{dk} = 0$. from closed loop char eq.

$$G_L(s) = \frac{K(s+4)}{s^2(s+3.6) + K(s+4)}$$

$$\text{so } s^2(s+3.6) + K(s+4) = 0$$

$$s^3 + 3.6s^2 + K(s+4) = 0 \rightarrow K = -\frac{s^3 + 3.6s^2}{s+4}$$

$$\text{so } \frac{dK}{ds} = -\frac{(s^3 + 3.6s^2)}{(s+4)^2} + \frac{1}{(s+4)} (3s^2 + 7.2s)$$

$$\Rightarrow -(s^3 + 3.6s^2) + (s+4)(3s^2 + 7.2s) = 0$$

$$\text{i.e. } -s^3 - 3.6s^2 + 3s^3 + 7.2s^2 + 1.2s^2 + 2.88s = 0$$

$$\text{i.e. } 2s^3 + 4.8s^2 + 2.88s = 0$$

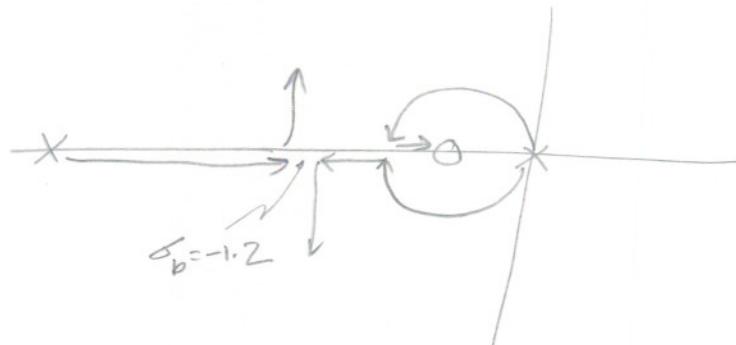
$$\text{or } s^2 + 2.4s + 1.44 = 0 \quad (s=0 \text{ is a solution})$$

$$(s+1.2)^2 = 0 \Rightarrow s = -1.2 \text{ (double root)}$$

double root means
 $\frac{d^2K}{ds^2} = 0$

so solutions are $s=0$ and $s=-1.2$

so $\sigma_b = -1.2$ since that is to left of zero where we are looking for break away, so loci looks like this so far



next find asymptotes



Step 7] find T_A and number of asymptotes.

$$\frac{\sum P_i - \sum Z_i}{3-1} = \frac{(0+0-3.6) - (-0.4)}{2} = \frac{-3.6+4}{2} = -1.6$$

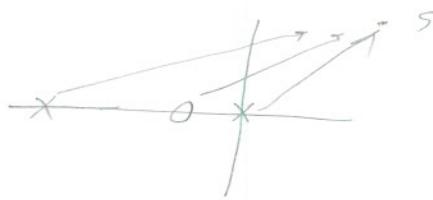
Note to TA: here I write $\sum P_i - \sum Z_i$ instead of ZodK

$\sum Z_i - \sum P_i$, so I do not have to negate poles/zeros values. It is the same answer, but this way is more clear to me.

for angles of asymptotes: put a test point 's' very far, we set

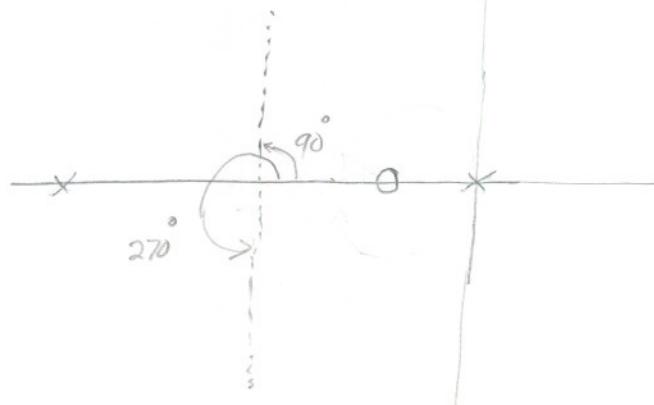
$$\underbrace{\theta_1 + \theta_2 + \theta_3}_{\substack{\nearrow \\ \text{from 3 poles}}} - \theta_4 = 180^\circ \pm n 360^\circ$$

\downarrow
from the zero



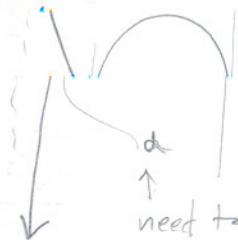
$$\therefore 2\theta = 180^\circ \pm n 360^\circ \Rightarrow \theta = 90^\circ \pm 180^\circ n$$

so angles are $[90^\circ, 270^\circ]$. so 2 asymptotes



next find angle of departures:

Step 2



need to find this angle.
how?

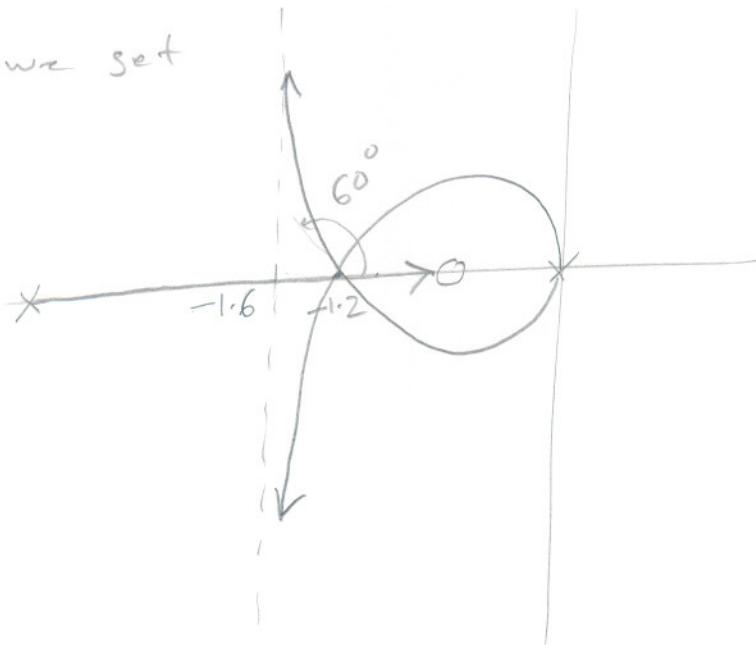
I am not sure how to find this angle.

according to book, since there are triple root of the characteristic equation there (at $s = -1.2$, the ζ_b point),

$$\text{then } \frac{180^\circ}{\# \text{ of roots at } \zeta_b} = \frac{180^\circ}{3} = 60^\circ.$$

(but I am not too clear on this point. Can you, please explain this to us more?)

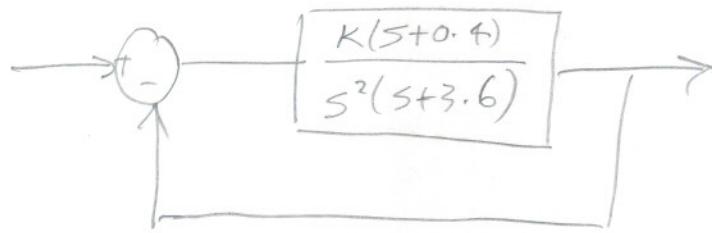
hence we get



Hw# 6

Problem A-6-7

ref to



Obtain equation for root Locus branches for the system. Show that loci branches cross real axis at breakaway point at angles $\pm 60^\circ$.

answer

since each point on loci must satisfy $\angle G(s) = 180^\circ \pm n360^\circ$

then let $s = \sigma + j\omega$, we set

$$\angle G(\sigma + j\omega) = 180^\circ \pm n360^\circ$$

$$\angle N(s) - \angle D(s) = 180^\circ \pm n360^\circ$$

$$\cancel{\angle K(\sigma + j\omega + 0.4)} - \cancel{\angle (\sigma + j\omega)^2(\sigma + j\omega + 3.6)} = 180^\circ \pm n360^\circ$$

Since K is (+ve constant), it has no effect on phase, so

$$\cancel{\angle \sigma + j\omega + 0.4} - \left[\cancel{\angle \sigma + j\omega} + \cancel{\angle \sigma + j\omega} + \cancel{\angle \sigma + j\omega + 3.6} \right] = 180^\circ \pm n360^\circ$$

$$\tan^{-1} \frac{\omega}{\sigma + 0.4} - \left[2 \tan^{-1} \left(\frac{\omega}{\sigma} \right) + \tan^{-1} \left(\frac{\omega}{\sigma + 3.6} \right) \right] = 180^\circ \pm n360^\circ$$

$$\tan^{-1} \frac{\omega}{\sigma + 0.4} - 2 \tan^{-1} \left(\frac{\omega}{\sigma} \right) - \tan^{-1} \left(\frac{\omega}{\sigma + 3.6} \right) = 180^\circ \pm n360^\circ$$

$$\text{Let } \tan^{-1} \frac{\omega}{\sqrt{4}} = A$$

$$\text{let } \tan^{-1} \frac{\omega}{\sigma} = B$$

$$\text{let } \tan^{-1} \frac{\omega}{\sqrt{3.6}} = C$$

so This can be written as

$$A - 2B - C = 180^\circ + n360$$

$$A - B = C + B + 180^\circ + n360$$

$$\text{Let } B + 180^\circ + n360 = D$$

$$\text{so } A - B = C + D$$

take tan of both sides

$$\tan(A-B) = \tan(C+D)$$

$$\text{but } \tan(A-B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}, \quad \tan(C+D) = \frac{\tan C - \tan D}{1 - \tan C \tan D}$$

hence

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\tan C - \tan D}{1 - \tan C \tan D} \quad \text{--- (1)}$$

$$\text{but } \tan D = \tan(B + 180^\circ) = \tan B \quad \because \tan(\alpha + 180^\circ) = \tan \alpha$$

so (1) becomes

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\tan C - \tan B}{1 - \tan C \tan B} \quad \text{--- (2)}$$

$$\text{but } \tan A = \frac{\omega}{\sqrt{4}}, \quad \tan B = \frac{\omega}{\sigma}, \quad \tan C = \frac{\omega}{\sqrt{3.6}}$$

so (2) becomes

$$\frac{\frac{\omega}{\sqrt{4}} + \frac{\omega}{\sigma}}{1 - \frac{\omega}{\sqrt{4}} \frac{\omega}{\sigma}} = \frac{\frac{\omega}{\sqrt{3.6}} - \frac{\omega}{\sigma}}{1 - \frac{\omega}{\sqrt{3.6}} \frac{\omega}{\sigma}} \quad \rightarrow$$

This Leads to

$$\omega(\sigma^3 + 2 \cdot 4\sigma^2 + 1.44\sigma + 1.6\omega^2 + \tau\omega^2) = 0$$

so $\omega = 0$ is a solution.

Another solution from

$$\omega^2(\sigma + 1.6) + \sigma(\sigma^2 + 2 \cdot 4\sigma + 1.44) = 0$$

using quadratic equation Leads to

$$\omega = (\sigma + 1.2) \sqrt{\frac{-\sigma}{\sigma + 1.6}}$$

and $\omega = -(\sigma + 1.2) \sqrt{\frac{-\sigma}{\sigma + 1.6}}$

so for $\omega = 0$, means imaginary part = 0 . i.e the real axis part of Loci .

in complex plan, equation of Loci is given by
(let me write $y = \omega$, $\sigma = x$ to make it more familiar)

$$y = \pm (x + 1.2) \sqrt{\frac{-x}{x + 1.6}}$$

Square both sides, we get

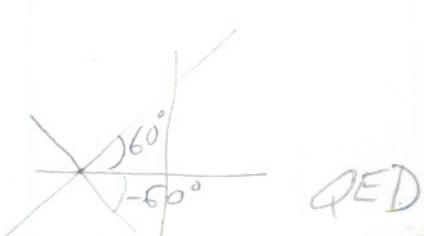
$$y^2 = (x + 1.2)^2 \left(\frac{-x}{x + 1.6} \right)$$

$$y^2 x + 1.6 y^2 = x^2 + 2.4x + 1.44$$

This is the equation of Loci in complex plan .

at $x = -1.2$, slope is $\left. \frac{dy}{dx} \right|_{x=-1.2}$

$$\text{but } \frac{dy}{dx} = \pm \sqrt{\frac{-x}{x+1.6}} \Rightarrow \left. \frac{dy}{dx} \right|_{x=-1.2} = \pm \sqrt{3}$$

i.e $\boxed{\text{slope} = \pm 60^\circ}$ so at breakaway we have  QED