University of California at Irvine
MAE106 Mechanical Systems Laboratory: Lecture 1

Part 1: Overview of the Class

If you work hard, you will leave this class with knowledge and practical experience in three interrelated areas:

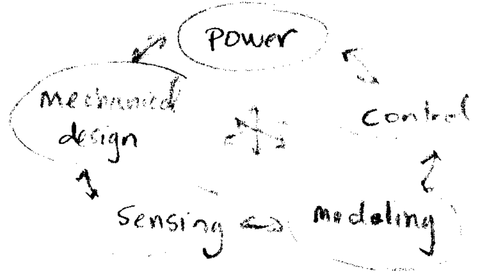
1. Physical intuition about how 1st and 2nd order linear, dynamical systems behave
    - You will be exposed to examples of common electrical, vibration, and robotic systems
    - Basic Idea: The dynamics of a wide variety of physical objects obey 1st and 2nd order linear, differential equations. These systems respond exponentially, sinusoidally, and exposoidally (OK, that's not a real word, but try to get the idea) in the time domain.
    - You will learn how to think about their behavior in both the time and frequency domains
2. Basic understanding of how feedback control works
    - Feedback is a common way to make cars, planes, robots, etc. respond like we want them too
    - You will learn about proportional feedback control (and derivative and integral control)
    - Basic idea: Measure error and try to reduce by changing the input to the controlled object
3. Familiarity with the components and tools for building mechatronic and robotic systems
    - Motors, potentiometers, tachometers, analog computational circuits (op-amps), electrical filters, power amplifiers, data acquisition systems, oscilloscopes, protoboards, ohmmeters

Part 2: Design Exercise

Final Project Competition: Build a robotic soccer player that can do two things:

1. kick a penalty kick
2. goal tend to block a penalty kick by another robot

Your robot will use a small motor. What questions do you need to know the answer to in order to build this robot?



Part 3: Review of Circuit Theory

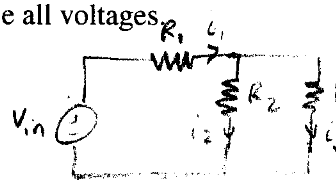
3.1 Linear circuit elements

Current: think of it as the flow of charge through a circuit element (such as a wire or resistor) Units: amps=coulombs/sec

Voltage: think of it as the electrical pressure that can cause charge carriers to flow

Current is always measured through something at a point; voltage is always measured between two points

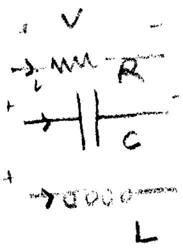
For this class, "ground" is an arbitrarily defined point on a circuit to which we reference all voltages.



Toolbox for circuit analysis

- Kirchoff's Current Law: Σ current in = Σ current out
- Kirchoff's Voltage Law: $\sum_{\text{Loop}}$ Voltage = 0 $\quad -V_i + V_1 + V_2 = 0$
- Power $P = Vi$ (stored or dissipated)
- Triad of linear circuit elements:

	i	$\frac{di}{dt}$
v	R	L
$\frac{dv}{dt}$	C	"Flux Capacitor



ohms (= $\frac{1}{\text{siemens}}$) (set gains & voltages)

farads (filters)

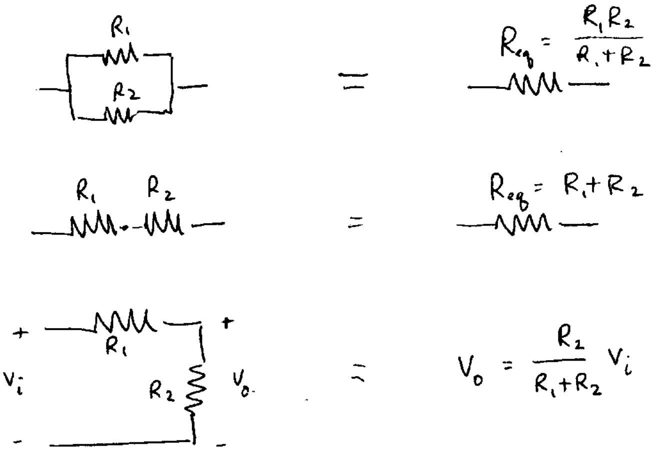
henrys (motors used to ignite spark plug in automobile engine)

$V = L\frac{di}{dt}$

Resistor Analysis Exercise:

Abstraction: the act of considering something as a general quality or characteristic apart from any concrete realities, specific object, or actual instance. It's the idea of a "black box"

ABSTRACTION, PATTERN RECOGNITION, + CIRCUIT ANALYSIS



$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$$

$$R_{eq} = R_1 + R_2$$

$$V_o = \frac{R_2}{R_1 + R_2} V_i$$

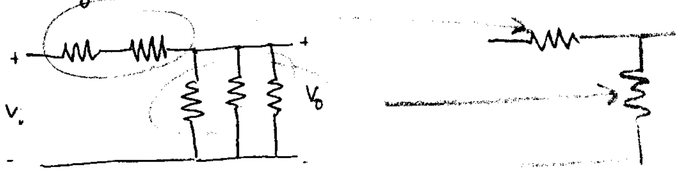
ALSO

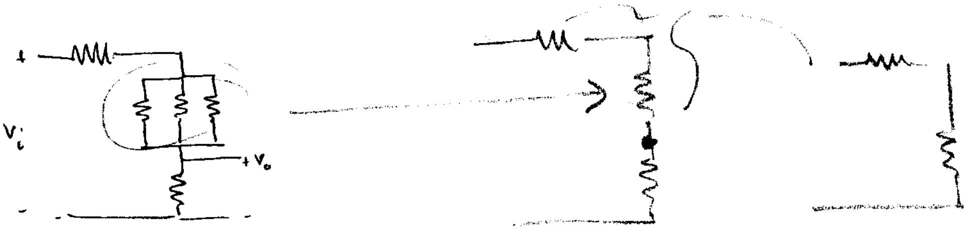
V = IR

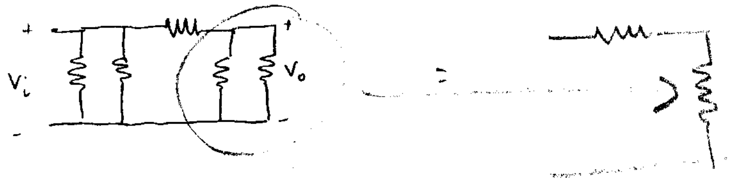
KCL

KVL

Using the above abstractions/rules, find V_0 for the following circuits:

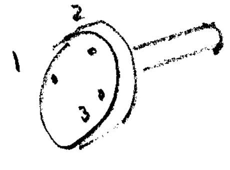


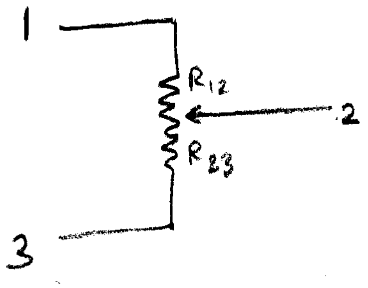




Potentiometers:

Typically used as voltage dividers. The two resistor values are changed by turning the pot.



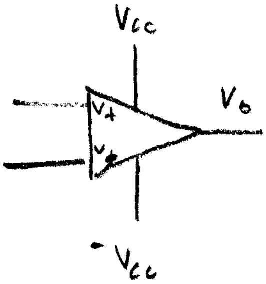


$R = R_{12} + R_{23} = [50 k\Omega, \text{ for example}]$

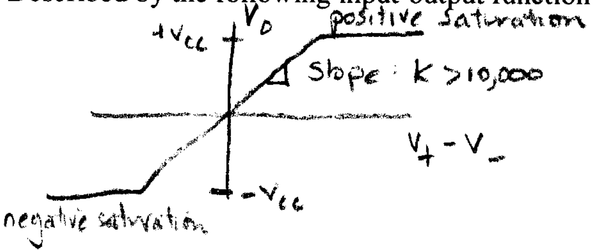
$$V_{out} = \frac{R_{23}}{R} V_{in}$$

3.2 Operational Amplifiers

- important building blocks for circuits; easy to use, cheap
- used to build filters, amplifiers, feedback controllers, computational circuits
- the "brains" in the analog control circuits that you will build for the class
- What are they? High gain, differential, linear voltage amplifiers
- Made of > 20 transistors plus resistors and capacitors
- Two input terminals, one output, two power supply lines (five pins total)
- Typically operate over a wide range of supply voltages
- By design, they have a high input resistance and a low output resistance



Described by the following input-output function:



in linear region

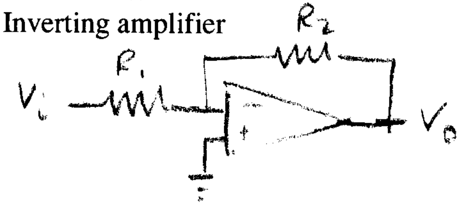
$V_0 = K(V_+ - V_-)$

Golden Rules of Op-amp Circuit design:

1. Input currents are zero (op amps are designed to have a high input resistance)
2. Input voltages are equal (If operating in linear region, and connected with negative feedback)

Four useful circuits:

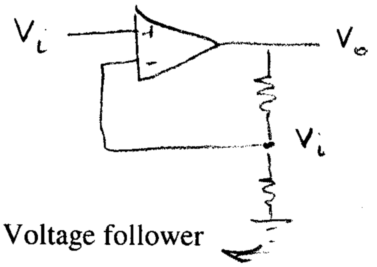
1. Inverting amplifier



What is V_o as a function of V_i?

KCL: $\frac{V_i}{R_1} + \frac{V_o}{R_2} = 0$ $V_o = \frac{-R_2}{R_1} V_i$

2. Non-inverting amplifier

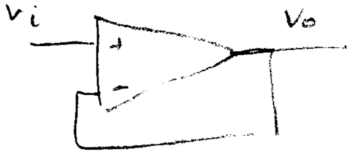


$\frac{V_o - V_i}{R_1} = \frac{V_i}{R_2} \Rightarrow V_o = \frac{R_1}{R_2} V_i + V_i$

$V_o = \left(\frac{R_1 + R_2}{R_2}\right) V_i$

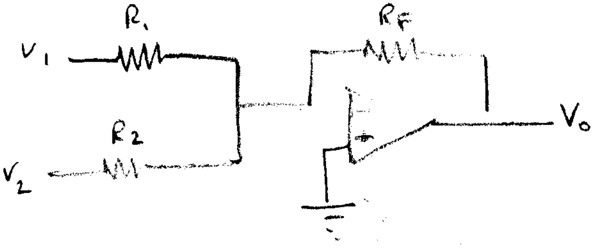
as $R_2 \rightarrow \infty$ what does this circuit do?

3. Voltage follower



$V_o = V_i$ Why? - high input impedance lets us connect circuit modules without altering their performance

4. Analog addition (subtraction also possible)



$\frac{V_1}{R_1} + \frac{V_2}{R_2} + \frac{V_o}{R_F} = 0$

$V_o = -R_F\left(\frac{1}{R_1} V_1 + \frac{1}{R_2} V_2\right)$

If $R_1 = R_2 = R$

$V_o = \frac{-R_F}{R}(V_1 + V_2)$

Note: feedback is always to V_- (negative feedback)
$|V_o| < |V_{cc}|$ and $i_{out} < i_{max}$ else op-amp saturates

3.3 Controlling power needed for devices like motors, light bulbs, etc.

Often we want to control a device that requires a lot of power (e.g. a motor) with signals that have very low power (e.g. an op amp or computer).

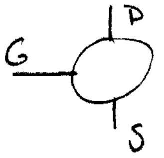
Small DC brushed motor: $V = 10$ V, $R = 2\ \Omega$ $i = \frac{V}{R} = 5$ amps

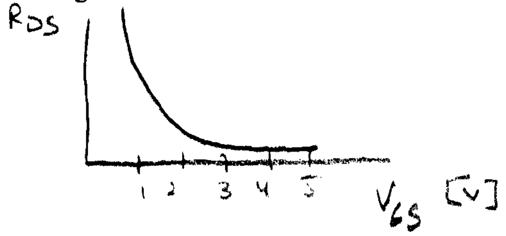
Typical Op-amp: $V = \pm 15$ V, $i_{max} < 20$ mamp

Solutions?

1. Power op-amp
2. Power transistor (e.g power MOSFET – simple and cheap)

Can think of a MOSFET as a voltage controlled resistor that can take a lot of current





Notes: Input resistance is very high (therefore effectively no current goes into gate)

Low-power MOSFETS are the "switches" used in computers (what is a switch?)

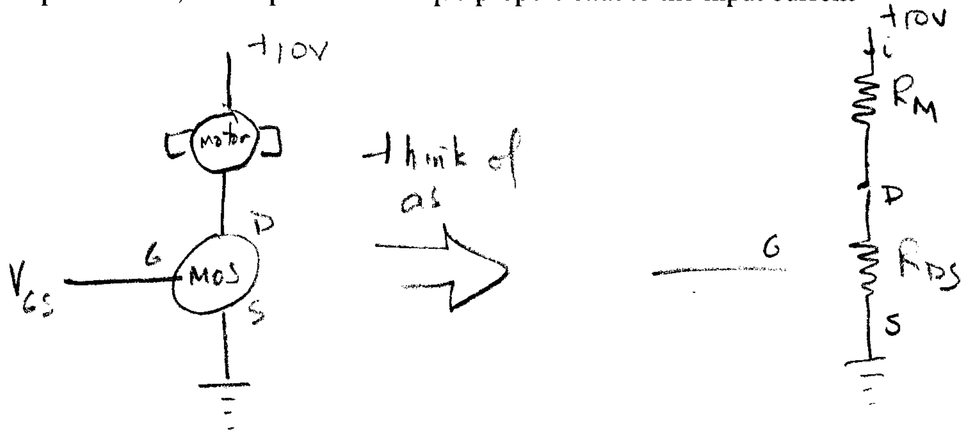
MOSFETS are very sensitive to static electricity – use a grounding strap when you handle them in lab

Example: use a power transistor to control a motor with a low-power computer output

Hints about motors:

A DC brushed motor spins at a speed proportional to the input voltage, if it is just turning an inertial load.

If you stop the motor, it will produce a torque proportional to the input current



If R_{DS} is big, no current flows

If R_{DS} is small (because V_{GS} is small) current flows

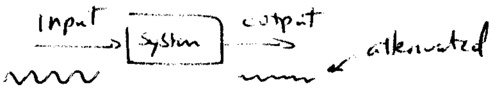
Thus, by controlling $\underbrace{V_{GS}}_{\text{low power, no current}}$, we can make the motor turn or not turn

MAE106 Mechanical Systems Laboratory: Time and Frequency Domain Notes

1. Why do engineers analyze systems in both the time and frequency domain?

Why the time domain? We live in the time domain.

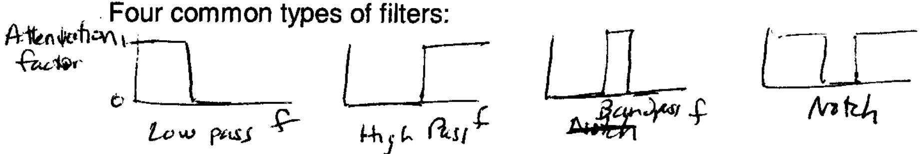
Typical questions: How does the system respond to a step input? Example: 0-60 mph
How does the system respond to a impulse input? Example: bump suspension
How fast does the system respond? (Useful #: time constant)
Does it overshoot ?
Does it oscillate ?



Why the frequency domain?

a. Intuition

Systems act like filters, responding differently to inputs at different frequencies
Four common types of filters:



b. Ease – sometimes its easier to solve differential equations in the frequency domain (Laplace Transform)

2. What is a transfer function and what is a frequency response?

A linear differential equation in the time domain becomes a transfer function in the freq. domain.

To see this take the Laplace transform of a differential equation: Recall $\mathcal{L}\left(\frac{dx}{dt}\right) = s\,\mathcal{L}(x(t))$

$\mathcal{L}(x_1+x_2) = \mathcal{L}(x_1) + \mathcal{L}(x_2)$

$\frac{dx}{dt} + ax = u$ Assumes initial conditions are zero

$sX(s) + aX(s) = U(s) \quad X(s) = \frac{1}{s+a}U(s) \quad \xrightarrow[x]{}$

$= H(s)\,U(s)$

FACT: The transfer function tells how a system responds to any input in the frequency domain. The output is just the input multiplied by the transfer function. $u(s) \rightarrow \boxed{H(s)} \xrightarrow{x(s) = H(s)u(s)}$

The transfer function also tells how a system responds to a sinusoidal input.

FACT: Using Laplace Transforms, it is possible to prove that: sine wave in ⇒ sine wave out ~~scaled~~ ~~shifted~~

The transfer function tells how much an input sine wave is scaled and shifted as a function of its frequency.

$\sin(\omega t) \rightarrow \boxed{G(s)} \rightarrow |G(j\omega)|\sin(\omega t + \phi_{G(j\omega)})$

Note $G(j\omega)$ is a complex variable, so it has a magnitude + phase

Knowing these two things means you know the frequency response of the system, which is characterized by the Magnitude and the phase responses

These facts are very useful when combined with two other facts:

FACT: Any signal can be represented as the sum of sinusoids. Fourier analysis

FACT: The response of a linear system to the sum of two inputs is the sum of the individual outputs.

THESE FACTS LET US THINK OF LINEAR SYSTEMS AS FILTERS.

$x_1 \rightarrow y_1$

$x_2 \rightarrow y_2$

$x_1 + x_2 \rightarrow y_1 + y_2$

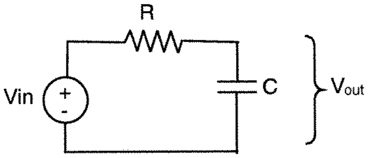
Linear System

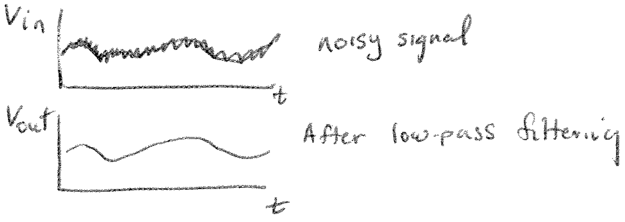
any signal $\rightarrow \Sigma$ sines $\rightarrow \boxed{G(s)} \rightarrow \Sigma$ scaled shifted sines $\rightarrow$ output signal

Mechanical Systems Laboratory: Lecture 3

Analysis of a 1st-order, Low-Pass Filter Circuit in the Time and Frequency Domains

The following circuit is a low-pass filter. It is useful to clean up signals with high frequency noise on it:





1. Time Domain Analysis

Let's analyze the response of this circuit to a step input

We'll use the method of undetermined coefficients to solve the differential equation. You can remember this very useful technique for linear, ordinary, differential equations using the following mnemonic:

1. Generals: set the forcing function = 0 and find the general solution to homogenous equation (don't evaluate it's coefficient yet)
2. are Particular: find the particular solution (assume particular soln is same form as forcing function)
3. about Initial Conditions: sum the homogenous and particular solutions and solve for the coefficient to the homogenous equation that satisfies the initial conditions.

KVL: $-V_i + iR + V_o = 0$

$i = C\frac{dV_o}{dt}$

$RC\frac{dV_o}{dt} + V_o = V_i$

$* \quad \frac{dV_o}{dt} = -\frac{1}{RC}V_o + \frac{1}{RC}V_i$

Assume: $V_o(0)=0$, $V_i = \begin{cases} 0 & t<0 \\ \text{constant} & t>0 \end{cases}$

Homog. $V_h = Ae^{-t/\tau}$ $\quad \tau = RC$

Part: $V_p = V_i$ (assume V_i is a constant)

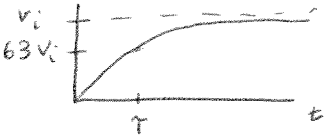
Total: $V_o = Ae^{-t/\tau} + V_i$ but $V(0) = 0 = A + V_i$

$\Rightarrow A = -V_i$

$V_o = V_i(1 - e^{-t/\tau})$

at $t = \tau$

$V_o = V_i(1-e^{-1}) = .63V_i$



Summary of important concepts:

- Method of undetermined coefficients for solving a differential equation.
- Time constant: a 1st order system has gone 63% of the way to its final value after one time constant – standard engineering technique for quantifying "how fast" a system responds.

2. Frequency Domain Analysis

Let's analyze how this system responds to a sinusoidal input. Remember: sine in ⇒ sine out (scaled and shifted), for a linear system. We will use three methods to find the scaling and shifting.

Method 1. Solve differential equation using method of undetermined coefficients (difficult) Assume $V_i = \sin\omega t$

Homogenous solution: $V_h = Ae^{-t/\tau}$ Note: as $t \to \infty$ $V_h \to 0$ ("Transient")

Particular solution: try $V_o = a\sin(\omega t + \phi)$ where a + ϕ are unknown

subst into $*$ = $a\omega\cos(\omega t+\phi) = -\frac{a}{RC}\sin(\omega t+\phi) + \frac{1}{RC}\sin\omega t$

Useful trig. identity: $A\cos(\theta) + B\sin(\theta) = \sqrt{A^2+B^2}\sin(\theta + \tan^{-1}(\frac{A}{B}))$

$a\omega\cos(\omega t+\phi) + \frac{a}{RC}\sin(\omega t+\phi) = \frac{1}{RC}\sin\omega t = \sqrt{(a\omega)^2 + (\frac{a}{RC})^2}\sin(\omega t + \phi + \tan^{-1}\omega RC)$

for the right-most equality to hold

$\frac{1}{RC} = \sqrt{(a\omega)^2 + (\frac{a}{RC})^2} \Rightarrow a = \frac{1}{\sqrt{1+(\omega RC)^2}}$ scaling

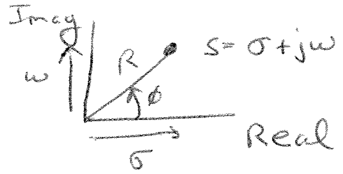
$\phi = -\tan^{-1}\omega RC$ phase shift

$V_o = a\sin(\omega t + \phi)$ with

Method 2: Take Laplace Transform of differential equation that describes circuit, find the transfer function, and solve for frequency response (easier than Method 1)

Brief review of complex variables:
Complex variables keep track of two pieces of information, real and imaginary part, or magnitude and phase

Can think of complex variables as a point in the complex plane.



Can write point in Cartesian or polar coordinates. $S = \sigma + j\omega$

$S = Re^{j\phi}$

To find the magnitude in Cartesian form:

$|S| = \sqrt{\sigma^2 + \omega^2} = R$

To find the phase in Cartesian form:

$\phi_S = \tan^{-1}\frac{\omega}{\sigma} = \phi$

Magnitude of two complex variables divided by each other: $\left|\frac{R_1 e^{j\phi_1}}{R_2 e^{j\phi_2}}\right| = \frac{|R_1|}{|R_2|}$

Phase of two complex variables divided by each other:

$= \frac{R_1}{R_2} e^{j(\phi_1 - \phi_2)}$

$\phi = \phi_1 - \phi_2$

$S = \sigma + j\omega$

$= R\cos\phi + jR\sin\phi$

$= Re^{j\phi}$

Euler's law $e^{j\phi} = \cos\phi + j\sin\phi$

(can derive by Taylor's expansion)

Now, find the transfer function and frequency response:

$\frac{dV_o}{dt} = \frac{-1}{RC}V_o + \frac{1}{RC}V_i \qquad sV_o = \frac{1}{RC}V_o + \frac{1}{RC}V_i$

$\frac{V_o}{V_i} = \frac{\frac{1}{RC}}{s + \frac{1}{RC}} = \frac{1}{1 + RCs} = G(s)$

$|G(s)|\Big|_{s=j\omega} = \frac{|1|}{|1 + RCj\omega|} = \frac{1}{\sqrt{1 + (RC\omega)^2}}$

$\measuredangle\phi(s)\Big|_{s=j\omega} = \tan^{-1}\frac{0}{1} - \tan^{-1}\frac{RC\omega}{1} = -\tan^{-1} RC\omega$

Method 3: Use "impedances" to find transfer function (easiest)

Circuit element	Time domain	Frequency domain	Impedance
Resistor	V(t) = RI(t)	V(s) = R I(s)	R
Capacitor	$V(t) = \frac{1}{C}\int i(t)$	$V(s) = \frac{1}{sC} I(s)$	$\frac{1}{sC}$
Inductor	V(t) = L di/dt	$V(s) = sL\, I(s)$	sL

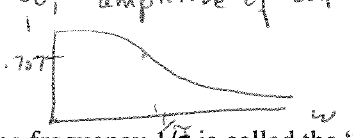
Note: All the usual circuit rules till hold in the frequency domain because of superposition (KVL, KCL, Op amp rules, voltage divider…). So, treat impedances like (frequency dependent) resistors in finding a circuit's transfer function.

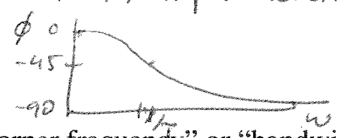
$V_o = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} V_i = \frac{1}{1 + RCs} V_i \qquad \frac{V_o}{V_i} = G(s) = \frac{1}{1 + RCs}$

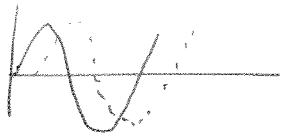
What do the magnitude response (i.e. scaling or attenuation factor) and phase shift response actually look like? Fill in the following chart:

	Magnitude or Scaling	Phase
Small ω	1	0
ω = 1/RC = 1/τ	$\sqrt{\frac{1}{2}} = .707$	-45°
ω ⇒ infinity	0	-90°

So, amplitude of output is .707 x input when $\omega = \frac{1}{\tau}$







The frequency 1/τ is called the "corner frequency" or "bandwidth" of the system. For this low-pass filter, input sinusoids with a frequency higher than the bandwidth are "filtered" or "attenuated".

Summary of important concepts:

- How to find a transfer function and the frequency response
- Impedances
- Corner frequency

Mechanical Systems Laboratory: Lecture 4
How DC Brushed Motors Work (Another example of a first-order system)

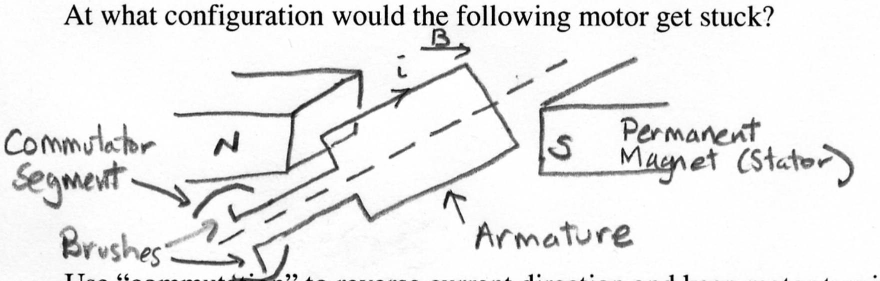
1. Introduction to DC Brushed Motors

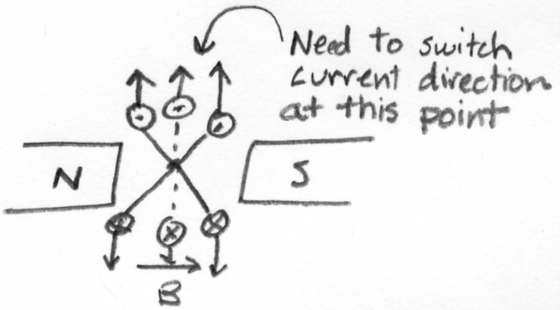
- very common for small jobs (toys, some appliances, robots)
- invented by Michael Faraday in the 1850's
- Operating principle:
  - apply voltage, motor spins
  - Polarity of voltage determines motor direction
  - Amplitude of voltage determines motor speed
- Other motor types: AC motors (washing machine), DC brushless motors, DC stepper motors

2. Physics of Operation

a. Makes use of Lorentz Force Law: $\vec{F} = i\vec{\ell} \times \vec{B}$
where F = force, l = unit vector in direction of current flow, B = magnetic flux, i = current into motor
i.e. current-carrying conductors placed in magnetic fields create forces

At what configuration would the following motor get stuck?





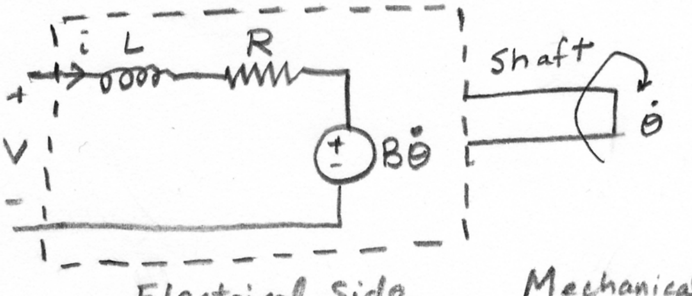
Use "commutation" to reverse current direction and keep motor turning
Adding enough commutator segments gives: $\tau = Bi$, where B = "torque constant"

b. Back EMF

- back EMF" (electromotive force or 'voltage')
- the voltage produced by motor as a result of its speed
- voltage is proportional to speed $V = B\dot{\theta}$
- physical basis: armature windings are an inductor
- as motor spins, get di/dt in armature
- $V = L di/dt \propto$ angular velocity
- Can use a motor as a velocity sensor (i.e. a "tachometer") by measuring voltage across terminals
- This is also the principle used by generators.
- Real tachometers have many armature coils to reduce voltage ripple

3. Mathematical Model of a DC Brushed Motor

A motor has a resistance and inductance associated with its coils.



Electrical Side
$V = L\frac{di}{dt} + Ri + B\dot{\theta}$

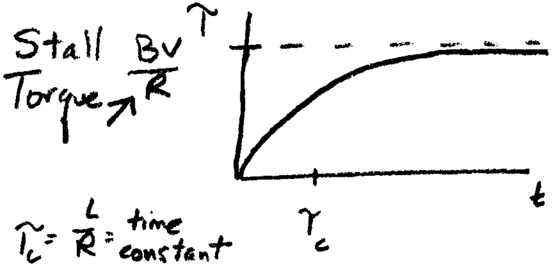
Mechanical Side
$\tau = Bi$

To see how the model predicts the motor behavior, consider two cases:

Case 1) Hold shaft fixed, apply constant voltage. What is the motor torque as a function of time?

Assume $i(0)=0$, shaft fixed $\Rightarrow \dot{\theta}=0$

$$\underbrace{V = L\frac{di}{dt} + Ri + \cancel{B\dot{\theta}}^{\,\to 0}}_{\text{Solve 1st order DEQ}}$$



1. General soln: $\frac{di}{dt} = \frac{-R}{L}i \Rightarrow i = Ke^{-t/\tau_c}$ $\quad \tau_c = \frac{L}{R}$ = time constant

2. Part soln: Assume i = constant $\equiv A$

$L\frac{dA}{dt} + RA = V \Rightarrow A = \frac{V}{R}$ $\quad$ 3. Total: $i = \frac{V}{R} + Ke^{-t/\tau_c}$

Find $K \Rightarrow i(0)=0 = \frac{V}{R} + K \Rightarrow K = -\frac{V}{R}$

Soln:

$i = \frac{V}{R}(1-e^{-t/\tau_c})$

Torque $= \tau = Bi$

$\tau = \frac{BV}{R}(1-e^{-t/\tau_c})$

Observations:

The stall torque = the torque you feel if you hold the motor shaft fixed
It takes time for a motor to develop torque (describable with a time constant)
After the transient response, the motor acts like a resistor

Case 2) Allow shaft to spin freely, apply constant voltage. What is the motor speed as a function of time?

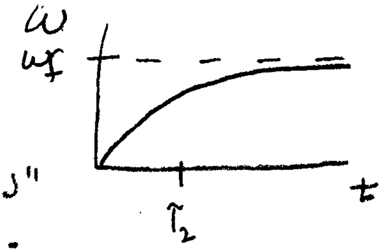
Assume shaft has inertia $\tau = J\ddot{\theta}$, assume $\frac{di}{dt} \approx 0$ (current reaches steady state)

$V = L\frac{di}{dt} + Ri + B\dot{\theta} \Rightarrow V = Ri + B\dot{\theta}$ $\quad$ recall $\tau = Bi \Rightarrow i = \frac{\tau}{B} = \frac{J\ddot{\theta}}{B}$

$\frac{RJ}{B}\ddot{\theta} + B\dot{\theta} = V$ $\quad$ let $\omega = \dot{\theta}$

$\frac{RJ}{B}\dot{\omega} + B\omega = V$ $\quad$ SOLUTION: $\omega = \frac{V}{B}(1-e^{-t/\tau_2})$ $\quad \tau_2 = \frac{RJ}{B^2}$

as $t \to \infty$, $\omega \to \omega_f = \frac{V}{B}$ "No Load Speed"



Observations:

No load speed is independent of inertia and proportional to voltage
Time constant of speed increase depends on inertia
Motor requires no power at no load speed (actually does because of friction)

at $\omega = \dot{\theta} = \omega_f$

$V = Ri + B\dot{\theta}$

$V = Ri + B\frac{V}{B} \Rightarrow i = 0$

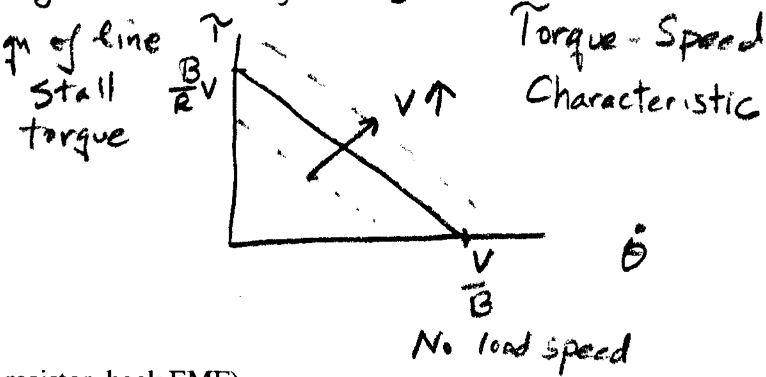
Summary: Torque-speed curve for a DC brushed motor

For V = constant (& $\frac{di}{dt}=0$) you get $V = Ri + B\dot{\theta}$, $i = \frac{\tau}{B}$

$V = \frac{R}{B}\tau + B\dot{\theta} \rightarrow$ can be written in eqn of line

$\tau = -\frac{B^2}{R}\dot{\theta} + \frac{B}{R}V$

$y = mx + b$



Important Ideas:

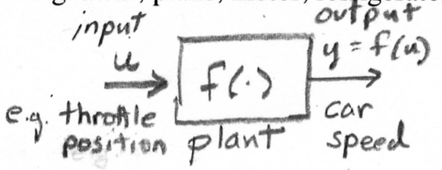
- Lorentz force law
- Commutation
- Back EMF
- Mathematical model of motor (inductor, resistor, back EMF)
- Exponential increase in torque if shaft is fixed; in speed if shaft is free to spin
- Torque-speed curve (no-load speed, stall torque)

Mechanical Systems Laboratory: Lecture 5
Basic Control Concepts; Example of Feedback Control of Motor Velocity

1. Basic Control Concepts

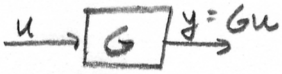
a. The problem of automatic control

Given a system with inputs and outputs (the "plant" – e.g. a car, plane, motor, refrigerator, …)
And a desired output y_d
Find an input u to give you the desired output y_d

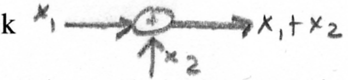


b. Block diagrams
Useful notation for visualizing control systems.
Two common blocks:

1) Gain block



2) Summer block

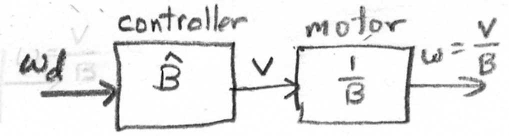


c. Two general approaches to designing the input "u"

Approach 1: Feedforward Control (or "Open Loop" Control)

Choose input to plant based on knowledge of plant (i.e. based on an "inverse model" of the plant).

Example: Control a motor's steady state velocity ω using voltage v as the input. What should v be such that $\omega = \omega_d$?



$V = \hat{B} w_d$

$w = \frac{V}{B} = \frac{\hat{B}}{B} w_d$

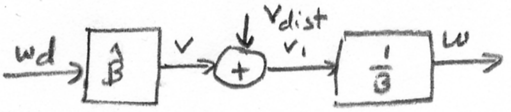
If $\hat{B} = B$, $w = w_d$

$\hat{\ } =$ an estimate

Shortcoming 1 of Feedforward Control: Need to have an accurate model of the plant

If $\hat{B} = .5B$, then $w = .5w_d$

Shortcoming 2 of Feedforward Control: Most systems have unpredicted "disturbances" that affect the output



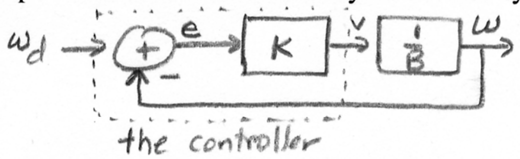
$w = \frac{V_1}{B} = \frac{V + V_{dist}}{B}$

$w = \frac{\hat{B}}{B} w_d + \frac{1}{B} V_{dist}$

If V_{dist} is big, $w \neq w_d$

Approach 2: Feedback Control (or "Closed Loop" Control)

Refers to measuring a system's output and "feeding it back" to change the input so $y = y_d$.
Also called "closed-loop control" because you "close the control loop" by "feeding back" the sensed output.
Usually, you subtract the desired output from the actual output to get an error signal, then apply an input to the system proportional to the error in the direction that reduces the error (negative feedback).
Example: Control a motor's steady state velocity ω using voltage v as the input.



$e = w_d - w$

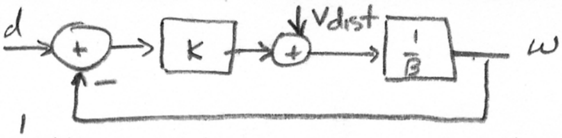
$V = Ke = K(w_d - w)$

$w = \frac{1}{B} V = \frac{K}{B}(w_d - w)$

$w\left(1 + \frac{K}{B}\right) = \frac{K}{B} w_d$

$\boxed{w = \frac{K}{B+K} w_d}$

For feedback control, you don't need an accurate model of the plant (you just need to know which way to "push" the plant to reduce error). Feedback control can also handle disturbances because it senses their effects on the output. If $K >> B$, then $w = w_d$



solve: $w = \frac{K}{K+B} w_d + \frac{1}{K+B} V_{dist}$ if $K >> B$, $w \simeq w_d$

What are two limitations of feedback control?

1) an error has to develop before controller acts
2) delay causes instability
3) requires a sensor

Many control systems use feedforward and feedback control together.

2. Example: Feedback Control of Motor Velocity (Lab 3)

In lab, you will build an op-amp circuit for controlling velocity of a motor using proportional feedback. To place this lab in a context, imagine that you are designing a control system for “Robbie the Rescue Robot”. The amplifiers that you use in lab give an output current (and thus motor torque) that is proportional to input voltage (“current amplifier”). So, a model of the motor and amplifier together is:

$$V \rightarrow \boxed{\frac{\alpha}{Js}} \xrightarrow{\omega}$$

$$\left.\begin{array}{l} \tau = J\dot{\omega} \text{ (inertial load)} \\ \tau = \underset{\substack{\uparrow \\ \text{voltage to amplifier}}}{\alpha V} \text{ (current amplifier)} \end{array}\right)$$

$$J\dot{\omega} = \alpha V$$

$$Js W = \alpha V$$

$$\frac{\omega}{V} = \frac{\alpha}{Js} = G(s)$$

Problem 1: Design a feedback control law relating V, ω, and ω_d, and verify that it works.

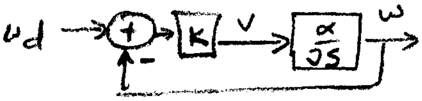
$$V = -K(\omega - \omega_d)$$

Intuitively: If motor turns too fast ($\omega > \omega_d$), apply a negative torque to slow it down. If motor turns too slowly ($\omega < \omega_d$), apply a positive torque to speed it up. Called a “proportional” or “P” control system since motor torque is proportional to error.

Problem 1a: Write the differential equation that describes the behavior of the controlled system

$$V = -K(\omega - \omega_d) = \frac{J}{\alpha}\dot{\omega}$$

Problem 1b: Draw a block diagram of the controlled system



Problem 1c: Find the transfer function for this system

$$\frac{J}{\alpha}s\,\omega + K\omega = K\omega_d \qquad \omega\left(K + \frac{Js}{\alpha}\right) = K\omega_d \qquad \frac{\omega}{\omega_d} = \frac{K}{K + \frac{J}{\alpha}s} = G(s) = \frac{1}{1 + \tau_1 s}$$

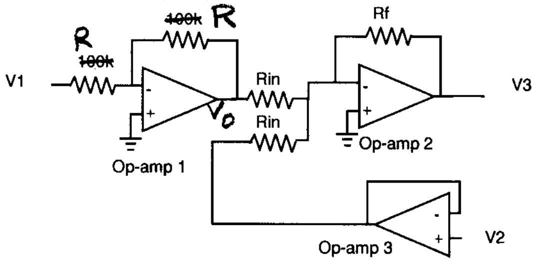
$$\tau_1 = \frac{J}{K\alpha}$$

Problem 1d: Discus the system’s behavior in the time and frequency domains

time domain → 1st order system $\omega = \omega_d(1 - e^{-t/\tau})$

frequency domain → 1st order low pass filter

Problem 1e: You will use the following circuit to implement P-control of the motor. What hardware would you connect to v1, v2, and v3?



$$\frac{V1}{R} = \frac{-V_0}{R} \Rightarrow V_0 = -V1$$

$$\frac{V_0}{R_{in}} + \frac{V_2}{R_{in}} = \frac{-V_3}{R_f} \Rightarrow \underbrace{V_3}_{\substack{\text{input to}\\ \text{amplifier}}} = -\underbrace{\frac{R_f}{R_{in}}}_{\substack{\text{feedback}\\ \text{gain}}}(\underset{\substack{\uparrow \\ \omega \\ \text{tach}}}{V_2} - \underset{\substack{\uparrow \\ \omega_d \\ \text{signal generator}}}{V_1})$$

Important Ideas: feedforward and feedback (basic idea and limitations), how to implement feedback control

Mechanical Systems Laboratory: Lecture 6
Integral Control; Introduction to Second Order Systems

1. Integral Control

In lab this week you are building an op-amp circuit for controlling velocity of a motor using proportional feedback. To place this lab in a context, we imagined in the last lecture that you are designing a velocity control system for "Robbie the Rescue Robot". We created a Proportional-type controller for Robbie, and found that the controlled system dynamics were as follows:

$$\underbrace{J\dot{\omega}}_{\text{dynamics}} = \underbrace{\tau = \alpha V}_{\text{current amplifier/motor}} \qquad \underset{\text{FB controller}}{V = -K(\omega - \omega_d)} \Longrightarrow \frac{J}{\alpha}\dot{\omega} + K\omega = K\omega_d$$

where v= voltage input to current amplifier that powers Robbies motors, ω = actual angular velocity of wheels, sensed with a tachometer, ω_d = desired angular velocity, K = proportional feedback gain, alpha = proportionality constant relating v (i.e. current amplifier input) to torque output from motor.

Note that the steady-state error for this system is zero:

$$\dot{\omega} = 0 \text{ (steady state)} \rightarrow \omega = \omega_d$$

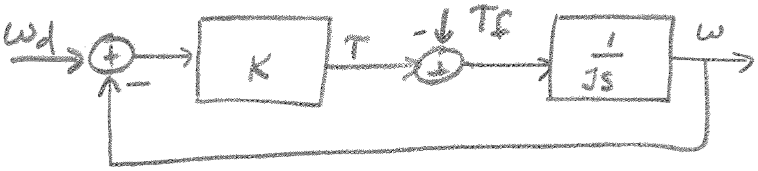
A more realistic model of Robbie's dynamics would include some friction in Robbie's wheels:

$$J\dot{\omega} = \tau - \tau_f \qquad \text{Assume } \tau_f = \text{constant (stiction)}$$

Let's assume that we control the torque to the motor directly, and express our control law in terms of torque. (Note, we actually control the current into the motor, but this is proportional to torque).

$$\tau = -K(\omega - \omega_d)$$

We can represent the combined system using a block diagram showing friction as disturbance.



Problem: Show that there is a steady-state error in velocity due to the friction.

$$\tau = -Ke \qquad e = \omega - \omega_d$$

$$J\dot{\omega} = -Ke - \tau_f \qquad \text{In steady state } \dot{\omega} = 0 \Rightarrow e = \frac{-\tau_f}{K}$$

KEY IDEA: We can get rid of this steady-state error by using a proportional plus integral (PI) controller:

$$\tau = -k_p e - K_I \int e\,dt \qquad \rightarrow J\ddot{e} + K_p\dot{e} + K_I e = 0 \qquad e = \omega - \omega_d$$

$$\text{steady-state: } \dot{e} = \ddot{e} = 0 \qquad \dot{e} = \dot{\omega}$$

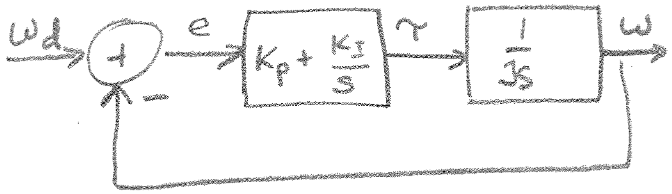
$$J\dot{\omega} = \tau - \tau_f = -K_p e - K_I \int e\,dt - \tau_f \qquad \Rightarrow e = 0 \qquad \ddot{e} = \ddot{\omega}$$

$$\frac{d}{dt} \quad J\ddot{\omega} = -K_p\dot{e} - K_I e \qquad \text{Not } \ddot{\omega} = \ddot{e} \text{ if } \omega_d = \text{constant}$$

How does I control work? (try to explain it to your neighbor in words).

Integral control works in the following way:
If error e(t) does not equal zero, then $\int e(t)dt$ increases with time, and eventually the torque (which is proportional to this integral) becomes high enough to overcome friction.

The block diagram for a P-I compensator is:



What is the transfer function for this system?

$$e = \omega_d - \omega$$

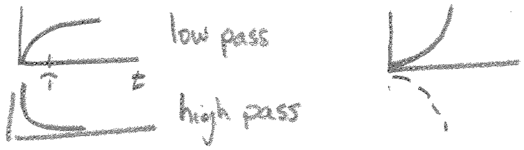
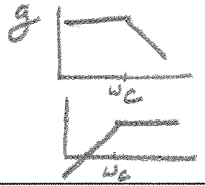
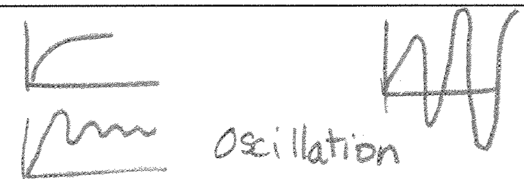
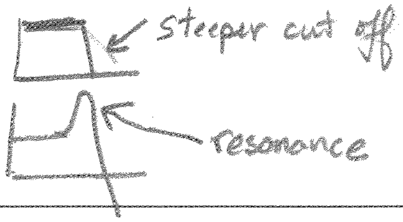
$$\tau = \left(K_p + \frac{K_I}{s}\right) e$$

$$\omega = \frac{1}{Js}\tau = \frac{1}{Js}\left(K_p + \frac{K_I}{s}\right)(\omega_d - \omega)$$

Solve for ω

$$\omega = \frac{K_p s + K_I}{Js^2 + K_p s + K_I}\,\omega_d$$

This is an example of second order system, which behaves differently than a first order system.

	Typical behaviors in time domain (step response)	Typical behaviors in frequency domain
First order system	Stable Unstable  low pass high pass	 low pass high pass
Second order system	 Oscillation	 steeper cut off resonance

Important Ideas: integral control can help remove steady state error. However, I-control adds dynamics to the system, which can lead to non-1st-order phenomena such as oscillation and resonance.

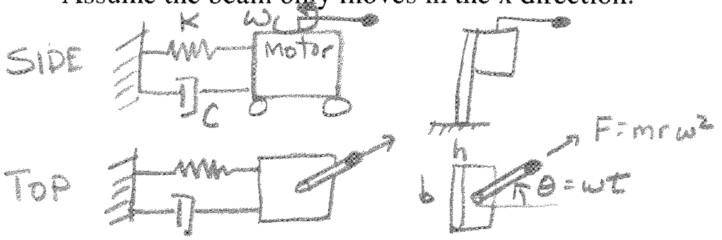
Mechanical Systems Laboratory: Lecture 7
Time and Frequency Response of Second Order Systems

1. A Common Second-Order System: A Mass-Spring-Damper System

In lab next week you will measure how a vibrating beam behaves in the time and frequency domains. The vibrating beam is an example of a system with a mass, some springiness, and some damping. Many physical systems have a mass, some springiness, and some damping, in different proportions. We can describe their behavior with a second order differential equation, and solve the equation to predict responses.

Modeling the Vibrating Beam

Assume the beam only moves in the x direction.



The force caused by the unbalanced load m in the x direction is: $F = mr\omega^2 \sin\theta = mr\omega^2 \sin(\omega t)$

So, we can use the unbalanced load to provide a sinusoidal force input into the beam.

What is K for the beam?

The load-deflection relationship for the beam (from any strength of materials book) is: $x = \frac{F\ell^3}{3EI}$

Where:

F = applied load

X = deflections of beam

E = modulus of elasticity

I = area moment of inertia of beam

For a spring: F=Kx, so K = $\frac{3EI}{\ell^3}$

So, a simple model is:

$M = M_{motor} + \overset{\text{mass of unbalanced load}}{m} + .236 M_{beam}$ } because beam has mass; can find using energy methods of vibration analysis

$F = A\sin\omega t \quad A = mr\omega^2$

ω = speed of motor

$c \approx 0$

The differential equation describing this system is:

$$M\ddot{x} = -kx - c\dot{x} + F$$

The transfer function for this system is:

$$(Ms^2 + cs + k)x = F \qquad \frac{x}{F} = H(s) = \frac{1}{Ms^2 + cs + K}$$

Note, for second order systems, we can write the denominator of the transfer function in a general form:

$$s^2 + 2\zeta\omega_n s + \omega_n^2$$

ω_n = "undamped natural frequency"

ζ = damping ratio

For the mass-spring-damper system, find the damping ratio and natural frequency

$$H(s) = \frac{\frac{1}{m}}{s^2 + \frac{c}{m}s + \frac{K}{m}} \Longrightarrow \omega_n = \sqrt{\frac{K}{m}} \Downarrow$$

$$2\zeta\omega_n = \frac{c}{m} \Rightarrow \zeta = \frac{c}{2\sqrt{KM}} \begin{matrix} \leftarrow \text{actual damping} \\ \leftarrow \text{"critical" damping} \end{matrix}$$

We will see if $\zeta < 1 \Rightarrow$ system is "underdamped" & oscillates

2. How does this system behave in the time domain?

In lab you will measure the transient response of the beam by "twanging" it. How does a system behave when you hit it with an impulse input?

Define $\delta(t) = 0$ for $t \neq 0$

$\int_{-\epsilon}^{\epsilon} \delta(t)\,dt = 1$ for $\epsilon > 0$

$\int_{-\infty}^{\infty} f(t)\delta(t)\,dt = f(0) \Rightarrow$ "sampling property of impulse"

L.T. of Impulse $\int_0^{\infty} \delta(t) e^{-st}\,dt = e^{-s(0)} = 1$

$$\begin{tikzcd} \delta(t) \arrow[r] \arrow[d, leftrightarrow] & \boxed{DEQ} \arrow[r] & y(t) \arrow[d, leftrightarrow] \\ 1 \arrow[r] & \boxed{H(s)} \arrow[r] & Y(s) = H(s) \times 1 = H(s) \end{tikzcd}$$

Thus, the inverse Laplace transform of the transfer function it the impulse response.

What is the impulse response of the vibrating beam?

Use a partial fraction expansion to find the inverse Laplace transform. Basic idea: write the transfer function as the sum of factors that we know how to take the Laplace transform of. Trick to find numerators: multiply by factor, chose s to set factor to zero.

$$H(s) = \frac{\frac{1}{m}}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\frac{1}{m}}{(s-p_1)(s-p_2)} = \frac{A}{s-p_1} + \frac{B}{s-p_2} \Rightarrow h(t) = Ae^{p_1 t} + Be^{p_2 t}$$

The poles of the transfer function are the zeros of the denominator, and they tell us a lot about the way the system behaves, because they became the exponents of exponentials in the time domain.

What are the poles of the vibrating beam?

quadratic eqn $\quad p_i = \dfrac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2} \Rightarrow \begin{array}{l} p_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} \\ p_2 = \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} \end{array}$

$\zeta^2 < 1 \Rightarrow$ poles are imaginary $\Rightarrow$ oscillation

Use partial fraction expansion to find A and B:

$$\left. \frac{\frac{1}{m}(s-p_1)}{(s-p_1)(s-p_2)} = \frac{A(s-p_1)}{(s-p_1)} + \frac{B(s-p_1)}{(s-p_2)} \right|_{s=p_1} \Longrightarrow A = \frac{\frac{1}{m}}{p_1 - p_2}$$

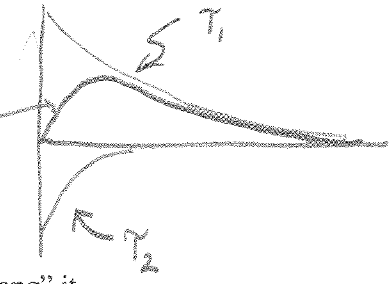
Likewise:

$$B = \frac{\frac{1}{m}}{p_2 - p_1} = -A$$

For $\zeta^2 > 1$ the poles are real, and the system does not oscillate when you "twang" it.

$$h(t) = Ae^{p_1 t} + Be^{p_2 t} = A\left(e^{-t/\tau_1} - e^{-t/\tau_2}\right)$$

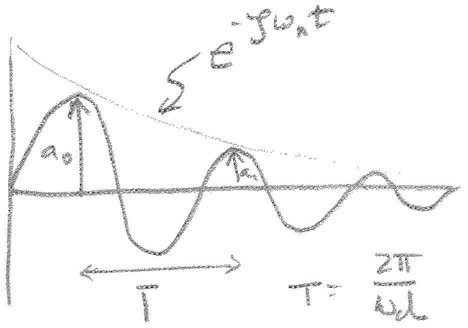
$\tau_1 = -\frac{1}{p_1} \quad \tau_2 = \frac{-1}{p_2}$



For $\zeta^2 < 1$ the poles are imaginary, and the system oscillates when you "twang" it.

$$p_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} = \zeta\omega_n + \underbrace{\omega_n\sqrt{1-\zeta^2}}_{\omega_d}\, j$$

$$h(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t$$



How do you measure damping given the impulse response? One way that you can estimate the damping is by using the "logarithmic damping method"

$$\frac{a_0}{a_n} = e^{-\zeta\omega_n T n}$$

$$\hat{\delta} \equiv \ln\frac{a_0}{a_n} \Rightarrow \text{solve for } \zeta = \frac{\hat{\delta}}{\sqrt{\hat{\delta}^2 + 4\pi^2 n^2}}$$

calculate $\hat{\delta} = \ln\frac{a_0}{a_n}$ ← measure (a_0), ← measure (a_n)

n = # of peaks (start counting from zero!)

3. Frequency response of the beam

How does the system behave if you apply a sinusoidal force input to it?

$$\sin\omega t \longrightarrow \boxed{H(s)} \longrightarrow |H(j\omega)|\sin(\omega t + \phi_H(j\omega))$$

Scaling:

Assume $C = 0$

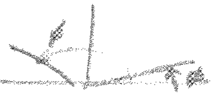
$M\ddot{x} + Kx = F \xrightarrow{\mathcal{L}} (Ms^2 + K)X = F$

$$H(s) = \frac{X}{F} = \frac{1}{Ms^2+K} = \frac{1/m}{s^2 + K/m} = \frac{\frac{1}{m}}{s^2+\omega_n^2} \qquad \omega_n = \sqrt{\frac{K}{m}}$$

$$H(j\omega) = \frac{\frac{1}{m}}{(j\omega)^2 + \omega_n^2} = \frac{\frac{1}{m}}{\omega_n^2 - \omega^2}$$

Phase shift:

$$\phi_H(j\omega) = 0 - \tan^{-1}\frac{0}{\omega_n^2 - \omega^2} = \boxed{\begin{cases} 0^\circ & \omega^2 < \omega_n^2 \\ 180^\circ & \omega^2 > \omega_n^2 \end{cases}}$$



$$\boxed{|H(j\omega)| = \frac{\frac{1}{m}}{|\omega_n^2 - \omega^2|}}$$

Plot on a log-log scale (makes curves into lines)

a) Asymptote 1 for $\omega \ll \omega_n$ $|H(j\omega)| \approx \frac{\frac{1}{m}}{\omega_n^2}$ = constant

b) Asymptote 2: for $\omega \gg \omega_n$ $|H(j\omega)| \approx \frac{\frac{1}{m}}{\omega^2} \Rightarrow 20\log|H(j\omega)| = 20\log\frac{1}{m} - 20\log\omega^2$

$$= 20\log\frac{1}{m} - 40\log\omega$$

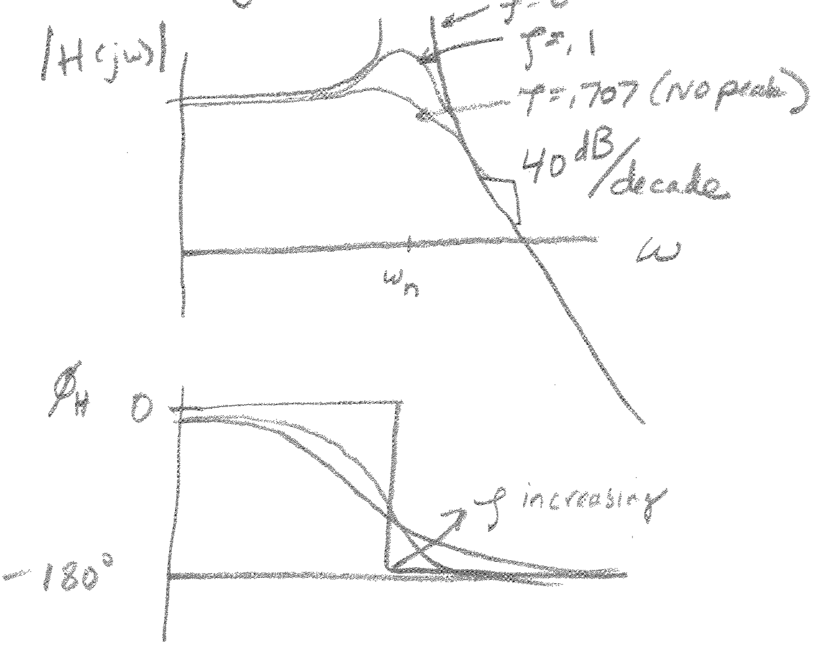
$$y = b + mx$$

c) $\omega = \omega_n \Rightarrow |H(j\omega)| = \infty$

Resonance

What are some uses of resonance?

- the military (star wars)
- communication (detecting 1 frequency)
- entertainment (musical instruments)
- medicine (shattering a kidney stone w/ ultrasound)
- the playground (swing set)



Mechanical Systems Laboratory: Lecture 8
Brief Review of Stability; PD Position Control of a Robot Arm

1. Brief Review of Stability

Stability refers to the concept of whether a system's performance "blows up" or converges to some value. What are some applications in which stability analysis is very important?

Airplanes, Spaceships, Active Suspensions for Cars, Surgery Robots

Three types of stability are:

stable, unstable, marginally stable (sustained oscillations)

The location of the poles of the transfer function determine the type of stability.
Why? Consider a second-order system:

$$H(s) = \frac{1}{s^2+as+b} = \frac{1}{(s-p_1)(s-p_2)} = \frac{A}{s-p_1} + \frac{B}{s-p_2} \quad p_i = \text{"poles"}$$

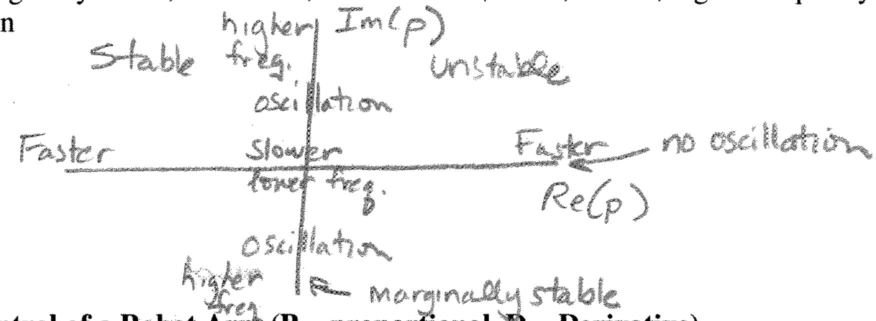
Remember, the inverse Laplace transform of the transfer function is the impulse response:

$$h(t) = Ae^{p_1 t} + Be^{p_2 t}$$

Case 1: Re{p_i} < 0	stable	exponential decay; if $\lvert Re(p_2)\rvert > Re(p_1)$ $e^{p_2 t}$ decays more quickly
Case 2: Re{p_i} = 0	marg. Stable	if $Im\{p_i\} \neq 0 \Rightarrow$ sustained oscillation
Case 3: Re{p_i} > 0	unstable	exponential blow-up (w/ oscillation if $Im\{p_i\} \neq 0$)

So the location of the poles in the complex plane determines the type of response of the system.

Exercise 1: Label the complex plane with the following words:
stable, unstable, marginally stable, oscillation, no oscillation, faster, slower, higher frequency oscillation, lower frequency oscillation



2. PD Position Control of a Robot Arm (P = proportional, D = Derivative)

Position control – most common industrial control system
Can you think of some applications? radar, robot arm, NC milling maching, manufacturing

Consider a one-joint robot arm:

Assume: 1) no friction or gravity; 2) we have a controller that can apply any torque that we want; 3) we can sense θ (for example, with a potentiometer)

Exercise 2: Design a proportional feedback controller to position the robot arm at $\theta=\theta_d$, find its transfer function, and analyze its stability

P- Control $\tau = -K_p(\theta - \theta_d)$

Robot Dynamics $\tau = J\ddot{\theta}$

Overall System Dynamics $J\ddot{\theta} = -K_p(\theta - \theta_d)$

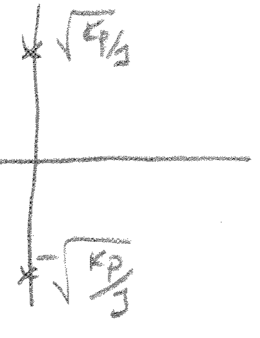
$$J\ddot{\theta} + K_p\theta = K_p\theta_d$$

Transfer Fun: $\dfrac{\theta(s)}{\theta_d(s)} = \dfrac{K_p}{Js^2+K_p} = \dfrac{K_p/J}{s^2+K_p/J}$

Poles: $s^2 = \frac{-K_p}{J} \Rightarrow s = \pm\sqrt{\frac{K_p}{J}}\,j$

Same form as undamped mass-spring system

Robot would oscillate forever



Exercise 3: Design a way to fix the problem. What kind of hardware would you need?

Add damping w/ control law: $\tau = -K_p(\theta - \theta_d) - K_v\dot{\theta}$] Proportional - Derivative Control

Two approaches to sensing angular velocity:

1) use a tachometer
2) differentiate position (e.g. using an op-amp circuit)

What are the dynamics and transfer function of the robot with the new controller?

$$J\ddot{\theta} = -K_p(\theta - \theta_d) - K_v\dot{\theta} \qquad \frac{\theta(s)}{\theta_d(s)} = H(s) = \frac{K_p}{Js^2 + K_v s + K_p}$$

$$J\ddot{\theta} + K_v\dot{\theta} + K_p\theta = K_p\theta_d$$

We choose K_p + K_v when we design our controller!

How are the gains K_p and K_v related to the natural frequency and damping ratio?

$$H(s) = \frac{K_p/J}{s^2 + \frac{K_v}{J}s + \frac{K_p}{J}} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$\omega_n = \sqrt{\frac{K_p}{J}}$ K_p determines ω_n

$\zeta = \frac{K_v}{2\sqrt{K_p J}}$ After choosing K_p based on desired ω_n, can set damping ratio w/ K_v

What is the step response of the system?

$\zeta = 1$ $\theta(t) = 1 - \cos(\omega_n t)$

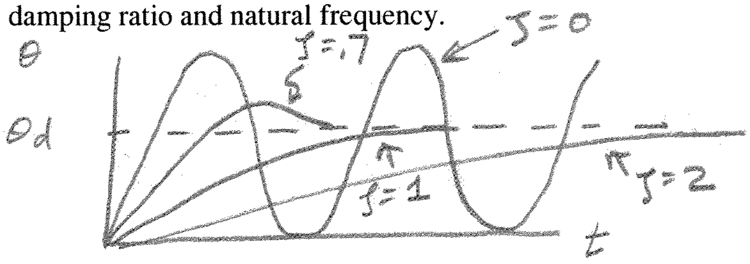
$0 < \zeta < 1$ $\theta(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta})$ $\omega_d = \omega_n\sqrt{1-\zeta^2}$

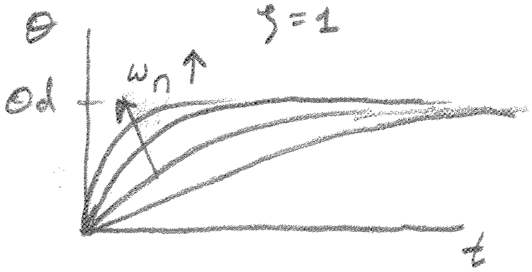
$\zeta = 1$ $\theta(t) = 1 - e^{-\omega_n t}(1 + \omega_n t)$

$\zeta > 1$ sum of two exponentials – see book; can neglect one exponential if $\zeta > 2$

Note: the damping ratio determines whether the system oscillates (i.e. whether the poles have an imaginary part)

Exercise 4: Plot the step response of the system of the system would look like for different values of the damping ratio and natural frequency.



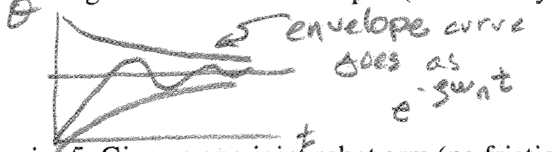


Notes: Overdamped systems are "sluggish"

Among systems responding without overshoot, critically damped systems exhibit the fastest response.

Underdamped systems with $0.5 < \zeta < .8$ get close to final value more rapidly than critically damped or overdamped systems.

The settling time of an underdamped (or critically damped) system is:



$e^{-t/\tau}$ is w/in 5% of final value after 3τ secs

$t_s \Big|_{5\% \text{ criterion}} \approx 3\tau = \frac{3}{\zeta\omega_n}$ (minimized at $\zeta = .68$)

Exercise 5: Given a one-joint robot arm (no friction, no gravity) with J = 1 kgm^2. Design a PD position controller such that the robot finishes 95% of a commanded step-function movement in .5 seconds, with no overshoot.

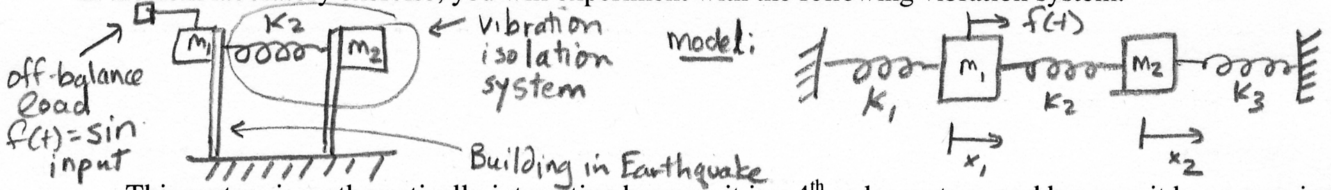
$t_s = \frac{3}{\zeta\omega_n} = .5$ $\zeta = 1$ (no overshoot) $\Rightarrow \omega_n = \frac{3}{.5} = 6.0$ rad/sec

$\omega_n = \sqrt{\frac{K_p}{J}} \Rightarrow K_p = \omega_n^2 J = 36 \frac{kg\,m^2}{s^2}$ $\zeta = \frac{K_v}{2\omega_n J} \Rightarrow K_v = 2\omega_n J\zeta = 12 \frac{kg\,m^2}{sec}$

Mechanical Systems Laboratory: Lecture 9
Systems with Two Modes of Vibration/Design of a Vibration Isolator

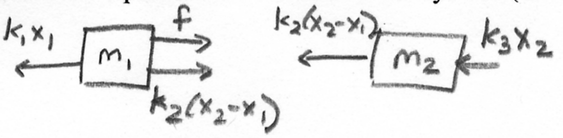
1. Experimental Apparatus and Relationship to Vibration Isolation

In the next laboratory exercise, you will experiment with the following vibration system:



- This system is mathematically interesting because it is a 4th order system, and because it has a zero in the transfer function.
- This system is practically interesting because it works much the same way that real vibration isolation systems do, such as ones used in washing machines or to stabilize a building in an earthquake. The key observation is that m1 doesn't move, even though it is being forced with a sinusoidal force, if m2 and k2 are chosen appropriately. The sinusoidal force could represent an off-balance load in a washing machine, or the forces from an earthquake on a building. Appropriate choice of m2 and k2 can stop the washing machine or building from shaking.

Find the equations of motion of the system (Hint: Use a free-body diagram and assume x2>x2>0)



$$m_1\ddot{x}_1 = f + k_2(x_2-x_1) - k_1x_1$$

$$m_2\ddot{x}_2 = -k_2(x_2-x_1) - k_3x_2$$

We can express these equations in matrix format:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix} \quad \text{or} \quad \underset{\nwarrow}{M}\ddot{\underline{x}} + \underset{\uparrow}{K}\underline{x} = \underline{f}$$

2x2 Matrices

The transfer function of the system is then expressed as follows:

$$(Ms^2+K)\underline{X} = \underline{f} \Rightarrow \underline{x} = (Ms^2+K)^{-1}\underline{f}$$

To find x(s), we need to find $(ms^2+K)^{-1}$

Fact: if $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} c & -b \\ -b & a \end{bmatrix}$$

$\det A = ac - b^2$

so

$$[Ms^2+K]^{-1} = \frac{1}{\underbrace{(m_1s^2+k_1+k_2)(m_2s^2+k_2+k_3)-k_2^2}_{\text{"DEN"}}}\begin{bmatrix} m_2s^2+k_2+k_3 & k_2 \\ k_2 & m_1s^2+k_1+k_2 \end{bmatrix}$$

Now, suppose f(t) = sin ωt and we want to find $x_1(t)$. To do this, we need to find the frequency response:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\begin{bmatrix} f \\ 0 \end{bmatrix} \Rightarrow x_1 = m_{11}f \Rightarrow \left.\frac{x_1}{f}\right|_{s=j\omega} = \frac{-m_2\omega^2+k_2+k_3}{(-m_1\omega^2+k_1+k_2)(-m_2\omega^2+k_2+k_3)-k_2^2}$$

At what input frequency will x1 not move?

when $-m_2\omega^2+k_2+k_3=0 \Rightarrow \frac{x_1}{f}=0$ (zero of transfer fcn) $\Rightarrow \omega_0 = \sqrt{\frac{k_2+k_3}{m_2}}$

What happens to m2 in this situation? $x_2 = m_{21}f = \frac{k_2}{DEN}f \Rightarrow m_2$ shakes, m_1 stays still

Assume k3 = 0 (like in a washing machine vibration isolation system). What is another name for the frequency at which the vibration isolation is achieved?

$\omega_0 = \sqrt{\frac{k_2}{m_2}} \Rightarrow$ resonant frequency of $\{k_2, m_2\}$

The system also has two resonant frequencies. Find the resonant frequencies.

"DEN" has form $a\omega^4 + b\omega^2 + c$

so $$\omega_1^2, \omega_2^2 = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$a = m_1 m_2$

$b = -(m_2(k_1+k_2) + m_1(k_2+k_3))$

$c = k_1 k_2 + k_1 k_3 + k_2 k_3$

How do the masses move at the resonant frequencies? We can gain insight by considering the case of free vibrations (i.e. f(t) = 0).

$M\ddot{\underline{x}} + K\underline{x} = 0$ (1)

Assume $\underline{x} = \underline{x}_0 \sin(\omega t + \phi)$ $\underline{x}_0 = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}$

plugging into (1):

$$(-M\omega^2 + K)\underline{x}_0 = 0 \Rightarrow \underbrace{(M^{-1}K - \omega^2 I)}_{B}\underline{x}_0 = 0$$

$\rightarrow$ If B is invertible, only soln is $\underline{x}_0 = B^{-1}\underline{0} = 0$

$\therefore$ for $\underline{x}_0 \neq 0$ to exist B must be singular

Note: this is an eigenvalue problem:

$$(M^{-1}K)\underbrace{x_0}_{\text{eigenvector}} = \underbrace{\omega^2}_{\text{eigenvalue}} x_0$$

For what values of ω is B singular?

$$B = M^{-1}K - \omega^2 I = \begin{bmatrix} \frac{k_1+k_2}{m_1} - \omega^2 & -\frac{k_2}{m_1} \\ -\frac{k_2}{m_2} & \frac{k_2+k_3}{m_2} - \omega^2 \end{bmatrix}$$

B singular $\Rightarrow$ det B = 0

$$\Rightarrow \left(\frac{k_1+k_2}{m_1} - \omega^2\right)\left(\frac{k_2+k_3}{m_2} - \omega^2\right) - \frac{k_2^2}{m_1 m_2} = 0$$

$$\Leftrightarrow \underbrace{m_1 m_2 \omega^4 - (m_2(k_1+k_2) + m_1(k_2+k_3))\omega^2 + k_1k_2 + k_1k_3 + k_2k_3 = 0}_{\text{"Characteristic Eqn"}}$$

Notes:

det(B) = 0 ⇔ denominator of transfer function = 0

roots of characteristic equation ⇔ poles of transfer function

free response frequencies ⇔ resonant frequencies

at resonant frequencies it is possible to have free response of non-zero amplitudes

At the resonant frequencies, what is x0? (the amplitude of the free response)

$$B\underline{x}_0 = 0 = \begin{bmatrix} \frac{k_1+k_2}{m_1} - \omega^2 & -\frac{k_2}{m_1} \\ -\frac{k_2}{m_2} & \frac{k_2+k_3}{m_2} - \omega^2 \end{bmatrix}\begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

Top Row says:

$$\left(\frac{k_1+k_2}{m_1} - \omega^2\right)x_{10} - \frac{k_2}{m_1}x_{20} = 0$$

$$\Rightarrow x_{10} = \frac{\frac{k_2}{m_1}}{\frac{k_1+k_2}{m_1} - \omega^2} x_{20}$$

Bottom Row says:

$$x_{20} = \frac{\frac{k_2}{m_2}}{\frac{k_2+k_3}{m_2} - \omega^2} x_{10}$$

Example: $m_1 = m_2 = k_1 = k_2 = k_3 = 1$

char eqn: $\omega^4 - 4\omega^2 + 3 = 0$

roots: $\omega^2 = \frac{4 \pm \sqrt{16-12}}{2} = 2 \pm 1 = 1$ or 3

at $\omega_1^2 = 1$ $x_{10} = \frac{1}{2-1}x_{20} = x_{20}$ IN phase

at $\omega_2^2 = 3$ $x_{10} = \frac{1}{2-3}x_{20} = -x_{20}$ 180° out of Phase

General free vibration contains both modes.

Isolation Freq:

$\omega_0 = \sqrt{\frac{k_2+k_3}{m_2}} = \sqrt{2}$

$\omega_1 < \omega_0 < \omega_2$

Mechanical Systems Laboratory: Lecture 10
Data Acquisition; Computer-Based Feedback Control

Note: These notes are derived from Ch. 8 Data Acquisition, Introduction to Mechatronics and Measurement Systems, 2nd Edition, David G. Alciatore and Michael B. Histand, McGraw-Hill 2003

1. Experimental Apparatus

For the next laboratory exercise, you will use a computer to control a motor. Up until now, you have used op-amp circuits as analog computers to implement the computations you need for feedback control. Another common way to implement controllers is digitally by using computers. A common set-up is:

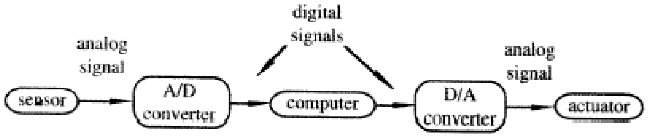


Figure 8.13 Computer control hardware.

The computer could be:

- a PC with a data acquisition card installed. A data acquisition card is sort of like a "video card", except it inputs and outputs arbitrary analog signals instead of a video signals. The Labjack is essentially a data acquisition card that communicates with the computer through the USB port.
- a microcontroller, which is a computer on a single chip. A digital signal processing chip is similar to a microcontroller.
- a programmable logic controller (PLC), which is a specialized industrial device for interfacing to analog and digital devices. PLC's are typically programmed with ladder logic, which is a graphical language for connecting inputs, outputs, and logic.
- Digital circuits, made with logic gates (e.g. AND, OR, NOT gates), or programmable logic arrays, which allow you to set-up arrays of logic gates.

2. Sampling, the Nyquist Frequency, and Aliasing

Many types of sensors (e.g. potentiometers, tachometers, accelerometers, force transducers) provide analog (i.e. continuous) voltage outputs, and many types of actuators (e.g. dc brushed motors) require analog inputs. Computers represent numbers using sequences of digital voltages (i.e. sequences of "bits"). Digital voltages (or "bits") can take only two discrete values, logical 0 (typically corresponding to 0 volts) and logical 1 (typically corresponding to 5 volts). Getting analog signals into digital forms usable by computers requires two processes: sampling and quantization.

Sampling refers to evaluating an analog signal at discrete instants in time. The sampling frequency (or sampling "rate") is how many times per second the signal is sampled.

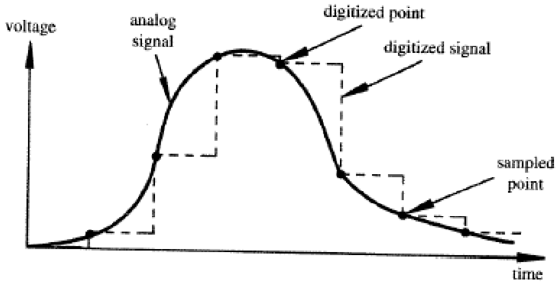


Figure 8.1 Analog signal and sampled equivalent.

The sampling theorem states that you must sample a signal at a frequency that is twice the maximum frequency in the signal (i.e. at the “Nyquist Frequency”), in order to preserve all of the information in the signal. If a signal is sampled at less than this frequency, “aliasing” happens. The result of aliasing is that a high frequency signal looks like a lower frequency signal.

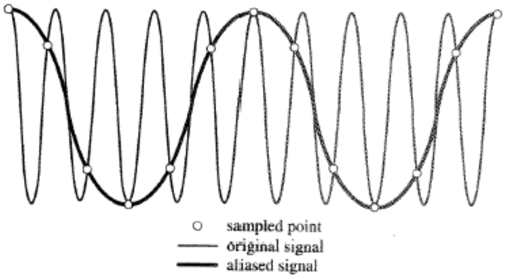
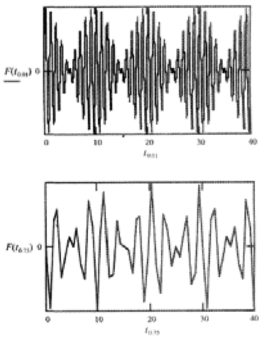
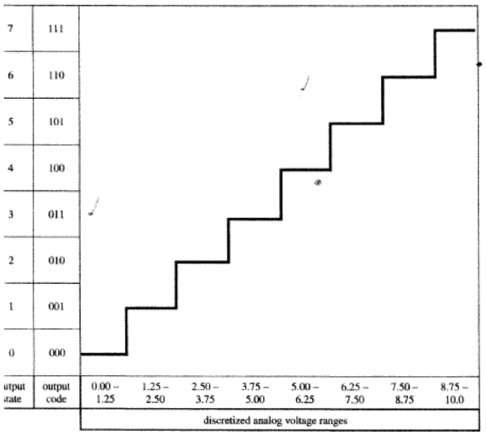


Figure 8.2 Aliasing.



3. Quantizing Theory

Quantizing transforms a continuous, analog input into a set of discrete output states. Coding is the assignment of a digital code word or number to each output state.



4. Analog-to-Digital Conversion (A/D)

An A/D converter quantizes an analog signal at some sampling rate, which is determined by a “trigger signal” from the computer. The resolution of the A/D converter is the number of bits that is uses to represent the analog value of the input. The number of possible states N is equal to the number of bit combinations that can be output from the converter: $N=2^n$. Most commercial A/D converters are 8, 10, or 12 bit devices that resolve 256, 1024, and 4096 output states, respectively. Here is a flash AD converter:

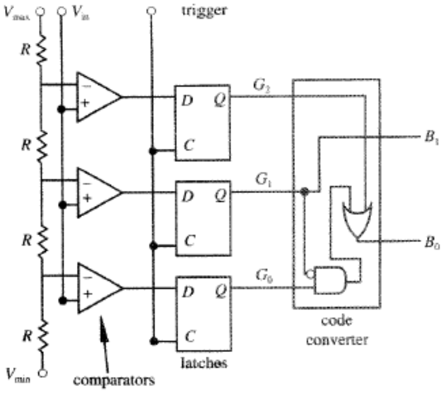


Figure 8.10 A/D flash converter.

Table 8.1 2-bit flash converter output

State	Code ($G_2G_1G_0$)	Binary (B_1B_0)	Voltage range
0	000	00	0–1
1	001	01	1–2
2	011	10	2–3
3	111	11	3–4

5. Digital-to-Analog (D/A) Conversion

A D/A converter takes the binary representation of a signal and converts it into an analog output signal.
A ladder D/A Converter works like this:

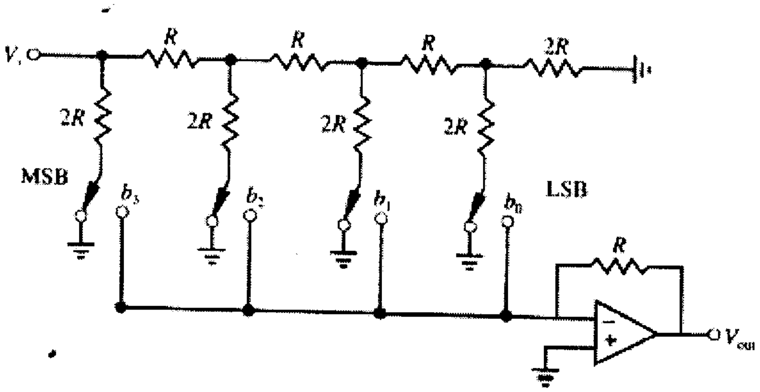


Figure 8.11 4-bit resistor ladder D/A converter.

7. Effect of Sampling Rate on Control Stability

Sampling introduces delays into a control system. If the sampling rate is high enough, the delay is negligible. But if sampling rate is low (e.g. < 100Hz for a robot), then the associated delay can make the control system unstable, especially for large feedback gains. Delay essentially causes "the right information" to be delivered at the wrong time. As an example, consider a proportional feedback control of a first-order system (such as the motor velocity control lab that you did). When there is no delay in this system, the system is stable for all positive values of the gain. What happens when we add delay?

Model of Motor

$$\underbrace{M}_{\text{torque}} = \overset{\text{inertia}}{J}\dot{\omega} + \overset{\text{friction}}{B}\overset{\text{angular velocity}}{\omega} \xrightarrow{\text{Transfer fcn}} \frac{\omega}{M} = \frac{1}{Js+B} = \frac{\frac{1}{B}}{\frac{J}{B}s+1} = \frac{A}{\underbrace{\tau s+1}}$$

$\tau = \frac{J}{B}$ ↑ time constant

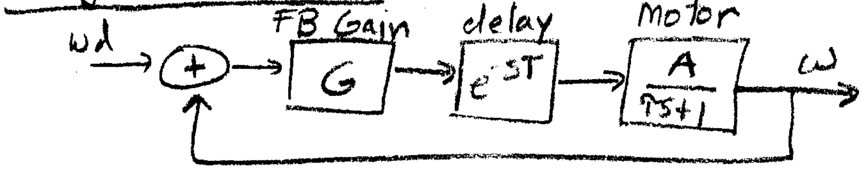
∴ motor acts like a low pass filter

How to Model Delay T due to Sampling?

Note: $\mathcal{L}(x(t-T)) = e^{-sT}\mathcal{L}(x(t))$

delay by T

Velocity Control System:



Solve for T.F. $\Rightarrow \omega = \frac{A}{\tau s+1}e^{-sT}G(\omega_d - \omega)$

Note: $e^{-sT} = 1 - sT + \frac{1}{2}(sT)^2 + \dots$ Taylor's series

$\approx 1 - sT$ if T small

$$\rightarrow \omega = \frac{A}{\tau s+1}(1-sT)G(\omega_d-\omega)$$

$$\Downarrow$$

$$\frac{\omega}{\omega_d} = \frac{AG(1-sT)}{\underbrace{1+AG+(\tau-AGT)s}_{1^{st}\text{ order system}}}$$

Pole at $s = \frac{-(1+AG)}{\tau - AGT}$

Pole stays in left half plane (& system remains stable) if

$\tau - AGT > 0 \Rightarrow G < \frac{\tau}{AT}$ Gain is limited!

Robomoths

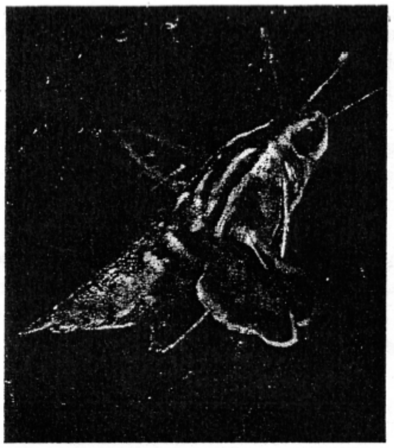
INSECTS are not nearly as biddable as dogs or horses. Although they can perform amazing feats of strength and dexterity on their own scale, that scale is so much smaller than humanity's that it is not surprising they have been overlooked. With rare exceptions, such as bees and silkworms, the insect world is a source of pests rather than of pets or pack animals.

In an age of miniaturisation, however, a few researchers are wondering if more insects might be harnessed to the service of man. One is John Hildebrand, a neurobiologist at the University of Arizona. As part of a project run by America's Defence Advanced Research Projects Agency (DARPA), he and his colleagues have been working with the giant sphinx moth to create a "biobot"—an animal that can be controlled electronically by a human. They have designed a radio transmitter small enough to attach to a sphinx moth without impairing its ability to fly. The next stage is to add a receiver to tell the moth where to go.

Moths may not be that bright, but Dr Hildebrand believes they can be manipulated in rather the same way as a donkey is by dangling a carrot in front of its head. The "carrot" he proposes is a sex pheromone—a mixture of chemicals that female sphinx moths give off to attract males. It is potent stuff. Previous research has shown that a few molecules are enough to attract a male's attention, and that, given a favourable wind, an amorous male can find a mate who is several kilometres away.

One of the team's ideas is to fit male sphinx moths with small, radio-controlled pheromone dispensers. A moth's pheromone-detectors are its antennae. It can work out where pheromone molecules are coming from by comparing the signals from each antenna, in the same way that a person works out the direction of a sound by comparing signals from each ear. A moth's senses could be subverted by puffing suitable molecules from a dispenser to steer it towards a chosen target.

That is a rather crude approach. Dr Hildebrand hopes to be more subtle. He has spent much of his career examining how a moth's nervous system responds to the pheromone, and he thinks he knows enough to steer a moth directly, without the need for the chemicals themselves. He plans to do it by attaching electrodes to the nerves involved and stimulating them appropriately—turning the moth into a genuine, radio-controlled biobot.



Now, where's my backpack?

That would be an interesting demonstration of mankind's powers over nature Could it also be useful? Brian Smith and colleagues at Ohio State University have recently shown that sphinx moths can be manipulated like dogs as well as donkeys. Pavlov's early experiments on reflexes trained dogs to salivate at the sound of a bell, by ringing one every time they were fed with meat. Dr Smith's team has mimicked Pavlov by training moths to stick their tongues out in response to a chemical called cyclohexanone, which was puffed at them while they were fed sugared water.

The reason that this trick might be useful—and the reason for DARPA's interest—is that cyclohexanone is a volatile component of TNT, an explosive often used in landmines. By releasing a swarm of trained moths over a minefield, and observing where they stuck their tongues out, it should be possible to locate mines without risk either to people or to expensive mine-detecting machinery.

It is hard to see if a moth is sticking its tongue out at a range of several hundred metres. But Dr Smith has thought of a way round that. He can sense when a moth is blowing a raspberry by attaching a wire to the muscle that controls the insect's tongue, and using it to transmit a signal via one of Dr Hildebrand's tiny electronic backpacks.

If lepidopteran mine detectors work they could be the start of a new indust The rate at which video cameras are being miniaturised means that they, too, may soon be light enough for insects to carry. That would have obvious military applications, even if one countermeasure is obvious, too: surrounding sensitive installations with giant candles.