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ME106 Course Notes from Prof. J Bobrow

Background Notes

This class has a combination of mathematical analysis and experimentation. It is important for you to understand the mathematical analysis if you wish to improve upon an existing design. Most designs have weaknesses that can be corrected if one understands both the theoretical and practical aspects of their operation. It is also important for you to be able to experimentally test a given design, otherwise, you would not understand its practical limitations. We begin with the most important mathematical concepts which will be used throughout the quarter.

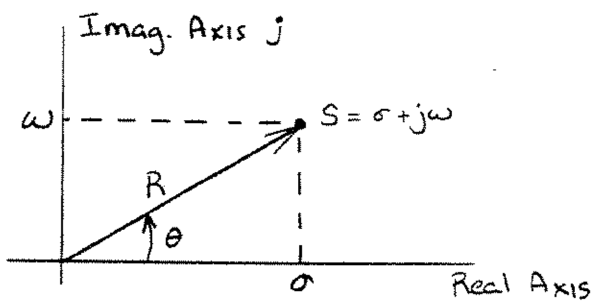
1 Complex Variables

Much of the analysis in this class will be in the *frequency domain* rather than *time domain.* A function $f(t)$ is transformed from time to frequency domain using the Laplace transform. In frequency domain analysis, the independent variable is complex, and has the form

$$s = \sigma + jw, \tag{1}$$

where both σ and ω are real numbers, and $j \equiv \sqrt{-1}$ or $j^2 = -1$. We see that s has a real part, σ, and an imaginary part ω. Often we write $\text{Re}(s) = \sigma$ and $\text{Im}(s) = \omega$. A function of a complex variable $F(s)$ is also complex, and we often need its real and imaginary parts $\text{Re}(F(s))$, and $\text{Im}(F(s))$.

It is helpful to think of a complex variable as a two dimensional vector in a plane, or *s-plane* as shown in figure below.



If we say that R is the length of the vector s, then $R^2 = \sigma^2 + \omega^2$, or equivalently, $R = |s|$. The beauty of this representation is that we can define the same point as

$$s = Re^{j\theta}, \text{ where } \theta = \tan^{-1}\left(\frac{\omega}{\sigma}\right). \tag{2}$$

To show that this representation works, recall the definition

$$e^{j\theta} = \cos\theta + j\sin\theta. \tag{3}$$

Hence, $s = Re^{j\theta} = R\cos\theta + jR\sin\theta = \sigma + j\omega$ from geometry shown in the figure. Sometimes we also write $s = R\angle\theta$ for shorthand. Also note that $\text{Re}(s) = \sigma$ and $\text{Im}(s) = \omega$ are both non-imaginary real numbers.

1.1 Multiplication and Division

Suppose we have two complex numbers s_1 and s_2, that we would like to multiply or divide. Using the above notation,

$$s_1 = R_1 e^{j\theta_1} = R_1 \angle \theta_1$$

$$s_2 = R_2 e^{j\theta_2} = R_2 \angle \theta_2,$$

then

$$s_1 s_2 = R_1 e^{j\theta_1} R_2 e^{j\theta_2} = R_1 R_1 e^{j(\theta_1+\theta_2)}$$

or

$$s_1 s_2 = R_1 \angle \theta_1 R_2 \angle \theta_2 = R_1 R_2 \angle (\theta_1 + \theta_2).$$

Similarly,

$$\frac{s_1}{s_2} = \frac{R_1 \angle \theta_1}{R_2 \angle \theta_2} = \frac{R_1}{R_2} \angle (\theta_1 - \theta_2)$$

since,

$$\frac{R_1 e^{j\theta_1}}{R_2 e^{j\theta_2}} = \frac{R_1}{R_2} e^{j(\theta_1-\theta_2)}.$$

Note that the hard way is to multiply components as

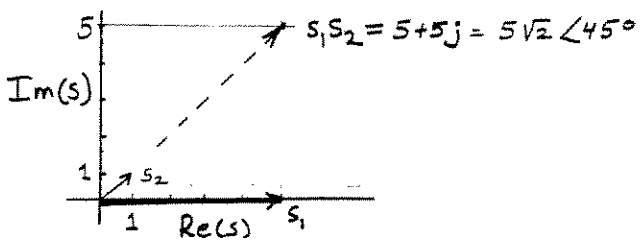
$$\begin{aligned} s_1 s_2 &= (\sigma_1 + j\omega_1)(\sigma_2 + j\omega_2) \\ &= \sigma_1\sigma_2 + j\omega_2\sigma_1 + j\omega_1\sigma_2 + j2\omega_1\omega_2 \\ &= (\sigma_1\sigma_2 - \omega_1\omega_2) + j(\omega_2\sigma_1 + \omega_1\sigma_2). \end{aligned}$$

Then the real part, $\mathrm{Re}(s_1 s_2) = \sigma_1\sigma_2 - \omega_1\omega_2 = R_1 R_2 \cos(\theta_1 + \theta_2)$, and the imaginary part, $\mathrm{Im}(s_1 s_2) = \omega_2\sigma_1 + \omega_1\sigma_2 = R_1 R_2 \sin(\theta_1 + \theta_2)$.

Example 1

multiplication $s_1 s_2$

Suppose $s_1 = 5 = 5\angle 0$, and $s_2 = 1 + j = \sqrt{1^2 + 1^2}\angle 45°$. This can be interpreted as multiplication of a vector by a scalar, so the direction should not change. Now, $s_1 s_2 = 5(1 + j) = 5 + 5j$. Also, $s_1 s_2 = (5\angle 0)(\sqrt{2}\angle 45°) = 5\sqrt{2}\angle 45°$. The sketch below shows that they are the same numbers.



Example 2

Suppose $s_1 = 3j = 3\angle 90°$, and $s_2 = -3j$. First note that $s_2 = 3\angle -90° = 3\angle 270°$, but that $s_2 \neq -3\angle 90°$ since the coefficient of the exponent must always be a positive number (why?). Then, $s_1 s_2 = (3j)(-3j) = 9$, or

$$s_1 s_2 = (3\angle 90°)(3\angle -90°) = 9\angle 0 = 9\angle 360° = 9.$$

1.2 Evaluating Functions of s Using Poles and Zeros (Skip this section on first read)

The most important use of the vector representation is for the evaluation of functions of s at certain points. A function of s, say $G(s)$, will also have an imaginary and a real part which will need to be computed for a given value of s. We will do this many times in later experiments. For instance, a second order rational polynomial in s might have the form

$$G(s) = \frac{as+b}{s^2+cs+d},$$

which can also be written in factored form as

$$G(s) = \frac{(s-s_o)k}{(s-s_1)(s-s_2)}$$

where $s_o = -\frac{b}{a}$, $k = a$, and $s_{1,2} = \frac{-c\pm\sqrt{c^2-4d}}{2}$.

The value s_o is called a *zero* of $G(s)$ since $|G(s_o)| = 0$. The values s_1 and s_2 are called *poles* of $G(s)$ since $|G(s_1)| = |G(s_2)| = \infty$. If we plot the points s_o, s_1, and s_2 on the s plane, we can evaluate $G(s)$ graphically.

For example, suppose

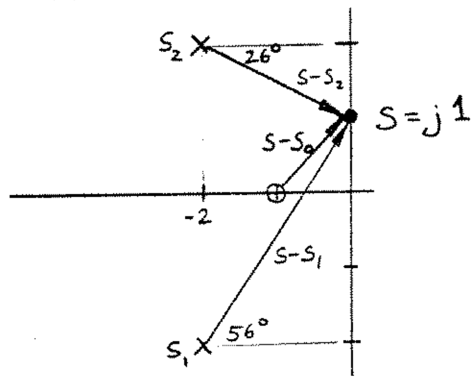
$$G(s) = \frac{s+1}{(s+2+2j)(s+2-2j)} = \frac{s+1}{(s+2)^2+4} = \frac{s+1}{s^2+4s+8},$$

which is of form

$$G(s) = \frac{s-s_o}{(s-s_1)(s-s_2)},$$

where $s_o = -1$, $s_1 = -2-2j$, $s_2 = -2+2j$.

Say we want to evaluate $G(j1)$, i.e. the value of G at $s = j$. We can either plug $s = j$ into $G(s)$ and do some algebra, or we can use the poles and zeros of $G(s)$, and graphically compute $G(j)$ from a *pole-zero* plot:



The figure shows that $s-s_o$ can also be thought of as a vector whose tip is at s and tail is at s_o. Then, at $s = j$, $s - s_o = j - -1 = 1+j$, $s - s_1 = j - (-2-2j) = 2+3j$, and $s - s_2 = j - (-2+2j) = 2-j$. Now using the exponential form of each factor we have:

$$G(s) = \frac{|s-s_o| \angle(s-s_o}{(|s-s_1| \angle(s-s_1)(|s-s_2| \angle(s-s_2)}$$

$$
\begin{aligned}
G(s=j) &= \frac{\sqrt{2}\langle 45^\circ}{\left(\sqrt{13}\langle\underbrace{\tan^{-1}\frac{3}{2}}_{56.31^\circ}\right)\left(\sqrt{5}\langle-\underbrace{\tan^{-1}\frac{1}{2}}_{26.56^\circ}\right)} \\
&= \sqrt{\frac{2}{13\cdot 5}}\langle 45^\circ - 56.31^\circ + 26.56^\circ \\
&= \sqrt{\frac{2}{65}}\langle 15.25^\circ
\end{aligned}
$$

Finally, we could have alternatively evaluated $G(s=j)$ algebraically as

$$
\begin{aligned}
G(j) = \left.\frac{s+1}{s^2+4s+8}\right|_{s=j} &= \frac{1+j}{-1+4j+8} = \frac{1+j}{7+4j} \\
&= \frac{1+j}{7+4j}\left(\frac{7-4j}{7-4j}\right) = \frac{7-4j+7j+4}{49+16} \\
&= \frac{11}{65} + \frac{3j}{65} = 0.169 + j0.046
\end{aligned}
$$

Note that the two methods are completely identical, since

$$
\sqrt{\frac{2}{65}}\langle 15.25^\circ = 0.1754\,(\cos\ 15.25 + j\sin\ 15.25) = 0.169 + j0.046.
$$

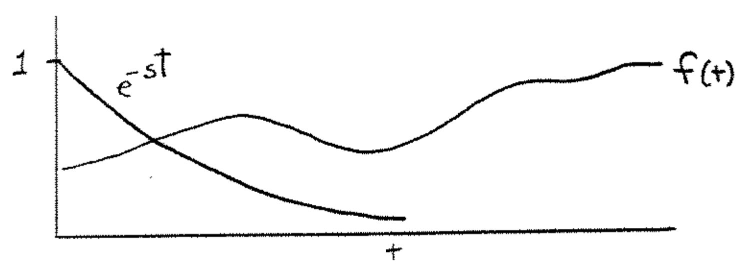
The most appropriate method to use to evaluate functions $G(s)$ at given points s will usually be clear from the problem.

2 Laplace Transforms

A function $f(t)$ can be transformed to the frequency domain using the Laplace Transform if the following integral exists for *some* value of s.

$$
\begin{aligned}
F(s) = L[f(t)] &\equiv \int_0^\infty f(t)e^{-st}dt \quad \text{(an improper integral).} && (4) \\
&= \lim_{T\to\infty}\int_0^T f(t)e^{-st}dt && (5)
\end{aligned}
$$

For instance, if $\mathrm{Re}(s) > 0$, e^{-st} and $f(t)$ might look like



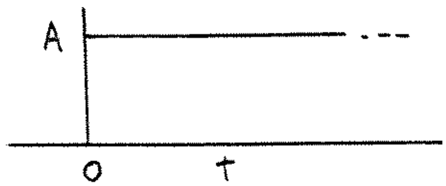
For $f(t)$ to be transformable, the product $f(t)e^{-st}$ must go to zero as $t\to\infty$ for the integral to exist. The mathematical condition is $|f(t)| < Me^{ct}$ for some

$M > 0$, $c > 0$ and all $t > 0$. For instance, if $f(t) = e^{t^2}$,

$$\int_0^\infty e^{t^2} e^{-st} dt = \infty \Rightarrow \text{no Laplace Transform.}$$

A few important examples of transformable functions are given below.

Step Function.



Let $f(t) = Au(t)$, where $u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases}$.

$$L[Au(t)] = \int_0^\infty Ae^{-st}dt = \frac{-A}{s}e^{-st}\Big|_0^\infty = -\frac{A}{s}\left[e^{-s\infty} - e^{-s0}\right] = \underline{\underline{\frac{A}{s}}} \qquad (6)$$

Note that t is integrated out of the function, and we are left with a new function of s only.

Exponential e^{-at}.

$$\begin{aligned} L\left[e^{-at}\right] &= \int_0^\infty e^{-at}e^{-st}dt = \int_0^\infty e^{-(s+a)t}dt \\ &= -\frac{1}{s+a}e^{-(s+a)t}\Big|_0^\infty = -\frac{1}{s+a}\left[e^{-\infty} - e^0\right] \\ &= \underline{\underline{\frac{1}{s+a}}} \leftarrow \text{you will see this one a lot!} \end{aligned} \qquad (7)$$

Sinusoid $\sin \omega t$.

First note that

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}, \qquad (8)$$

which can be verified using the fact that $e^{j\theta} = \cos\theta + j\sin\theta$. Also note that

$$\begin{aligned} L[f_1(t) + f_2(t)] &= \int_0^\infty (f_1(t) + f_2(t))e^{-st}dt = \int_0^\infty f_1(t)e^{-st}dt + \int_0^\infty f_2(t)e^{-st}dt \\ &= L[f_1(t)] + L[f_2(t)], \end{aligned}$$

so, using the transform of the exponential in (7),

$$\begin{aligned} L[\sin\omega t] &= \frac{1}{2j}\left(L\left[e^{j\omega t}\right] - L\left[e^{-j\omega t}\right]\right) \\ &= \frac{1}{2j}\left(\frac{1}{s-jw} - \frac{1}{s+jw}\right) = \frac{1}{2j}\left(\frac{(s+jw)-(s-jw)}{(s-jw)(s+jw)}\right) \\ &= \underline{\underline{\frac{\omega}{s^2+\omega^2}}}. \end{aligned} \qquad (9)$$

These transforms and other common ones encountered in linear system theory are tabulated in the attached tables from Ogata "Modern Control Engineering."

2.1 Differentiation using Laplace Transforms

The most important property of Laplace Transforms is the relationship between $L\left[\frac{df}{dt}\right]$ and $L\left[f\left(t\right)\right]$, where $f(t)$ is *any* differentiable function of time. It is:

$$L\left[\frac{df}{dt}\right] = sL\left[f\left(t\right)\right] - f\left(0\right) = sF\left(s\right) - f\left(0\right). \tag{10}$$

To show this, use the definition of the transform (4) on the derivative $L\left[\frac{df}{dt}\right] = \int\limits_0^\infty \frac{df}{dt}e^{-st}dt$, and integrate this expression by parts: $\int udv = uv - \int vdu$. To do this, choose $u = e^{-st}$, and $dv = \frac{df}{dt}dt = df$. Then

$$\begin{aligned} L\left[\frac{df}{dt}\right] &= \int\limits_0^\infty \frac{df}{dt}\,e^{-st}dt \\ &= e^{-st}f(t)\big|_0^\infty - \int\limits_0^\infty f(t)(-se^{-st})dt \\ &= e^{-s\infty}f\left(\infty\right) - f\left(0\right) + s\int\limits_0^\infty f\left(t\right)e^{-st}dt \\ L\left[\frac{df}{dt}\right] &= sL\left[f(t)\right] - f(0) \end{aligned} \tag{11}$$

We will see this many times as $L[\dot{x}] = sX(s) - x(0)$, where $x(t)$ is used in place of $f(t)$, and $\dot{x}$ was used for $\frac{df}{dt}$. Note that we can obtain $L\left[\ddot{x}\right]$ recursively as follows. Let $\dot{x} = v$, so $\dot{v} = \ddot{x}$. Then

$$L\left[\ddot{x}\right] = L[\dot{v}] = sV(s) - v(0),$$

but

$$V(s) = L\left[v\right] = L\left[\dot{x}\right] = sX(s) - x\left(0\right)$$

so

$$\underline{L\left[\ddot{x}\right] = s^2X(s) - sx\left(0\right) - \dot{x}\left(0\right)} \tag{12}$$

Example

If we are given that the derivative of $\sin\omega t$ is $\omega\cos\omega t$, we can use (10) to derive $L\left[\cos\omega t\right]$ from $L\left[\sin\omega t\right]$. Recall that for $f\left(t\right) = \sin\omega t$, its transform is $F\left(s\right) = \frac{\omega}{s^2+\omega^2}$ as derived before. Then because $\dot{f}\left(t\right) = \omega\cos\omega t$, we have

$$\begin{aligned} L[\cos\omega t] &= \frac{1}{\omega}L[\dot{f}] \\ &= \frac{1}{\omega}\left(sF\left(s\right) - f(0)\right) \\ &= \frac{1}{\omega}s\left(\frac{\omega}{s^2+\omega^2}\right) \\ &= \underline{\underline{\frac{s}{s^2+w^2}}}, \end{aligned} \tag{13}$$

where we have used the fact that $f(0) = \sin \omega 0 = 0$.

In a manner similar to finding $L[\frac{df}{dt}]$, we can derive $L\left[\int f(t)\right]$, which is

$$L\left[\int f(t)dt\right] = \frac{F(s)}{s} + \frac{1}{s}\int f\,dt\,|_{t=0}$$

2.2 Solving Differential Equations

One big use of Laplace Transforms is to manipulate and solve differential equations. A very common first order example is to solve

$$\dot{x} = ax + b,$$

where a, b are constants and $x(0) = x_o$. First take the Laplace Transform of both sides of the differential equation

$$sX - x_0 = aX + \frac{b}{s},$$

where we have assumed that b is applied at $t = 0$ (so its really $bu(t)$). Now solve for the unknown X

$$X(s-a) = \frac{b}{s} + x_0 = \frac{b + x_0 s}{s}$$

or

$$X = \frac{b + x_0 s}{s(s-a)}.$$

The final step is to determine $x(t)$ given $X(s)$. Since there is no expression of this form in our table of transform pairs, we can use a *partial fraction* expansion to find an equivalent form. That is,

$$X(s) = \frac{b + x_0 s}{s(s-a)} = \frac{A}{s} + \frac{B}{s-a}, \tag{14}$$

where A and B are unknowns that we can determine. Once we know A and B, the terms on the right hand side are in the transform table, and hence their time domain counterparts are known. To determine A and B, note that this equation is true for any s, so multiply both sides of (14) by s and let $s = 0$ to get A.

$$\left(\frac{b + x_o s}{s(s-a)}\right)s = A + \left(\frac{B}{s-a}\right)s$$

$$A = \left(\frac{b + x_o s}{s(s-a)}\right)s\,|_{s=0} = -\frac{b}{a.}$$

Similarly, multiply both sides of (14) by $s - a$ and set $s = a$ to find B

$$B = \left(\frac{b + x_o s}{s(s-a)}\right)(s-a)\,|_{s=a} = \frac{b + x_0 a}{a},$$

so

$$X(s) = \frac{-b/a}{s} + \frac{(b + x_0 a)/a}{s-a}.$$

Table 1-1 Laplace Transform Pairs

	$f(t)$	$F(s)$
1	Unit impulse $\delta(t)$	1
2	Unit step $1(t)$	$\frac{1}{s}$
3	t	$\frac{1}{s^2}$
4	$\frac{t^{n-1}}{(n-1)!} \quad (n = 1,2,3, \ldots)$	$\frac{1}{s^n}$
5	$t^n \quad (n = 1,2,3, \ldots)$	$\frac{n!}{s^{n+1}}$
6	e^{-at}	$\frac{1}{s+a}$
7	te^{-at}	$\frac{1}{(s+a)^2}$
8	$\frac{1}{(n-1)!} t^{n-1} e^{-at} \quad (n = 1,2,3, \ldots)$	$\frac{1}{(s+a)^n}$
9	$t^n e^{-at} \quad (n = 1,2,3, \ldots)$	$\frac{n!}{(s+a)^{n+1}}$
10	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
11	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
12	$\sinh \omega t$	$\frac{\omega}{s^2-\omega^2}$
13	$\cosh \omega t$	$\frac{s}{s^2-\omega^2}$
14	$\frac{1}{a}(1-e^{-at})$	$\frac{1}{s(s+a)}$
15	$\frac{1}{b-a}(e^{-at}-e^{-bt})$	$\frac{1}{(s+a)(s+b)}$
16	$\frac{1}{b-a}(be^{-bt}-ae^{-at})$	$\frac{s}{(s+a)(s+b)}$
17	$\frac{1}{ab}\left[1+\frac{1}{a-b}(be^{-at}-ae^{-bt})\right]$	$\frac{1}{s(s+a)(s+b)}$

Table 1–1 Cont'd

18	$\frac{1}{a^2}(1 - e^{-at} - ate^{-at})$	$\frac{1}{s(s+a)^2}$
19	$\frac{1}{a^2}(at - 1 + e^{-at})$	$\frac{1}{s^2(s+a)}$
20	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$
21	$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2 + \omega^2}$
22	$\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}\, t$	$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
23	$-\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin (\omega_n \sqrt{1-\zeta^2}\, t - \phi)$ $\phi = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}$	$\frac{s}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
24	$1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin (\omega_n \sqrt{1-\zeta^2}\, t + \phi)$ $\phi = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}$	$\frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$
25	$1 - \cos \omega t$	$\frac{\omega^2}{s(s^2 + \omega^2)}$
26	$\omega t - \sin \omega t$	$\frac{\omega^3}{s^2(s^2 + \omega^2)}$
27	$\sin \omega t - \omega t \cos \omega t$	$\frac{2\omega^3}{(s^2 + \omega^2)^2}$
28	$\frac{1}{2\omega} t \sin \omega t$	$\frac{s}{(s^2 + \omega^2)^2}$
29	$t \cos \omega t$	$\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$
30	$\frac{1}{\omega_2^2 - \omega_1^2} (\cos \omega_1 t - \cos \omega_2 t) \qquad (\omega_1^2 \neq \omega_2^2)$	$\frac{s}{(s^2 + \omega_1^2)(s^2 + \omega_2^2)}$
31	$\frac{1}{2\omega} (\sin \omega t + \omega t \cos \omega t)$	$\frac{s^2}{(s^2 + \omega^2)^2}$

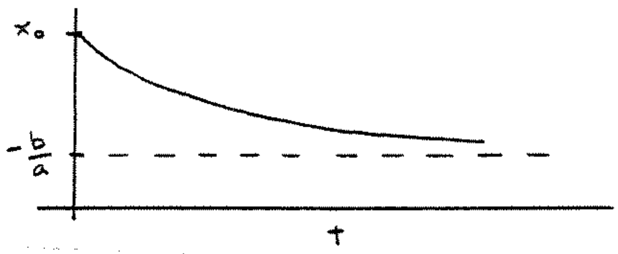
Table 1–2 Properties of Laplace Transforms

1	$\mathscr{L}[Af(t)] = AF(s)$
2	$\mathscr{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$
3	$\mathscr{L}_\pm\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0\pm)$
4	$\mathscr{L}_\pm\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s) - sf(0\pm) - \dot{f}(0\pm)$
5	$\mathscr{L}_\pm\left[\frac{d^n}{dt^n}f(t)\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k} \overset{(k-1)}{f}(0\pm)$ where $\overset{(k-1)}{f}(t) = \frac{d^{k-1}}{dt^{k-1}}f(t)$
6	$\mathscr{L}_\pm\left[\int f(t)\,dt\right] = \frac{F(s)}{s} + \frac{\left[\int f(t)\,dt\right]_{t=0\pm}}{s}$
7	$\mathscr{L}_\pm\left[\int\int f(t)\,dt\,dt\right] = \frac{F(s)}{s^2} + \frac{\left[\int f(t)\,dt\right]_{t=0\pm}}{s^2} + \frac{\left[\int\int f(t)\,dt\,dt\right]_{t=0\pm}}{s}$
8	$\mathscr{L}_\pm\left[\int \cdots \int f(t)(dt)^n\right] = \frac{F(s)}{s^n} + \sum_{k=1}^{n} \frac{1}{s^{n-k+1}}\left[\int \cdots \int f(t)(dt)^k\right]_{t=0\pm}$
9	$\mathscr{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$
10	$\int_0^\infty f(t)\,dt = \lim_{s\to 0} F(s) \qquad \text{if } \int_0^\infty f(t)\,dt \text{ exists}$
11	$\mathscr{L}[e^{-at}f(t)] = F(s+a)$
12	$\mathscr{L}[f(t-\alpha)1(t-\alpha)] = e^{-\alpha s}F(s) \qquad \alpha \geq 0$
13	$\mathscr{L}[tf(t)] = -\frac{dF(s)}{ds}$
14	$\mathscr{L}[t^2f(t)] = \frac{d^2}{ds^2}F(s)$
15	$\mathscr{L}[t^nf(t)] = (-1)^n\frac{d^n}{ds^n}F(s) \qquad n = 1,2,3,\ldots$
16	$\mathscr{L}\left[\frac{1}{t}f(t)\right] = \int_s^\infty F(s)\,ds$
17	$\mathscr{L}\left[f\left(\frac{t}{a}\right)\right] = aF(as)$

Now, using the table,

$$x(t) = -\frac{b}{a}u(t) + \frac{b + x_o a}{a}e^{at}.$$

Note that the initial conditions are satisfied, $x(0) = -\frac{b}{a} + \frac{b+x_0 a}{a} = x_0$, and if $a < 0$, the final value of x is $x(\infty) = -\frac{b}{a}$ (See the figure). If $a > 0$, $x(\infty) = \infty$ since $e^{at} \to \infty$ in this case.

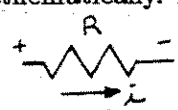


Also note that the poles of this system are $s = 0$, and $s = a$. They characterize the response of the system since the two time dependent terms in the inverse transform, $u(t)$ and e^{at}, arise because of these poles. That is why the expression that determines the roots of the denominator of $X(s)$, i.e. $s(s-a) = 0$ is called the *characteristic equation*.

3 Electronic Elements, Impedance, and Block Diagrams

3.1 Resistor

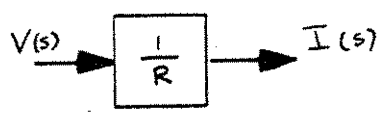
Our first experiment uses capacitors and resistors, which have dynamic properties that we need to understand mathematically. A *resistor* has the symbol



the voltage across the resistor is proportional to current, or

$$v = Ri. \tag{15}$$

The Laplace Transform of this expression is $V(s) = RI(s)$. A block diagram can be used to describe the same relation. For this case it is



The block diagram shows $\frac{L[output]}{L[input]} = \frac{I(s)}{V(s)} = \frac{1}{R}$ in the box, where all the initial conditions are assumed to be zero (for the case of a resistor the initial conditions do not matter). For a general system a rational polynomial will be in the box called the *transfer function*. Multiplication of the input by the transfer function gives the output.

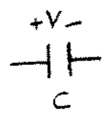
Again, a

$$\textbf{Transfer Function} \equiv \frac{L[\text{output}]}{L[\text{input}]}, \tag{16}$$

assuming zero initial conditions. It is up to you to choose what the input and the output of the system are. The choice is usually clear from the problem that is being solved.

3.2 Capacitor

A *capacitor* has the symbol

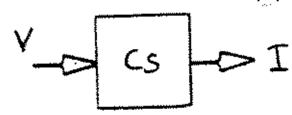


with the voltage across the capacitor proportional to the charge on its plates, or $q = \int i dt = Cv$. Differentiating this expression with respect to time gives

$$i = C\frac{dv}{dt}. \tag{17}$$

Taking the Laplace Transform of both sides with zero initial conditions, $I(s) = CsV(s)$ or

$$V(s) = \frac{1}{Cs}I(s). \quad \equiv$$

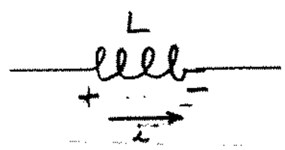


The coefficient of $I(s)$ is called the *impedance.* For a capacitor the impedance is $\frac{1}{Cs}$, for a resistor it is just R. The impedance is often labeled as $Z(s)$, so

$$V(s) = Z(s)I(s). \tag{18}$$

3.3 Inductor

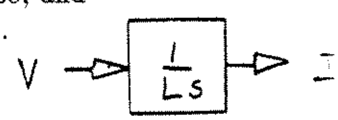
An *inductor* has the symbol



with the voltage across it proportional to the change in current through it, or

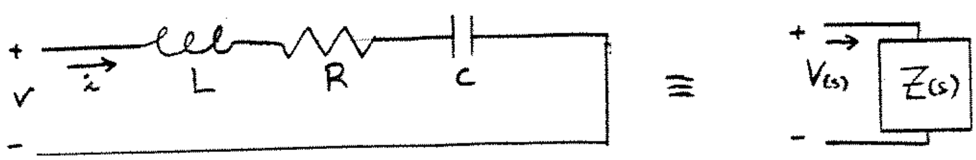
$$v = L\frac{di}{dt}. \tag{19}$$

Taking the Laplace Transform gives $V(s) = LsI(s)$, so the impedance is Ls, and if we assume voltage is the input, the transfer function for an inductor is $\frac{1}{Ls}$.



3.4 Combinations of Elements

To analyze a series LRC circuit as shown,



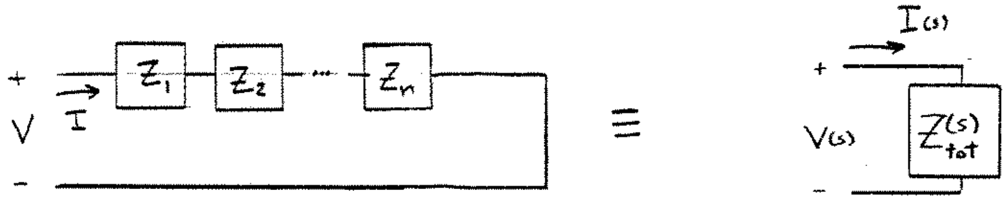
sum the transformed voltages around the loop to obtain

$$V = RI + LsI + \frac{I}{Cs} = (R + Ls + \frac{1}{Cs})I.$$

We see that the coefficient of I, or the impedance is

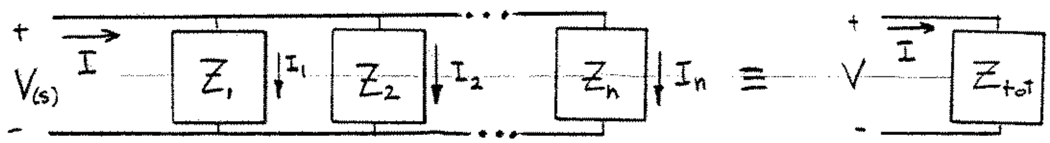
$$Z(s) = (R + Ls + \frac{1}{Cs}).$$

In general, for series circuits



$$Z_{tot} = \sum_{i=1}^{n} Z_i.$$

For parallel circuits,



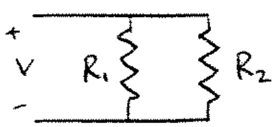
$V(s) = Z_{tot}I(s)$ and $I_i = \frac{V}{Z_i}$ for each branch.

$$I = \sum_{i=1}^{n} I_i = \sum_{i=1}^{n} \frac{V}{Z_i} = V \sum_{i=1}^{n} \frac{1}{Z_i},$$

so $V = \frac{1}{\sum_{i=1}^{n} \frac{1}{Z_i}} I$ and $Z_{tot} = \frac{1}{\sum_{i=1}^{n} \frac{1}{Z_i}}$.

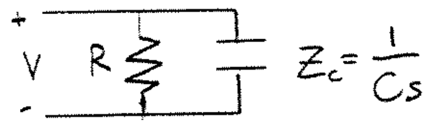
Example 1

Two resistors in parallel:



$$Z_{tot} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} = \frac{1}{\frac{R_2 + R_1}{R_1 R_2}} = \frac{R_1 R_2}{R_1 + R_2}.$$

Example 2
A resistor and capacitor in parallel:



$$Z_{tot} = \frac{R\frac{1}{cs}}{R + \frac{1}{cs}} = \frac{R}{1 + Rcs}$$

3.5 Semiconductors, the Diode and the MOSFET

The following notes were taken from Radio Shack's *Getting Started in Electronics.* You may need to purchase a MOSFET for use with two experiments, and possibly your project. For more detailed information on these elements, refer to Horowitz and Hill: *The Art of Electronics,* or Millman and Halkias: *Integrated Electronics.*

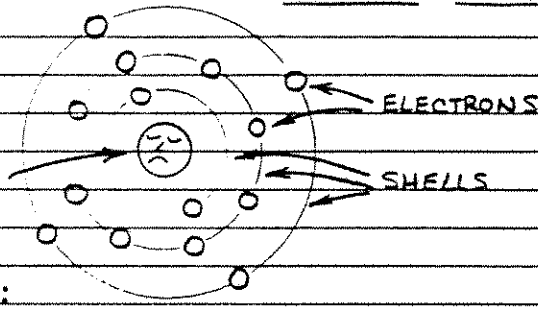
3. SEMICONDUCTORS

THE MOST EXCITING AND IMPORTANT ELECTRONIC COMPONENTS ARE MADE FROM CRYSTALS CALLED SEMICONDUCTORS. DEPENDING ON CERTAIN CONDITIONS, A SEMICONDUCTOR CAN ACT LIKE A CONDUCTOR OR AN INSULATOR.

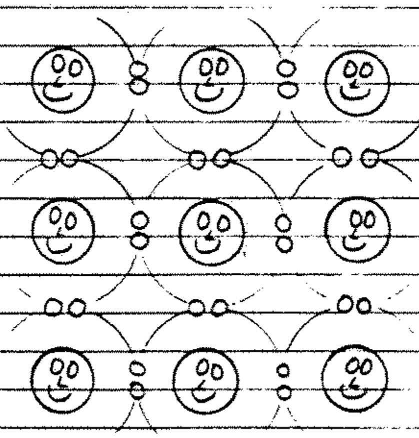
SILICON

THERE ARE MANY DIFFERENT SEMICONDUCTING MATERIALS, BUT SILICON, THE MAIN INGREDIENT OF SAND, IS THE MOST POPULAR.

SILICON ATOM



A SILICON ATOM HAS BUT FOUR ELECTRONS IN ITS OUTERMOST SHELL, BUT IT WOULD LIKE TO HAVE EIGHT. THEREFORE, A SILICON ATOM WILL LINK UP WITH FOUR OF ITS NEIGHBORS TO SHARE ELECTRONS:

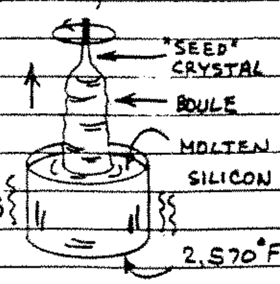


A CLUSTER OF SILICON ATOMS SHARING OUTER ELECTRONS FORMS A REGULAR ARRANGEMENT CALLED A CRYSTAL.

← THIS IS A MAGNIFIED VIEW OF A SILICON CRYSTAL. TO KEEP THINGS SIMPLE, ONLY THE OUTER ELECTRONS OF EACH ATOM ARE SHOWN.

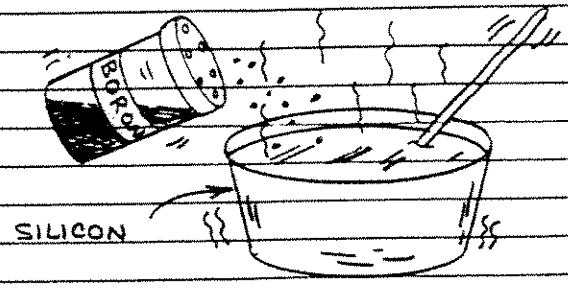
SILICON FORMS 27.7 % OF THE EARTH'S CRUST! ONLY OXYGEN IS MORE COMMON. IT'S NEVER FOUND IN THE PURE STATE. WHEN PURIFIED, IT'S DARK GRAY IN COLOR.

SILICON AND DIAMOND SHARE THE SAME CRYSTAL STRUCTURE AND OTHER PROPERTIES. BUT SILICON IS NOT TRANSPARENT.



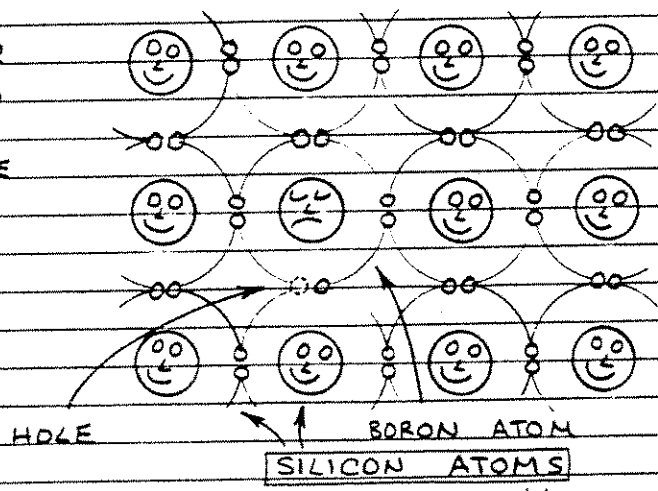
SILICON CAN BE GROWN INTO BIG CRYSTALS. IT'S CUT INTO WAFERS FOR MAKING ELECTRONIC PARTS.

☐ SILICON RECIPES — PURE SILICON ISN'T VERY USEFUL. THAT'S WHY SILICON MAKERS SPICE UP THEIR SILICON RECIPES WITH A DASH OF PHOSPHORUS, BORON OR OTHER GOODIES. THIS IS CALLED DOPING THE SILICON. WHEN GROWN INTO CRYSTALS, DOPED SILICON HAS VERY USEFUL ELECTRONIC PROPERTIES!

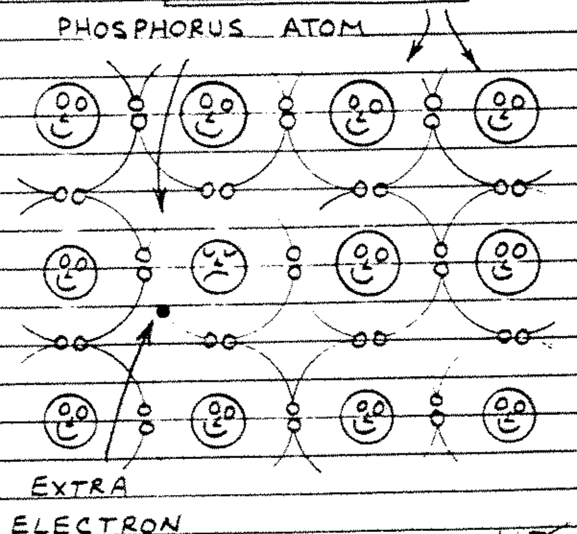


☐ P & N SPICED SILICON LOAF — BORON, PHOSPHORUS AND CERTAIN OTHER ATOMS CAN JOIN WITH SILICON ATOMS TO FORM CRYSTALS. HERS'S THE CATCH: A BORON ATOM HAS ONLY THREE ELECTRON IN ITS OUTER SHELL. AND A PHOSPHORUS ATOM HAS FIVE ELECTRONS IN ITS OUTER SHELL. SILICON WITH EXTRA PHOSPHORUS ELECTRONS IS CALLED N-TYPE SILICON (N = NEGATIVE). SILICON WITH ELECTRON DEFICIENT BORON ATOMS IS CALLED P-TYPE SILICON (P = POSITIVE).

☐ P-TYPE SILICON — A BORON ATOM IN A CLUSTER OF SILICON ATOMS LEAVES A VACANT ELECTRON OPENING CALLED A HOLE. IT'S POSSIBLE FOR AN ELECTRON FROM A NEARBY ATOM TO "FALL" INTO THE HOLE. THEREFORE, THE HOLE HAS MOVED TO A NEW LOCATION. REMEMBER, HOLES CAN MOVE THROUGH SILICON (JUST AS BUBBLES MOVE THROUGH WATER).



☐ N-TYPE SILICON — A PHOSPHORUS ATOM IN A CLUSTER OF SILICON ATOMS DONATES AN EXTRA ELECTRON. THIS EXTRA ELECTRON CAN MOVE THROUGH THE CRYSTAL WITH COMPARATIVE EASE. IN OTHER WORDS, N-TYPE SILICON CAN CARRY AN ELECTRICAL CURRENT. BUT SO CAN P-TYPE SILICON! HOLES "CARRY" THE CURRENT.



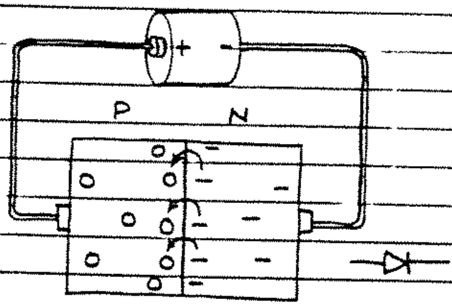
THE DIODE

BOTH P-TYPE AND N-TYPE SILICON CONDUCT ELECTRICITY. THE RESISTANCE OF BOTH TYPES IS DETERMINED BY THE PROPORTION OF HOLES OR SURPLUS ELECTRONS. THEREFORE BOTH TYPES CAN FUNCTION AS RESISTORS. AND THEY WILL CONDUCT ELECTRICITY IN ANY DIRECTION.

BY FORMING SOME P-TYPE SILICON IN A CHIP OF N-TYPE SILICON, ELECTRONS WILL FLOW THROUGH THE SILICON IN ONLY ONE DIRECTION. THIS IS THE PRINCIPLE OF THE DIODE. THE P-N INTERFACE IS CALLED THE PN JUNCTION.

☐ HOW THE DIODE WORKS — HERE'S A SIMPLIFIED EXPLANATION OF HOW A DIODE CONDUCTS ELECTRICITY IN ONE DIRECTION (FORWARD) WHILE BLOCKING THE FLOW OF CURRENT IN THE OPPOSITE DIRECTION (REVERSE).

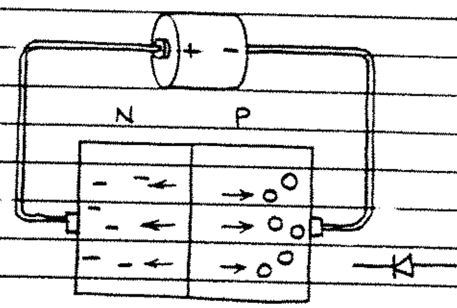
FORWARD BIAS



← ELECTRON FLOW

→ HOLE FLOW

REVERSE BIAS

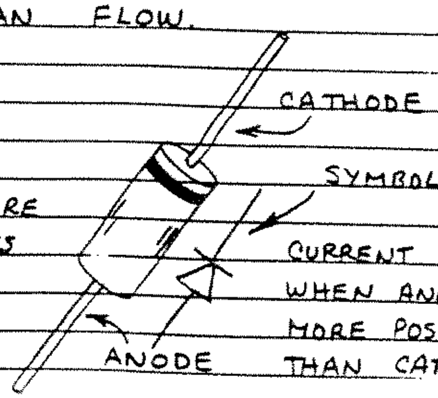


NO CURRENT FLOW

HERE THE CHARGE FROM THE BATTERY REPELS HOLES AND ELECTRONS TOWARD THE JUNCTION. IF THE VOLTAGE EXCEEDS 0.6-VOLT (SILICON), THEN ELECTRONS WILL CROSS THE JUNCTION AND COMBINE WITH HOLES. A CURRENT THEN FLOWS.

HERE THE CHARGE FROM THE BATTERY ATTRACTS HOLES AND ELECTRONS AWAY FROM THE JUNCTION. THEREFORE, NO CURRENT CAN FLOW.

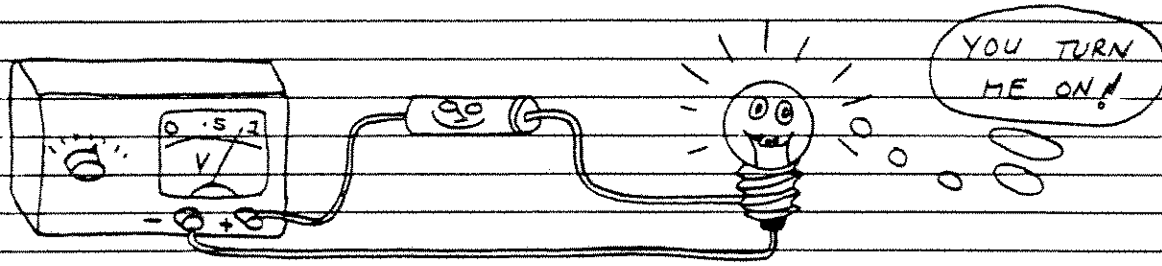
☐ A TYPICAL DIODE — DIODES ARE COMMONLY ENCLOSED IN SMALL GLASS CYLINDERS. A DARK BAND MARKS THE CATHODE TERMINAL. THE OPPOSITE TERMINAL IS THE ANODE

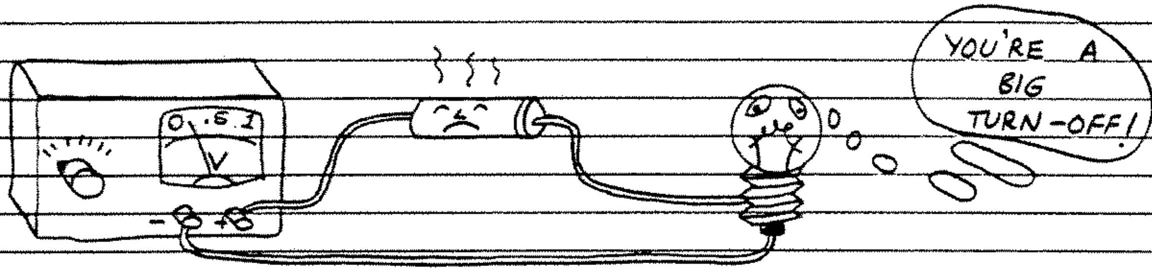


CURRENT FLOWS WHEN ANODE IS MORE POSITIVE THAN CATHODE.

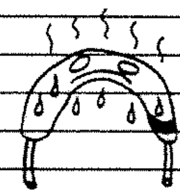
☐ DIODE OPERATION — YOU ALREADY KNOW A DIODE IS LIKE AN ELECTRONIC ONE-WAY VALVE. IT'S IMPORTANT TO UNDERSTAND SOME ADDITIONAL ASPECTS OF DIODE OPERATION. HERE ARE SOME KEY ONES:

1. A DIODE WILL NOT CONDUCT UNTIL THE FORWARD VOLTAGE REACHES A CERTAIN THRESHOLD POINT. FOR SILICON DIODES THIS VOLTAGE IS ABOUT 0.6-VOLT.

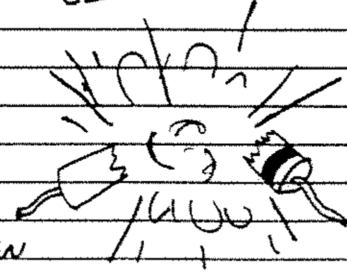




2. IF THE FORWARD CURRENT BECOMES EXCESSIVE, THE SEMICONDUCTOR CHIP MAY CRACK OR MELT! AND THE CONTACTS MAY SEPARATE. IF THE CHIP MELTS, THE DIODE MAY SUDDENLY CONDUCT IN BOTH DIRECTIONS. THE RESULTING HEAT MAY VAPORIZE THE CHIP!

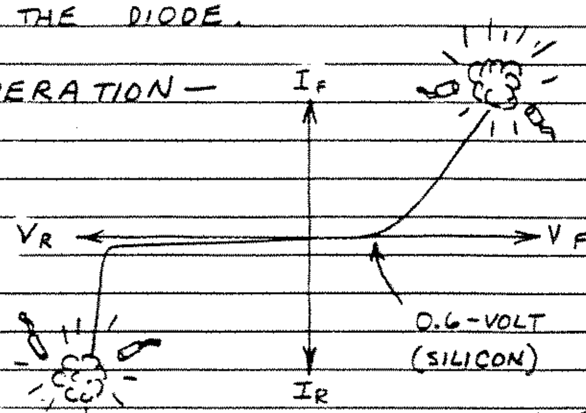


3. TOO MUCH REVERSE VOLTAGE WILL CAUSE A DIODE TO CONDUCT IN THE WRONG DIRECTION. SINCE THIS VOLTAGE IS FAIRLY HIGH, THE SUDDEN CURRENT SURGE MAY ZAP THE DIODE.



☐ SUMMING UP DIODE OPERATION — THIS GRAPH SUMS UP DIODE OPERATION. (IT'S APPROXIMATE.)

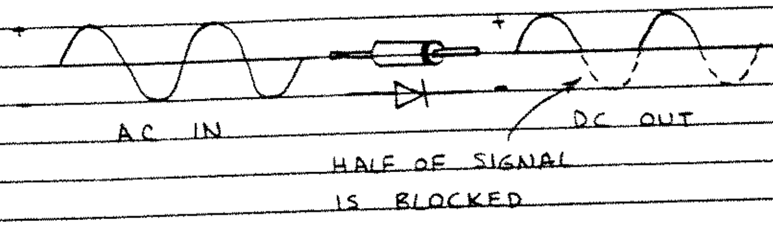
V_F = FORWARD VOLTAGE
V_R = REVERSE VOLTAGE
I_F = FORWARD CURRENT
I_R = REVERSE CURRENT



HOW DIODES ARE USED

IN CHAPTER 9 YOU'LL SEE HOW VARIOUS TYPES OF DIODES ARE USED IN MANY APPLICATIONS. FOR NOW HERE ARE TWO OF THE MOST IMPORTANT ROLES FOR SMALL SIGNAL DIODES AND RECTIFIERS:

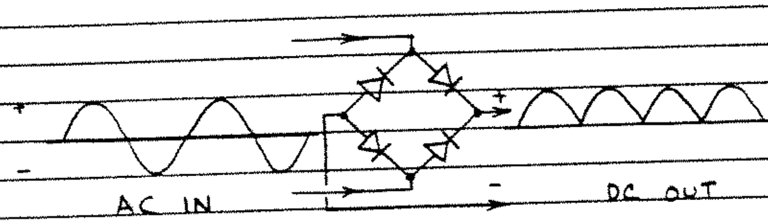
☐ HALF-WAVE RECTIFIER



AN UNDULATING (AC) SIGNAL (OR VOLTAGE) IS RECTIFIED INTO A SINGLE POLARITY (DC) SIGNAL (OR VOLTAGE).

☐ FULL-WAVE RECTIFIER

P. 101



THIS 4-DIODE "NETWORK" (OR BRIDGE RECTIFIER) RECTIFIES BOTH HALVES OF AN AC SIGNAL.

MORE ABOUT THE DIRECTION OF CURRENT FLOW

AN ELECTRICAL CURRENT IS THE MOVEMENT OF ELECTRONS THROUGH A CONDUCTOR OR SEMICONDUCTOR. SINCE ELECTRONS MOVE FROM A NEGATIVELY CHARGED TO A POSITIVELY CHARGED REGION, WHY DOES THE ARROWHEAD IN A DIODE SYMBOL POINT IN THE OPPOSITE DIRECTION? THERE ARE TWO REASONS:

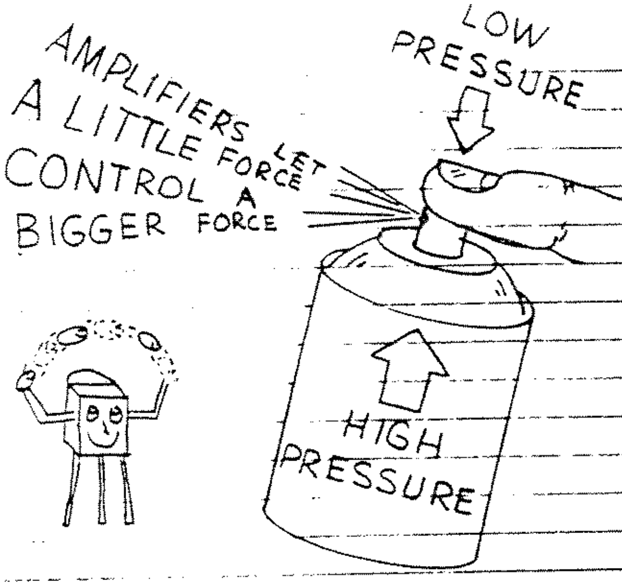
1. BEGINNING WITH BENJAMIN FRANKLIN, IT WAS TRADITIONALLY ASSUMED ELECTRICITY FLOWS FROM A POSITIVELY CHARGED TO A NEGATIVELY CHARGED REGION. THE DISCOVERY OF THE ELECTRON CORRECTED THAT. (BUT MOST ELECTRICAL CIRCUIT DIAGRAMS TODAY STILL FOLLOW THE OLD TRADITION IN WHICH THE POSITIVE POWER SUPPLY CONNECTION IS PLACED ABOVE THE NEGATIVE CONNECTION AS IF GRAVITY SOMEHOW INFLUENCES THE FLOW OF A CURRENT.)

2. IN A SEMICONDUCTOR, AS SHOWN ON PAGE 44, HOLES FLOW IN THE DIRECTION OPPOSITE THAT OF ELECTRON FLOW. IT'S THEREFORE COMMON TO REFER TO POSITIVE CURRENT FLOW IN SEMICONDUCTORS.

FOR ACCURACY, IN THIS BOOK "CURRENT FLOW" REFERS TO ELECTRON FLOW. BUT WE'RE STUCK WITH SYMBOLS THAT INDICATE HOLE FLOW.

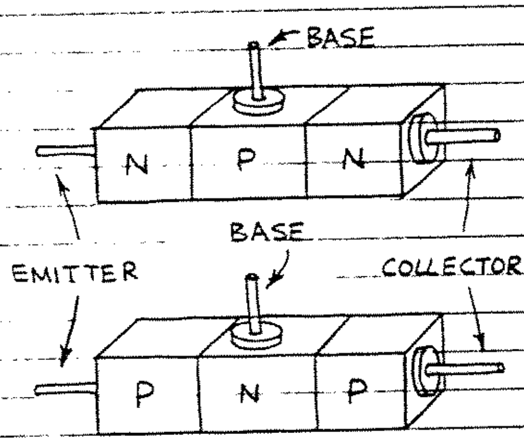
THE TRANSISTOR

TRANSISTORS ARE SEMICONDUCTOR DEVICES WITH THREE LEADS. A VERY SMALL CURRENT OR VOLTAGE AT ONE LEAD CAN CONTROL A MUCH LARGER CURRENT FLOWING THROUGH THE OTHER TWO LEADS. THIS MEANS TRANSISTORS CAN BE USED AS AMPLIFIERS AND SWITCHES. THERE ARE TWO MAIN FAMILIES OF TRANSISTORS: BIPOLAR AND FIELD-EFFECT.

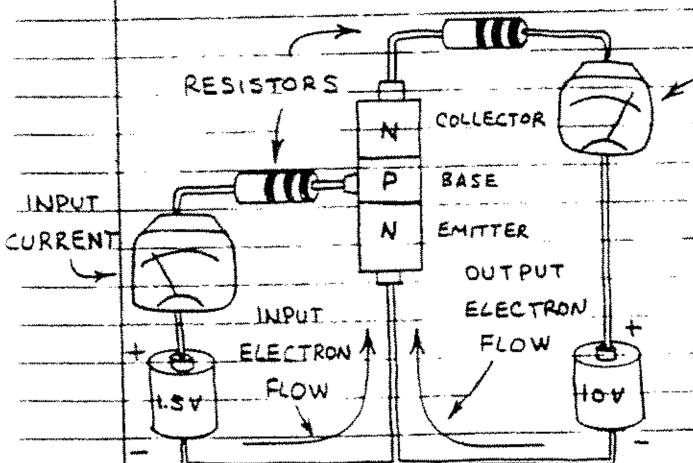


BIPOLAR TRANSISTORS

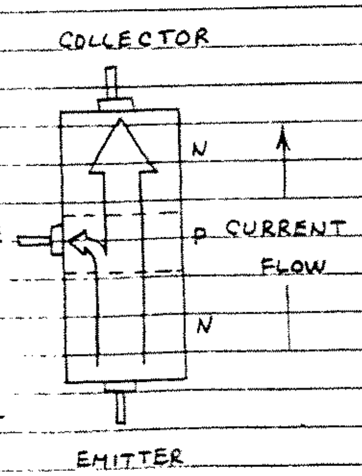
ADD A SECOND JUNCTION TO A PN JUNCTION DIODE AND YOU GET A 3-LAYER SILICON SANDWICH. THE SANDWICH CAN BE EITHER NPN OR PNP. EITHER WAY, THE MIDDLE LAYER ACTS LIKE A FAUCET OR GATE THAT CONTROLS THE CURRENT MOVING THROUGH THE THREE LAYERS.



☐ BIPOLAR TRANSISTOR OPERATION — THE THREE LAYERS OF A BIPOLAR TRANSISTOR ARE THE EMITTER, BASE AND COLLECTOR. THE BASE IS VERY THIN AND HAS FEWER DOPING ATOMS THAN THE EMITTER AND COLLECTOR. THEREFORE A VERY SMALL EMITTER-BASE CURRENT WILL CAUSE A MUCH LARGER EMITTER-COLLECTOR CURRENT TO FLOW.



THE RESISTORS PROTECT THE TRANSISTOR FROM TOO MUCH CURRENT (WHICH CAN CAUSE EXCESSIVE HEAT).

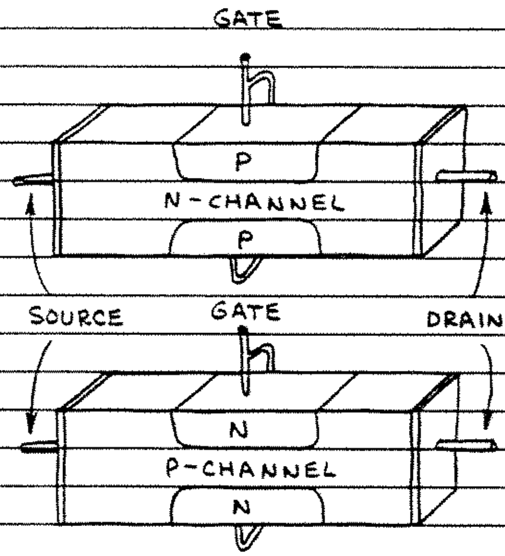


FIELD-EFFECT TRANSISTORS

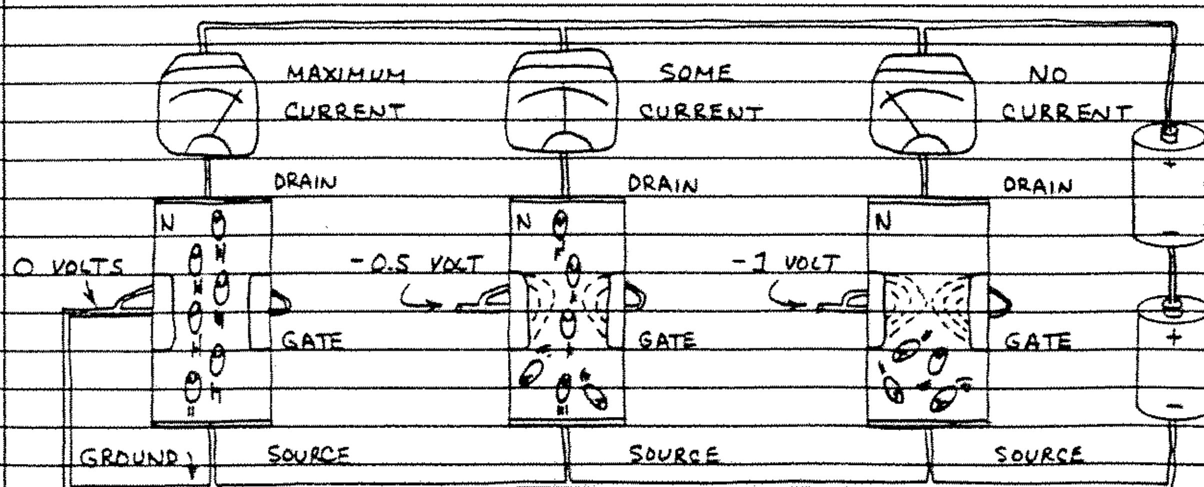
FIELD-EFFECT TRANSISTORS (OR FETs) HAVE BECOME MORE IMPORTANT THAN BIPOLAR TRANSISTORS. THEY ARE EASY TO MAKE AND REQUIRE LESS SILICON. THERE ARE TWO MAJOR FET FAMILIES, JUNCTION AND METAL-OXIDE-SEMICONDUCTOR. IN BOTH KINDS AN OUTPUT CURRENT IS CONTROLLED BY A SMALL INPUT VOLTAGE AND PRACTICALLY NO INPUT CURRENT!

JUNCTION FETs

THE TWO MAIN KINDS OF FETs ARE N-CHANNEL AND P-CHANNEL. THE CHANNEL IS LIKE A SILICON RESISTOR THAT CONDUCTS CURRENT MOVING FROM THE SOURCE TO THE DRAIN. A VOLTAGE AT THE GATE INCREASES THE CHANNEL RESISTANCE AND REDUCES THE DRAIN-SOURCE CURRENT. THEREFORE THE FET CAN BE USED AS AN AMPLIFIER OR A SWITCH.

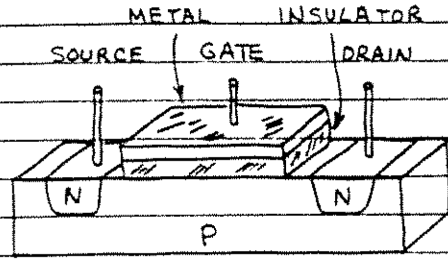


☐ JUNCTION FET OPERATION — THE ARRANGEMENT BELOW SHOWS HOW AN N-CHANNEL FET WORKS. A NEGATIVE GATE VOLTAGE CREATES TWO HIGH RESISTANCE REGIONS (THE FIELD) IN THE CHANNEL ADJACENT TO THE P-TYPE SILICON. MORE GATE VOLTAGE WILL CAUSE THE FIELDS TO MERGE TOGETHER AND COMPLETELY BLOCK THE CURRENT. THE GATE-CHANNEL RESISTANCE IS VERY HIGH.

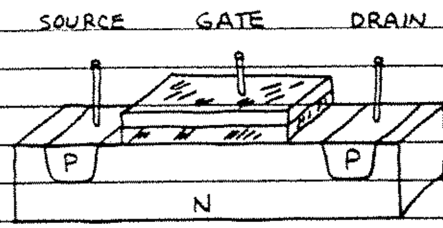


METAL-OXIDE-SEMICONDUCTOR FETs

THE METAL-OXIDE-SEMICONDUCTOR FET (OR MOSFET) HAS BECOME THE MOST IMPORTANT TRANSISTOR. MOST MICROCOMPUTER AND MEMORY INTEGRATED CIRCUITS ARE ARRAYS OF THOUSANDS OF MOSFETs ON A SMALL SLIVER OF SILICON. WHY? MOSFETs ARE EASY TO MAKE, THEY CAN BE VERY SMALL, AND SOME MOSFET CIRCUITS CONSUME NEGLIGIBLE POWER. NEW KINDS OF POWER MOSFETs ARE ALSO VERY USEFUL.

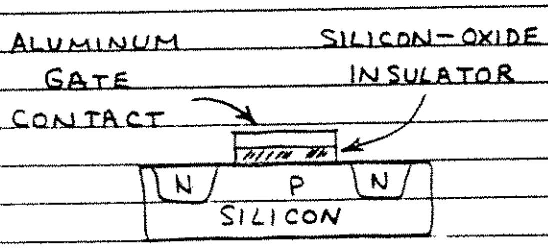


N-MOSFET

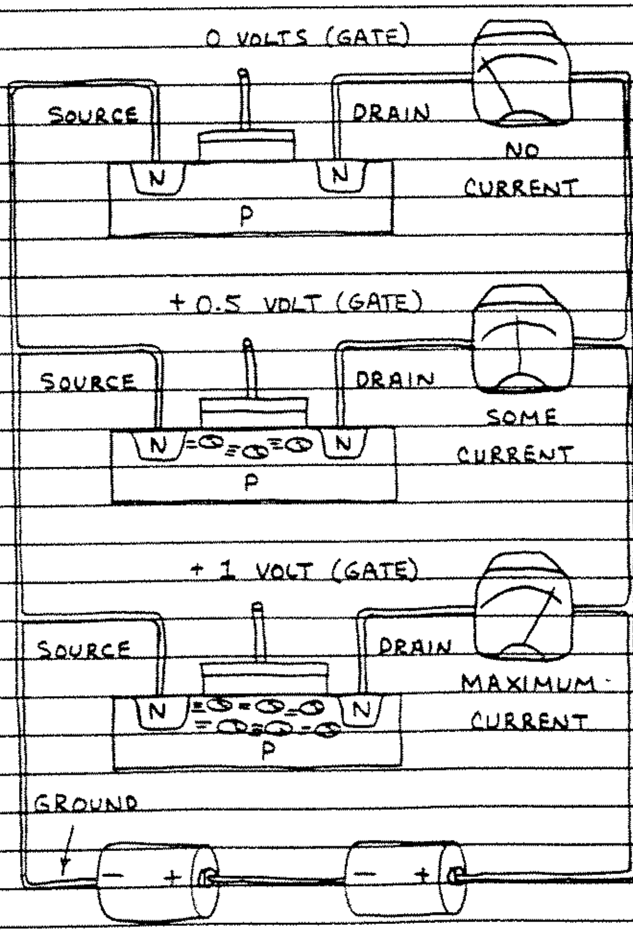


P-MOSFET

☐ MOSFET OPERATION — ALL MOSFETs ARE N-TYPE OR P-TYPE. UNLIKE THE JUNCTION FET, THE GATE OF A MOSFET HAS NO ELECTRICAL CONTACT WITH THE SOURCE AND DRAIN. A GLASS-LIKE LAYER OF SILICON-DIOXIDE (AN INSULATOR) SEPARATES THE GATE'S METAL CONTACT FROM THE REST OF THE TRANSISTOR.



A POSITIVE GATE VOLTAGE ATTRACTS ELECTRONS TO THE REGION BELOW THE GATE. THIS CREATES A THIN N-TYPE CHANNEL IN THE P-TYPE SILICON BETWEEN THE SOURCE AND DRAIN. CURRENT CAN THEN FLOW THROUGH THE CHANNEL. THE GATE VOLTAGE DETERMINES THE RESISTANCE OF THE CHANNEL.



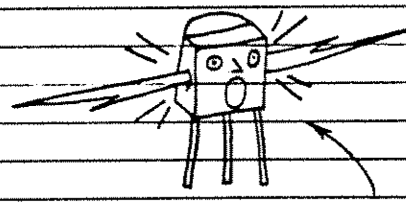
☐ MORE ABOUT MOSFETS — THE INPUT RESISTANCE OF THE MOSFET IS THE HIGHEST OF ANY TRANSISTOR. THIS AND OTHER FACTORS GIVE MOSFETS IMPORTANT ADVANTAGES:

1. THE GATE-CHANNEL RESISTANCE IS ALMOST INFINITE (TYPICALLY 1,000,000,000,000,000-OHMS). THIS MEANS THE GATE PULLS NO CURRENT FROM EXTERNAL CIRCUITS. (WELL, IT MAY BORROW A FEW TRILLIONTHS OF AN AMPERE.)

2. MOSFETS CAN FUNCTION AS VOLTAGE-CONTROLLED VARIABLE RESISTORS. THE GATE VOLTAGE CONTROLS CHANNEL RESISTANCE.

3. NEW KINDS OF MOSFETS CAN SWITCH VERY HIGH CURRENTS IN A FEW BILLIONTHS OF A SECOND.

☐ CAUTION — BECAUSE THE GLASS-LIKE SILICON OXIDE LAYER BELOW THE GATE IS SO THIN, IT CAN BE PIERCED BY TOO MUCH VOLTAGE OR EVEN STATIC ELECTRICITY. EVEN THE STATIC CHARGE GENERATED BY CLOTHING OR A CELLOPHANE WRAPPER CAN ZAP THE GATE OF A MOSFET!

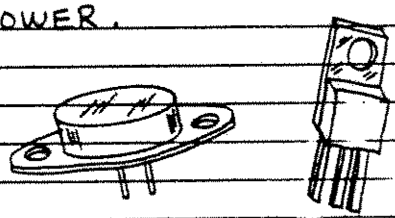


ZAPPED MOSFET

CAUTION SYMBOL

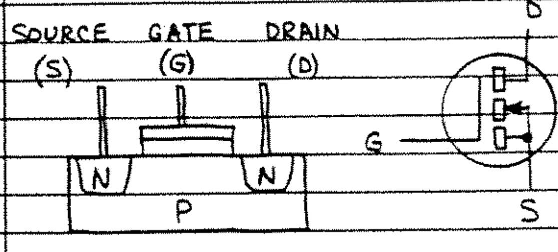
☐ KINDS OF MOSFETS — LIKE JFETS, MOSFETS INSTALLED IN SMALL METAL OR PLASTIC PACKAGES ARE USED TO GIVE AMPLIFIERS AN ULTRA-HIGH INPUT RESISTANCE. THEY ARE ALSO USED AS VOLTAGE CONTROLLED RESISTORS AND SWITCHES. THE MOST IMPORTANT CATEGORY HAS BECOME:

POWER.

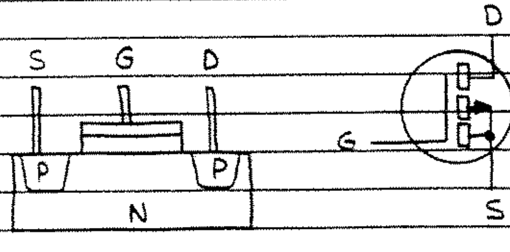


POWER MOSFETS ALLOW A FEW VOLTS TO SWITCH OR AMPLIFY MANY AMPERES AT VERY FAST SPEEDS.

☐ MOSFET SYMBOLS — THESE ARE THE MOST COMMON.



N-MOSFET



P-MOSFET

Circuit Analysis Review

Main Electrical Quantities:

1) **Charge** — 1 Coulomb (C) = 6.25×10^{18} electrons
   1 electron (e) = 1.6×10^{-19} C

2) **Current** — $I = dQ/dt$
   1 Amp (A) = 1 C/sec

3) **Voltage** — V = energy/charge ; potential difference
   1 Volt (V) = 1 Joule (J)/C

4) **Power** — P = work/time
   1 Watt (W) = 1 J/sec

5) **Resistance** (Impedance (Z) for AC) — $R = V/I$
   1 Ohm (Ω) = 1 V/A

6) **Conductance** (Admittance (Y) for AC) — $G = 1/R$
   1 Mho ($\mho$) = 1 A/V
   (1 Siemen = 1 Mho)

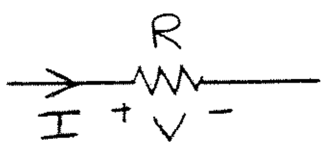
Ohm's Law

$$V = IR$$
$$I = V/R$$
$$R = V/I$$

Power Equations

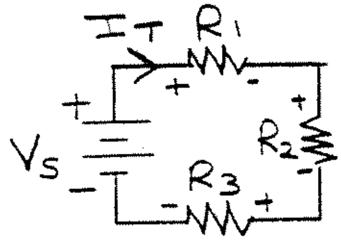
$$P = VI = I^2R = V^2/R$$

Conventional Current Polarity



Series Resistive Circuits

$$R_T = R_1 + R_2 + R_3$$
$$P_T = P_1 + P_2 + P_3$$



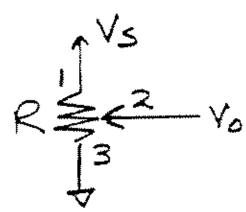
$$I_T = I_{R_1} = I_{R_2} = I_{R_3} = V_S/R_T$$

$$V_{R_1} = \underbrace{I_T R_1}_{\text{Ohm's Law}} = \underbrace{V_S\left(\frac{R_1}{R_T}\right)}_{\text{Voltage Divider Eq.}}$$

Kirchhoff's Voltage Law: Σ voltages in closed loop = 0
(KVL)
or Net Voltage supplied = Σ voltage drops

$$V_S = V_{R_1} + V_{R_2} + V_{R_3}$$

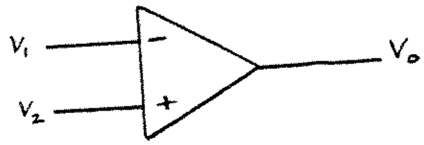
Potentiometer:



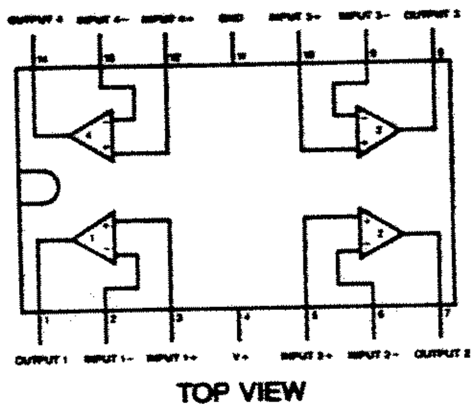
$$R = R_{12} + R_{23}$$
$$V_o = V_S\left(\frac{R_{23}}{R}\right)$$

1 Operational-Amplifiers

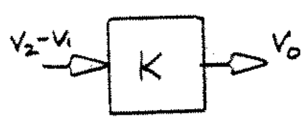
Operational amplifiers are the most common elements of analog circuits. Nearly all electronic controllers use some op-amp circuit to provide the desired signal conditioning. We will use these devices for many of our experiments. The schematic symbol is



You can buy electronic chips with four op-amps on each chip for about $1.25 each. We use a such a quad op-amp chip in the lab called an LM324. The four amplifiers are connected to the output leads as:



Power must be input to the chip through a positive supply voltage at pin 4, and a negative voltage or ground at pin 11. Circuits that use op-amps are sometimes called *active filters* because power may be added to the circuit by the amplifier. Circuits that don't use op-amps, such as R-C networks are called *passive filters* because they can only absorb power. The voltages v_1, v_2, and v_0 in the schematic diagram are related by the block diagram



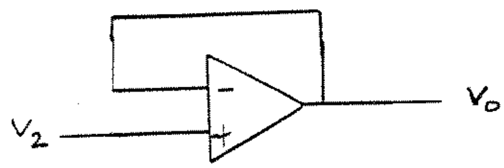
or

$$v_0 = K(v_2 - v_1), \tag{1}$$

where K is a large gain ($K \approx 1 \times 10^5$). In reality, the op-amp has some dynamics which means that K is a function of frequency, or $K = K(s)$, but we assume it acts as a pure gain, so the dynamics are neglected.

1.1 A Voltage Follower

Suppose we hook v_0 up to v_1



hence $v_1 = v_0$ and, using (1)

$$v_0 = K(v_2 - v_0)$$

or

$$v_0 = \frac{K}{(1+K)} v_2 \approx v_2,$$

since $\frac{K}{(1+K)} \approx 1$ for large values of K. This is called a *voltage follower* circuit. With this circuit $v_0(t) = v_2(t)$ for all ~~but~~ voltages except very quickly time varying signals. As explained below, this simple circuit is used in almost every electronic instrument because the output $v_2(t)$ draws no current from the input $v_0(t)$. For fast signal variations, the dynamics of the amplifier gain $K(s)$ can not be ignored. You will test this in Experiment 2.

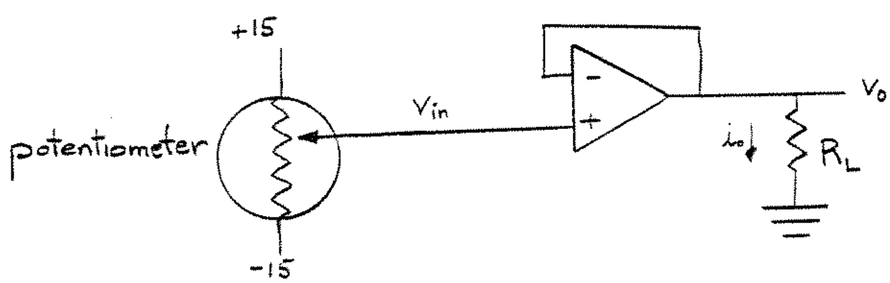
The inputs to the op-amp v_1 and v_2 are applied to the base of internal field effect transistors (FET) which require no current at their base to operate, they only use the field produced by the electric potential. Hence, no current flows into the + and - inputs. An analysis similar to what we used for the voltage follower will show that whenever there is any feedback path from v_0 to v_1, the amplifier forces v_1 to equal v_2. The feedback path can be through any network. Thus, to analyze op-amp circuits that have negative feedback (a path from v_0 to v_1), we always make the following basic assumptions.

1.2 Three Op-Amp Assumptions.

Use these assumptions whenever negative feedback is present in an op-amp circuit.

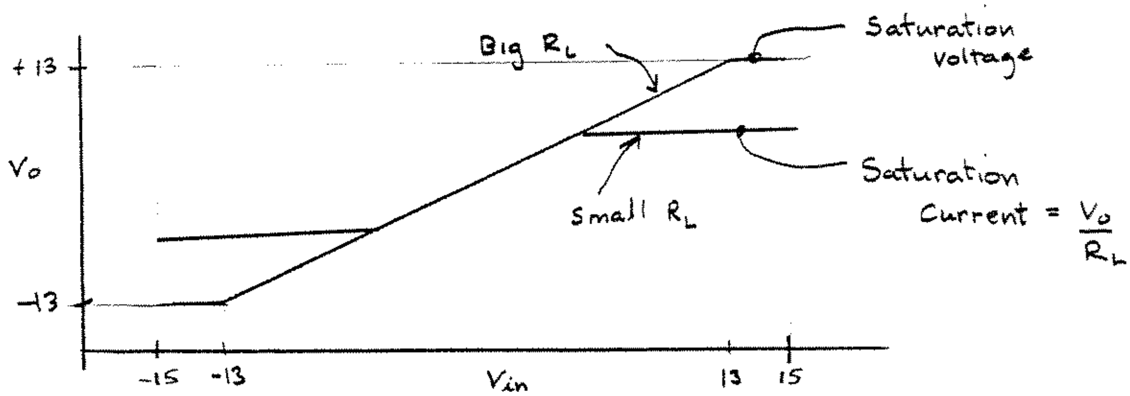
1. $v_1 = v_2$.
2. No current flows into the + or - inputs.
3. The op-amp is an ideal gain, or $K(s) = K$, a constant

For the lab experiment we make the circuit:



If the op-amp is working correctly $v_o = v_{in}$ regardless of R_L. The output current is $i_o = \frac{v_o}{R_L}$. Our op-amps can only output about $i_o = 0.020$ amp before the internal transistors *saturate*, or reach their current limit. This means that for small loads (high R_L) everything works fine, and for high loads (low R_L) the operational amplifier no longer amplifies as it should. You will find the saturation current in the second experiment.

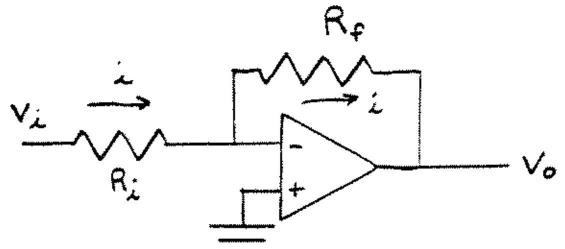
A plot of v_{out} versus v_{in} for some load R_L might look like



The voltage follower circuit acts like an *ideal voltage source* as long as i_o is below the saturation current. Note that the input voltage v_{in} is not affected by changes in i_o and R_L, because no current can flow into the op-amp. That is why the voltage follower is also called a *buffer* or an *isolation amplifier.*

1.3 An Inverting Gain

Another useful circuit is the *inverting gain.*



Using op-amp assumptions 1 and 2, we see that

$$\frac{v_i}{R_i} = i = \frac{(0 - v_o)}{R_f}$$

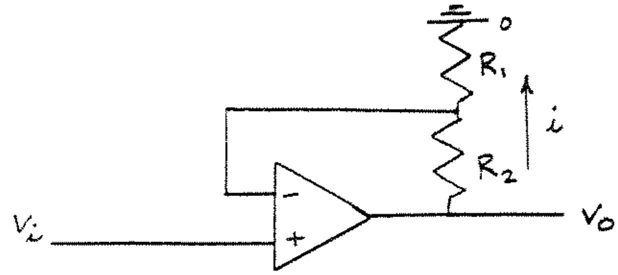
or

$$v_o = -v_i \frac{R_f}{R_i}$$

In reality this equation remains true for any v_i that causes v_o to be less than some *saturation voltage* that is determined by the voltage used to provide power to the op-amps. The operational amplifier can not do its job of maintaining a linear input-output voltage relationship if the input voltage is trying to produce an output voltage is beyond the saturation level.

1.4 A Noninverting Gain.

A circuit performs the same function as the last, but with no minus sign is the *noninverting gain.*



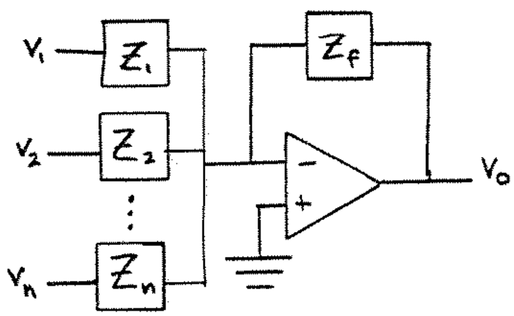
The voltage at the minus input is the voltage between the two resistors, which is $v_o \frac{R_1}{(R_1+R_2)}$. Then, using assumption 1,

$$v_o = \frac{(R_1 + R_2)}{R_1} v_i .$$

Note that the *gain* of this circuit is always greater than one, while for the previous circuit this is not the case.

1.5 A General Circuit.

One general op-amp circuit that is commonly used has the form



where the $Z_i(s)$ are general impedances of networks of capacitors and/or inductors. Using assumptions 1 and 2 we have

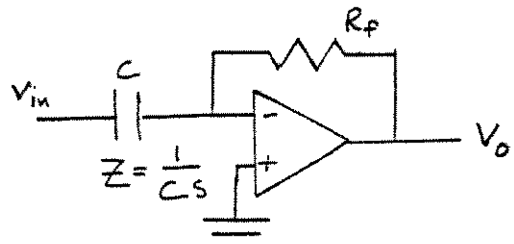
$$-\frac{V_o(s)}{Z_f(s)} = \frac{V_1}{Z_1} + \frac{V_2}{Z_2} + \cdots + \frac{V_n(s)}{Z_n(s)}$$

or

$$V_o(s) = -Z_f \sum_{i=1}^{n} \frac{V_i(s)}{Z_i(s)}. \tag{2}$$

1.6 Example: A Differentiator.

For instance, if there is just one input to a capacitor,



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then (2) gives,

$$V_o(s) = -R_f C s V_{in}(s).$$

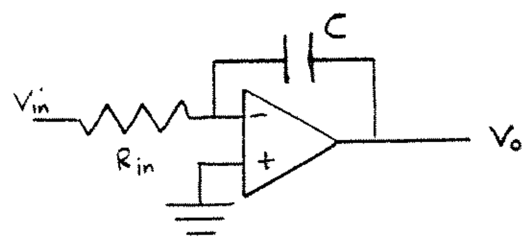
Note that the above equation is in the frequency domain. In the time domain, the same equation is

$$v_o = -R_f C \frac{dv_{in}}{dt}. \tag{3}$$

Hence, this circuit takes the time derivative of a *any* input voltage as a function of time, provided that it is differentiable. If it is not differentiable, then noise is output.

1.7 Example: An Integrator.

If we swap the resistor and the capacitor of the above circuit we get an integrator.



(2) gives,

$$V_o(s) = -\frac{1}{Cs}\frac{V_{in}(s)}{R_{in}}.$$

In the time domain, the same equation is

$$v_o = -\frac{1}{R_{in}C}\int v_{in}\,dt.$$

Because of the integration property of this circuit, if we input a square wave to this system, we get out a triangle wave.

1.8 The Concept of Lead and Lag

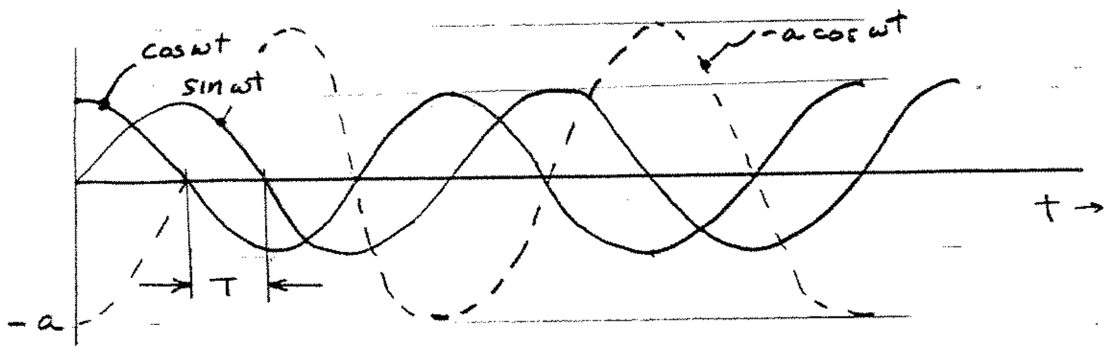
Suppose a sin wave is input from a function generator into the above differentiator circuit. That is,

$$v_{in} = a \sin \omega t,$$

then, the above analysis in (3) shows that

$$v_o = -R_f C a \omega \cos \omega t$$

We see that, although the output has the same frequency as the input, the amplitude of the output depends on the frequency, and the *phase* of the output is different than the input. To understand the phase relationship, consider the three waves sketched below.



The function $\cos \omega t$ is said to *lead* $\sin \omega t$ by 90°. The sin wave is behind (or *lags*) the cos wave by the time corresponding to $\omega T = \pi/2 (= 90^\circ)$. If it were shifted ahead in time by T seconds it is said to be *in phase* with the cos wave. This is stated mathematically as

$$\cos \omega t = \sin(\omega t + \frac{\pi}{2}).$$

The wave coming out of the circuit above is a negative cos wave, which lags the input sin wave by 90°. Finally, note that it is also correct to say that this wave *leads* the input by 270°, since $\sin \theta = \sin(\theta \pm 2\pi)$.

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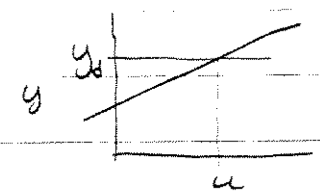
The concept of feedback

Feedback is a measurement of a system's output. You can use this feedback to change the input to the system to make it do what you want it to. For instance, suppose

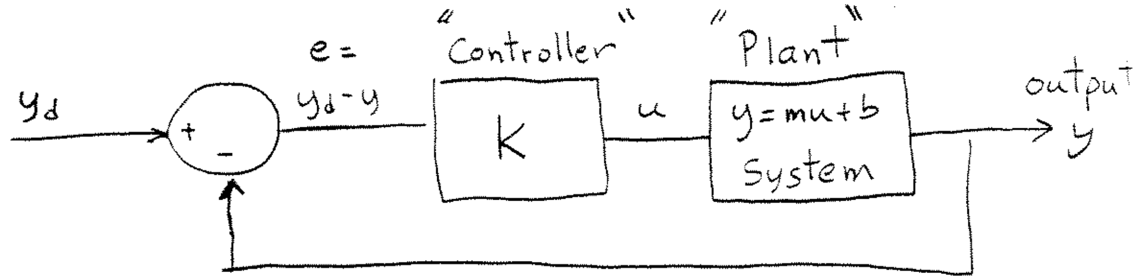


$$y \cong mu + b \tag{1}$$

where y is the output of some system, and u is the input. Let m and b be highly uncertain constants. They might change with time or temperature. What u makes the output y close to y_d (d means desired)?

Feedback can be used to make $y \to y_d$ for any m or b!

Block diagram:



Let $u = K(y_d - y)$.

From (1)

$$y = mK(y_d - y) + b$$

$$y(1 + mK) = mK y_d + mK b$$

$$(2) \qquad y = \frac{mk\, y_d}{(1+mk)} + \frac{mk\, b}{(1+mk)}.$$

By letting $K \to$ big, $y \to y_d$. In the limit

$$\lim_{K \to \infty} y = y_d$$

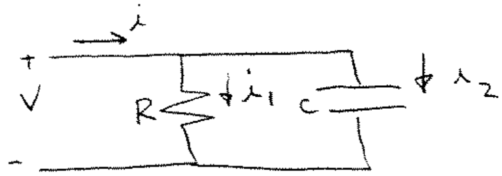
So you don't even need to know m or b!

Notes For Lab 2

The impedance, and transfer funtions are usually used to solve for the <u>forced</u> response of the system. Recall initial conditions are set to zero.

For the transient response, or free response, we need to include initial conditions.

In Exp 2, we have



We need $V(t)$, given $i=0$ and $V(0) = V_o$.

$i = i_1 + i_2 \qquad i_1 = \frac{V}{R} \qquad \int i_2 dt = CV \Rightarrow i_2 = C\frac{dV}{dt}$

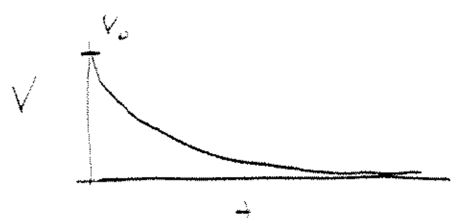
$0 = \frac{V}{R} + C\frac{dV}{dt}$

$\underline{\frac{dV}{dt} = -\frac{1}{RC}V}$

This is like $\dot{x} = ax + b$
with $b = 0 \quad a = -\frac{1}{RC}$
Solution $x(t) = e^{at}x(0)$

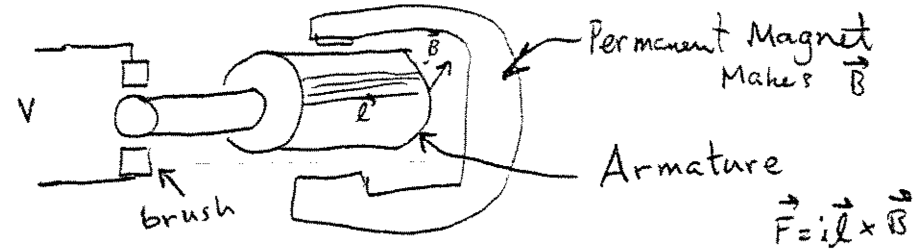
$$\downarrow$$

$V(t) = e^{-\frac{t}{RC}}V(0)$



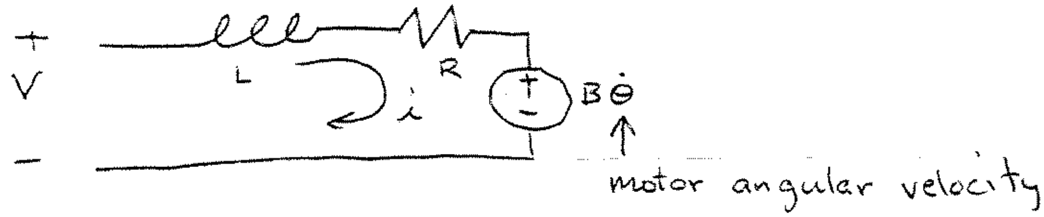
$\frac{1}{RC}$ is the <u>time constant</u>

Notes for Lab 3



DC Motors

We use DC motors in many experiments and projects; the math model is



So $V = L\frac{di}{dt} + Ri + B\dot{\theta}$. Also, Torque = Ki for all $\dot{\theta}$.

- If you hold the motor shaft fixed $\dot{\theta} = 0$, and apply a constant voltage $\frac{di}{dt} \to 0$ (Why?),

So the motor acts like a pure resistor, with $V = Ri$

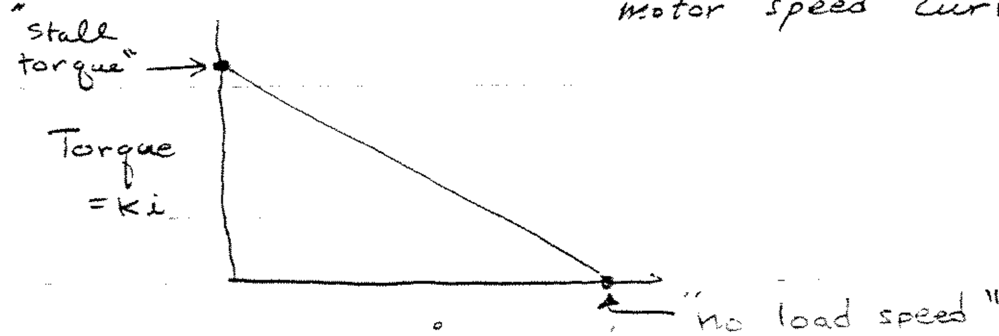
The torque you feel is called the "stall torque" and $T_{stall} = K i_{stall} = K\frac{V}{R}$

- If the shaft can spin freely, then Torque = small, and i = small, so the motor speeds up until

$i \cong 0$ $\frac{di}{dt} = 0$ and

$V \cong B\dot{\theta}$ called the "no load speed"

In summary, for $\frac{di}{dt} = 0$, and V = constant, you get the motor speed curve:



First Order Systems

Notes for Lab 4

Many systems have the block diagram

which corresponds to the differential equation:

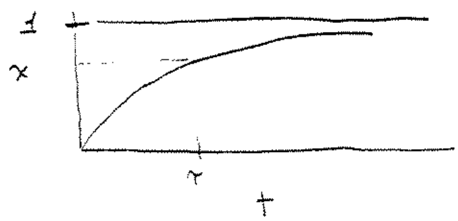
$$\dot{x} = -\frac{1}{\tau}x + \frac{u(t)}{\tau}.$$

This is a first order system with forcing function $u(t)$ and <u>time constant</u> τ. If we apply a unit step function $U(s) = \frac{1}{s}$ to the system with zero initial conditions,

$$X = \frac{1}{s(\tau s+1)} = \frac{1}{s} - \frac{1}{s+\frac{1}{\tau}}$$

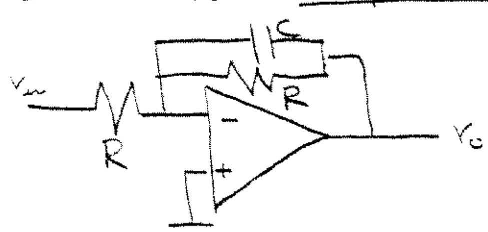
or,

$$x(t) = 1\,u(t) - e^{-\frac{t}{\tau}}.$$



when $t = \tau$ seconds, the output $x(\tau) = 1 - e^{-1} = .632$, which is 63% of the way to the final value $x(\infty) = 1$. Hence τ seconds is a measure of how fast the system responds to changes in the input. τ can be many hours for thermal systems or micro seconds for electical system

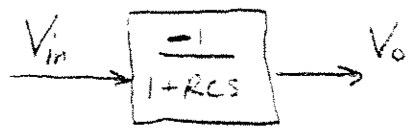
In experiment 2 we wire a <u>low pass filter</u> as



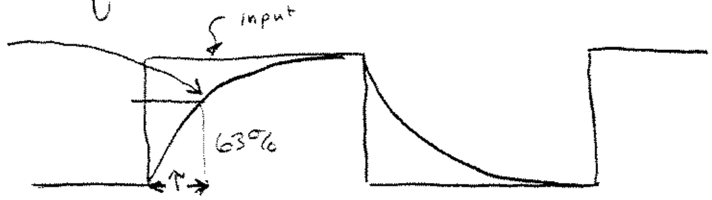
$$Z_F = \frac{R}{1+RCs}$$

$$Z_{in} = R$$

$$V_o = -\frac{Z_F}{Z_{in}}V_{in} = \frac{-1}{1+RCs}V_{in}$$



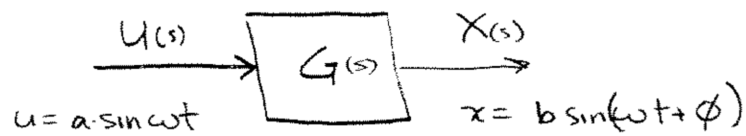
If we input a square wave to the circuit, the output will look like

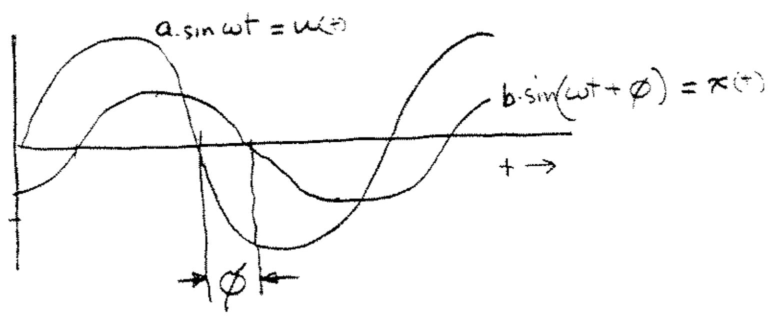


so we can measure τ from the output.
By making this measurement we have identified the system using time-domain measurements. We can also use frequency domain methods as follows.

Frequency Response

The most important concept of this class is the notion of frequency response of a dynamic system. If we input a sine wave to a stable linear system, the output will be a sine wave of the same frequency with a different amplitude and phase.





For the signals shown $x(t)$ lags $u(t)$ by $\phi°$; and this means that $\phi < 0$.

We can predict the output amplitude b and phase ϕ if we have a mathematical model of the system. Consider a general n^{th} order linear system

$$\frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1} x}{dt^{n-1}} + \cdots a_0 x = b_{n-1}\frac{d^{n-1} u}{dt^{n-1}} + b_{n-2}\frac{d^{n-2} u}{dt^{n-2}} \cdots b_0 u$$

where the a_i and b_i are constant coefficients. Now take the Laplace Transform

$$X\underbrace{\left(s^n + a_{n-1}s^{n-1} + \cdots a_0\right)}_{\overset{\parallel}{A(s)}} = U\underbrace{\left(b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots b_0\right)}_{\overset{\parallel}{B(s)}} + IC(s),$$

where $IC(s)$ is a polynomial in s that arises from the initial conditions. Solve of $X(s)$,

$$X = \frac{B(s)}{A(s)}U + \frac{IC(s)}{A(s)}$$

$A(s)$ is called the <u>characteristic</u> <u>polynomial</u> of the system. If we solve for the n roots of $A(s)$, $s_1, s_2 \cdots s_n$, we can write it in factored form as

$$A(s) = (s-s_1)(s-s_2)\cdots(s-s_n).$$

A partial fraction expansion of $X(s)$ with $U(s)=0$ will then be a function of the form

$$x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} \cdots c_n e^{s_n t} \qquad \left(\mathcal{L}\left[e^{-at}\right] = \frac{1}{s+a}\right)$$

So if $Re(s_i) < 0$ the output $x(t)$ decays exponetially when

The system $\frac{B(s)}{A(s)}$ is said to be <u>stable</u>. Equivalently, if all the roots of the characteristic polynomial $A(s)$ lie in the left half s plane the system is stable.

Now, assume we have a stable system, and we apply the input

$$u(t) = a \cdot \sin \omega t$$

or

$$U(s) = \frac{a\omega}{s^2+\omega^2} \; .$$

Then

$$X(s) = \frac{B(s)}{A(s)} \frac{a\omega}{s^2+\omega^2} + \underbrace{\frac{IC(s)}{A(s)}}_{\text{decays to zero}}$$

For any input, the contribution of the response due to the initial conditions decays to zero since $A(s)$ is stable.
Now look at the forced part of the response

$$X(s) = \frac{B(s)}{A(s)} \frac{a\omega}{s^2+\omega^2} = \frac{K_1}{(s+j\omega)} + \frac{k_2}{(s-j\omega)} + \frac{k_3}{(s-s_1)} + \frac{k_4}{(s-s_2)} + \cdots$$

roots of $s^2+\omega^2$ — roots of $A(s)$

Take the inverse Laplace Transform

$$x(t) = K_1 e^{-j\omega t} + K_2 e^{j\omega t} + \underbrace{k_3 e^{s_1 t}}_{\to 0} + \underbrace{k_4 e^{s_2 t}}_{\to 0} + \cdots$$

The terms $e^{s_i t}$ decay to zero since $Re(s_i) < 0$.

$$K_1 = \frac{A(s)}{B(s)} \frac{a\omega}{s^2+\omega^2} (s+j\omega) \Bigg|_{s=-j\omega} = \frac{A(-j\omega)}{B(-j\omega)} \frac{a\omega}{-2j\omega} = -\frac{a}{2j} G(-j\omega)$$

where $G(s) = \frac{B(s)}{A(s)}$.

Similarly,

$$K_2 = \frac{a}{2j} G(j\omega)$$

Now express $G(j\omega) = |G(j\omega)| e^{j\phi}$, and note that $G(-j\omega)$ is the complex conjugate of $G(j\omega)$ or $G(-j\omega) = |G(j\omega)| e^{-j\phi}$

So

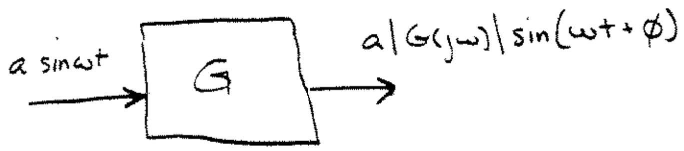
$$x(t) = \underbrace{-\frac{a}{2j} |G| e^{-j\phi}}_{K_1} e^{-j\omega t} + \underbrace{\frac{a}{2j} |G| e^{j\phi}}_{K_2} e^{j\omega t}$$

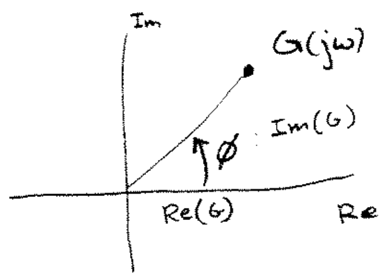
$$= \frac{a}{2j} |G| \left(- e^{-j(\omega + \phi)t} + e^{j(\omega+\phi)t} \right).$$

but $\sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$, so

$$x(t) = a\,|G(j\omega)| \sin(\omega t + \phi).$$

In summary,





where $\phi = \angle G(j\omega) = \tan^{-1}\left(\frac{\mathrm{Im}\, G(j\omega)}{\mathrm{Re}\, G(j\omega)}\right)$.

Hence $|G(j\omega)| = \frac{\text{amplitude of output}}{\text{amplitude of input}} \equiv$ Amplitude ratio.

Consider the system of experiment 2.

$$G(j\omega) = \frac{1}{\tau j\omega + 1} = \frac{1 \angle 0}{\sqrt{1+(\tau\omega)^2} \angle \tan^{-1} \tau\omega}$$

$$|G(j\omega)| = \frac{1}{\sqrt{1+\tau\omega^2}} \qquad \angle G(j\omega) = 0 - \angle \tan^{-1} \tau\omega$$

At low frequencies $\omega \ll \frac{1}{\tau}$

$$|G| \cong 1 \qquad \phi \cong 0$$ (hence the term <u>low pass</u> <u>filter</u>)

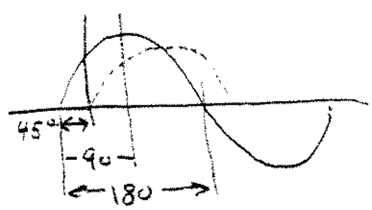
At high frequencies $\omega \gg \frac{1}{\tau}$

$$|G| \longrightarrow 0 \qquad \phi \longrightarrow -90°$$

When $\omega = \frac{1}{\tau}$

$$|G(\tfrac{j}{\tau})| = \frac{1}{\sqrt{2}} \qquad \phi = -\tan^{-1}(1) = -45°$$.

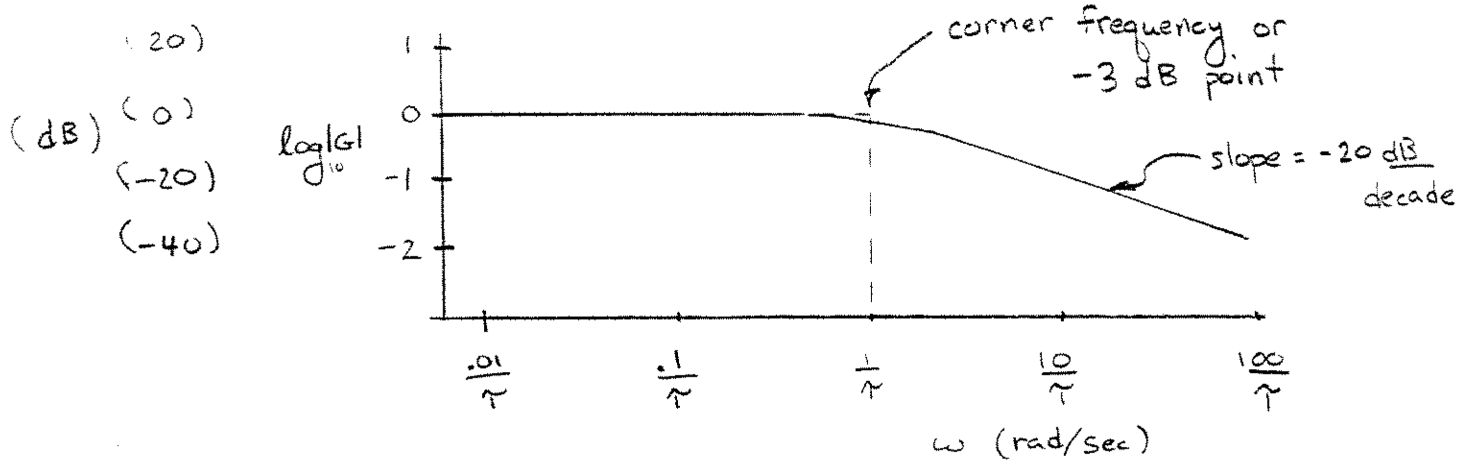
So the amplitude of the output wave when $\omega = \frac{1}{\tau}$ is .707 x (amplitude of the input wave), and the output <u>lags</u> the input by 45°.



The frequency $\omega = \frac{1}{\tau}$ is called the <u>corner</u> <u>Frequency</u> or <u>Bandwidth</u> of this system.

Input Frequencies higher than the bandwidth are filtered or attenuated.

We can also plot $|G(j\omega)|$ and ϕ on a <u>Bode Plot</u>. This is a log-log plot of $\log_{10}|G(j\omega)|$ versus ω. For our example, the magnitude plot is:



Bode Plot of $G = \frac{1}{\tau s + 1}$

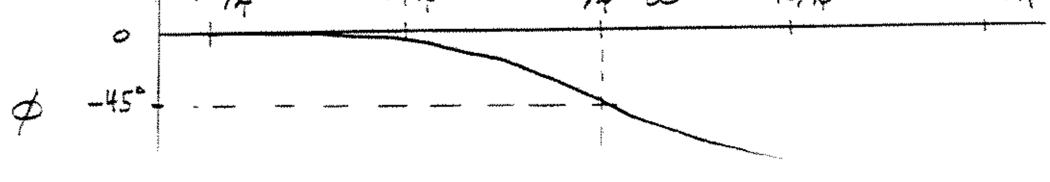
Sometimes the units of <u>decibels</u> (dB) are used for the $\log|G|$ axis. The number of dB can be computed for $|G|$ by

$$\#\ \text{dB} \equiv 20 \log_{10} |G(j\omega)|.$$

At the corner frequency $\omega = \frac{1}{\tau}$, so $20 \log|G| = 20 \log(.707) = -3$ dB so this frequency is refered to as the <u>-3dB point</u>

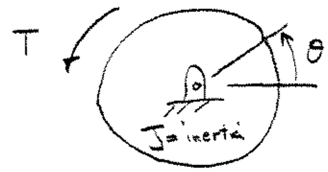
A decibel is a unit used originally to mesure the intensity of sound named after Alexander G. Bell. (decibel ≡ 10 * Bell, - Bell ≡ $\log_{10} V^2$ the log of the power)

The phase angle ϕ is also usually plotted, for our case it is:



MOTOR VELOCITY CONTROL

An interesting system with a transfer function of $\frac{1}{\tau s+1}$ is a motor velocity control system. The relation between angular velocity and motor torque is



$T = J\ddot{\theta}$, where T is the torque applied to the motor by the magnetic field, J is mass moment of inertia of the rotating portion of the motor and all attached gears and linkage, and $\ddot{\theta}$ is the angular acceleration.

Suppose you want to control the angular velocity $\dot{\theta} \equiv \Omega$ very accurately, and you can apply any torque you want T to the motor shaft. A good <u>control</u> <u>law</u> is

$$T = -K(\Omega - \Omega_d)$$

where K is a gain (constant) and Ω_d is the desired angular velocity. Intuitively if $\Omega > \Omega_d$ the motor is turning too fast, so we apply a negative torque to slow it down. If $\Omega < \Omega_d$ we apply a positive torque to speed it up.

This control law works very well for this case, and is called a proportional or "P" control system, since the motor torque is proportional to the angular velocity error $(\Omega - \Omega_d)$.

We can show why this control law works mathematically as follows.

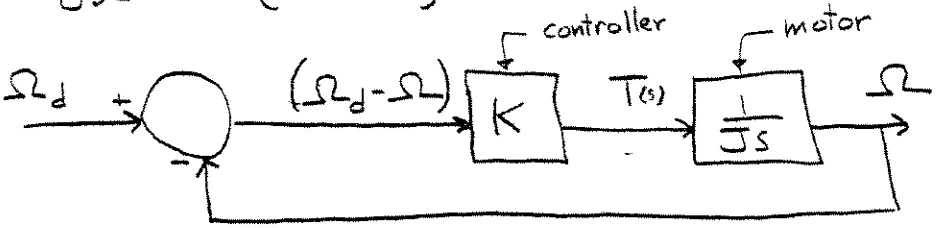
Motor dynamics: $T = J\dot{\Omega}$ $\quad (\dot{\Omega} = \ddot{\theta})$

Controller: $T = K(\Omega - \Omega_d)$ (also called the "compensator")

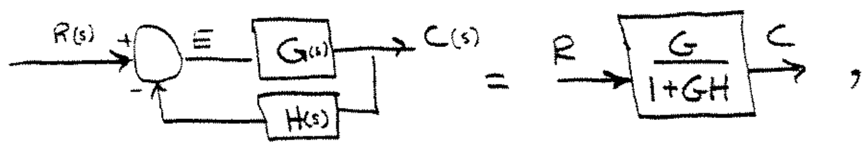
equation of combined system: $J\dot{\Omega} = K(\Omega - \Omega_d)$.

Laplace Transform: $Js\Omega = K(\Omega - \Omega_d)$.

Block Diagram:



Or, using block diagram algebra,

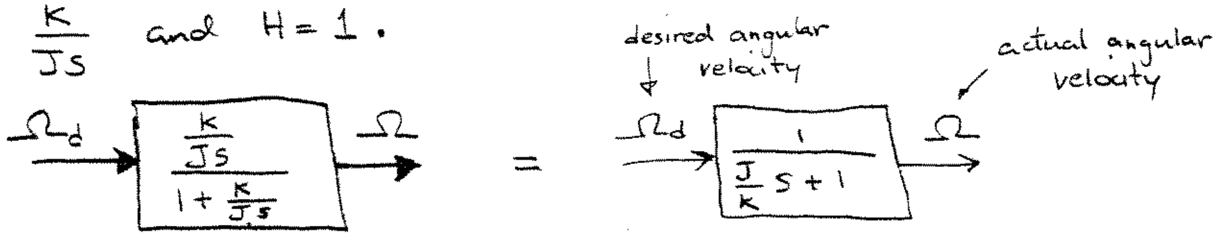


since,

$$\left.\begin{aligned} E &= R - CH \\ C &= EG \end{aligned}\right\} \; C = (R - CH)G \Rightarrow C(1+GH) = RG \text{ or } \frac{C}{R} = \frac{G}{1+GH}$$

$\frac{G}{1+GH}$ is called the closed loop system

For our case, the closed loop system is found by setting $G = \frac{K}{Js}$ and $H = 1$.



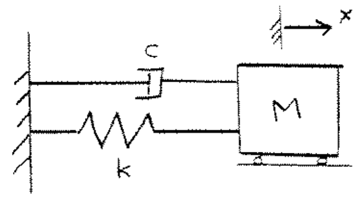
So the time constant for the closed loop system is $\tau = \frac{J}{K}$ seconds.

To reduce τ, just increase the gain K.

Notes for Lab 5

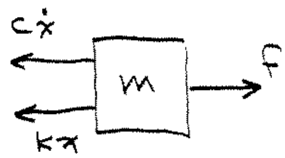
Second Order Systems

- A very important differential equation in engineering is second order with constant coefficients. This equation arises all the time in dynamics, for instance



Units
f : lb
k : lb/in
c : $\frac{\text{lb-sec}}{\text{in}}$
m : $\frac{\text{weight}}{g} = \frac{\text{lb}}{386.4}$

where x is the displacement of the mass from the undeformed spring position, and $f(t)$ is an applied force. We find the equation of motion for this system by drawing a free-body diagram of the forces acting on m, assuming the system is not at rest ($x \neq 0$, $\dot{x} \neq 0$). If we assume $x > 0$, $\dot{x} > 0$ then the positive forces are labeled as



so ΣF_x gives

$$m\ddot{x} = f - c\dot{x} - kx \quad , \text{ or } \quad \underline{m\ddot{x} + c\dot{x} + kx = f}$$

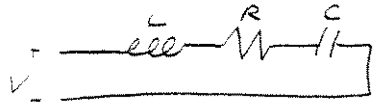
The Laplace transform of this equation with zero initial conditions is

$$X(ms^2 + cs + k) = F(s),$$

or, the block diagram is

$$\frac{X}{F} = \frac{1}{ms^2 + cs + k} \qquad F \longrightarrow \boxed{\frac{1}{ms^2+cs+k}} \longrightarrow X \quad .$$

Another example of a second order system is an RLC circuit



where we saw in the first lecture that

$$V = \left[R + Ls + \frac{1}{Cs} \right] I \quad \text{or,} \quad \frac{I}{V} = \frac{Cs}{LCs^2 + RCs + 1}$$

$$V \longrightarrow \boxed{\frac{Cs}{LCs^2 + RCs + 1}} \longrightarrow I \; .$$

Both of these systems denominators can be expressed in the form

(1) $$s^2 + 2\zeta\omega_n s + \omega_n^2 \, .$$

For the mass-spring oscillator, divide the numerator and denominator by m to make the coefficient of s^2 be 1. Hence

$$2\zeta\omega_n = \frac{c}{m} \qquad \text{and} \qquad \omega_n^2 = \frac{k}{m},$$

ω_n is called the <u>undamped natural frequency</u>

ζ is called the <u>damping ratio</u>

The roots of (1) are

$$s_i = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \, .$$

The interesting situation is when $\zeta < 1$. If $\zeta \geq 1$ we have two real roots, and the response $x(t)$ will be the sum

Step Response

If a step force is applied to the mass-spring system $f(t) = A\,u(t)$, $F(s) = \frac{A}{s}$, the response is

$$X = \frac{A/m}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}.$$

From the transform table we know if $\zeta < 1$:

$$F(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} \Rightarrow f(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n\sqrt{1-\zeta^2}\,t + \phi)$$

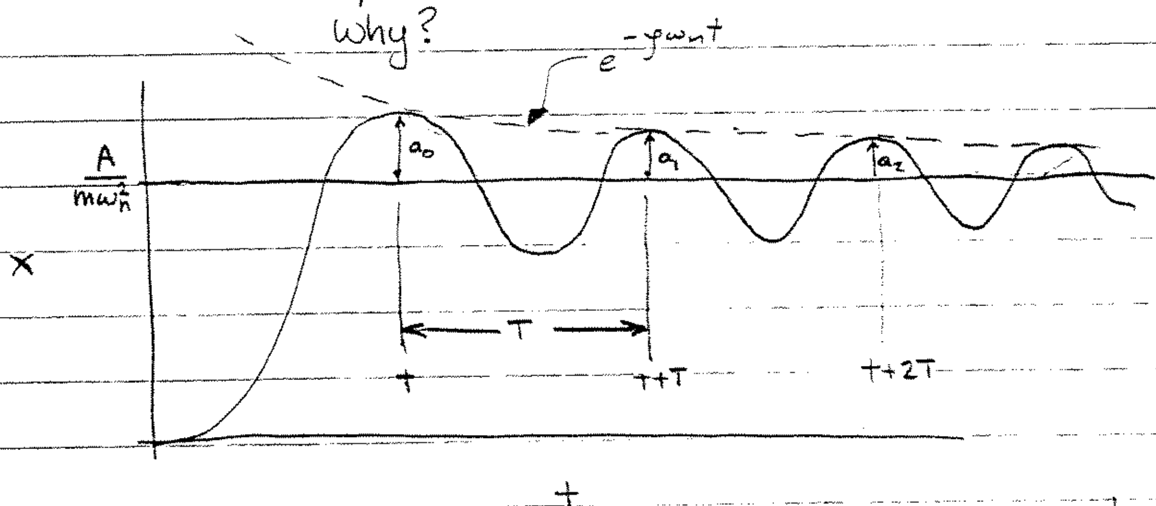
$$\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

(You will derive this relation for the lab work!)

So for our system

$$(2) \qquad x(t) = \frac{A}{m\omega_n^2}\left[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\underbrace{\omega_n\sqrt{1-\zeta^2}}_{\omega_d}\,t + \phi)\right]$$

why?



We see that the response decays exponentially with $e^{-\zeta\omega_n t}$ and $\tau = \frac{1}{\zeta\omega_n}$ is a measure of how quickly the system reaches

The quantity $\omega_n\sqrt{1-\zeta^2} \equiv \omega_d$ is called the <u>damped natural</u> frequency. For small damping $\zeta = \frac{c}{2\omega_n m}$ is small,

say $\zeta < .1$, so $\omega_n \cong \omega_d$.

In experiment 3, you determine ζ and ω_n experimentally. One method is to look at the free, unforced, response of the system on the oscilloscope. The form of the solution is the same as (2) with no steady-state term (the 1). If you measure the amplitude of succesive peaks, $a_0, a_1, a_2 \cdots a_n$ you can estimate ζ as follows.

The period between peaks, T, is the time it takes $\sin(\omega_d t + \phi)$ to oscillate once, so

$$\left[\omega_d(t+T)+\phi\right] - \left[\omega_d t + \phi\right] = 2\pi$$

$$\omega_d T = 2\pi$$

$$T = \frac{2\pi}{\omega_d}$$

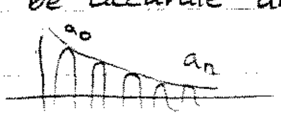
The ratio $\frac{a_0}{a_1}$, which you can measure, is

Called the <u>logarithmic decrem</u>

$$\frac{a_0}{a_1} = \frac{e^{-\zeta\omega_n t}}{e^{-\zeta\omega_n(t+T)}} = e^{\zeta\omega_n T}$$

so $$\ln\frac{a_0}{a_1} = \zeta\omega_n T = \zeta\omega_n\frac{2\pi}{\omega_d} = \zeta\omega_n\frac{2\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\zeta 2\pi}{\sqrt{1-\zeta^2}} \equiv \delta$$

If you want to be accurate about it, measure $\frac{a_0}{a_n}$



Then $\quad \frac{a_0}{a_n} = e^{\zeta \omega_n T n}$

$\hat{\delta} \equiv \ln \frac{a_0}{a_n} = \frac{\zeta 2\pi n}{\sqrt{1-\zeta^2}}$, $\quad$ or, solving for ζ,

$$(3) \qquad \zeta = \frac{\hat{\delta}}{\sqrt{4\pi^2 n^2 + \hat{\delta}^2}} \approx \frac{\hat{\delta}}{2\pi n}$$

So to compute ζ, measure a_0, n, a_n, then solve for $\hat{\delta}$ and use (3).

Frequency Response

Another important concept is the frequency response of a second order system. Suppose we apply a sine wave to the system $G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$. If the input is

$$a \sin \omega t,$$

then the output is

$$a\,|G(j\omega)|\,\sin(\omega t + \phi) \qquad \phi = \angle G(j\omega).$$

Now write $G(j\omega)$ as

$$G(j\omega) = \frac{1}{\frac{(j\omega)^2}{\omega_n^2} + \frac{2\zeta j\omega}{\omega_n} + 1}$$

$= \frac{1}{(\ldots)^2 \ldots}$

so
$$|G(j\omega)| = \frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$

and
$$\phi = 0 - \tan^{-1}\left[\frac{\frac{2\zeta\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right]$$

The maximum amplitude response occurs when

(4) $$\omega_r = \omega_n\sqrt{1-2\zeta^2} \qquad \zeta \le .707,$$

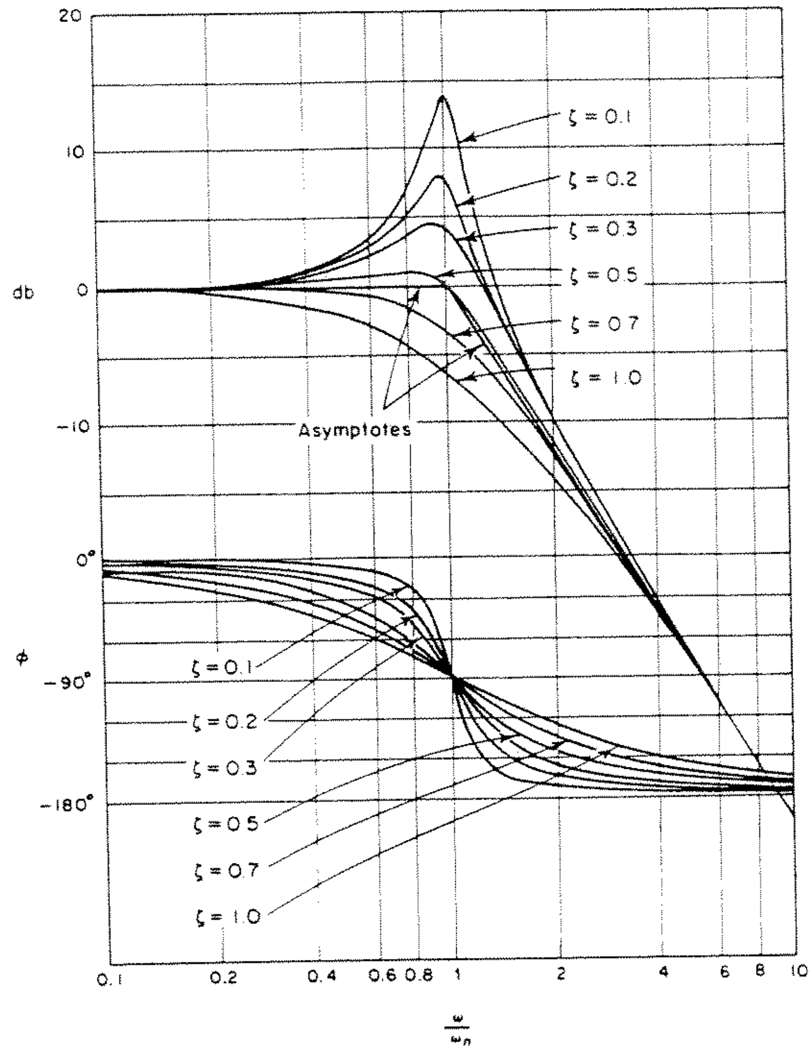
which is called the <u>resonant frequency</u>.

(5) $$|G(j\omega_r)| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$ is the amplitude ratio

at the resonant frequency. You will derive (4) and (5) for the lab write up. In general $\omega_n \neq \omega_d \neq \omega_r$ but they are usually very close.

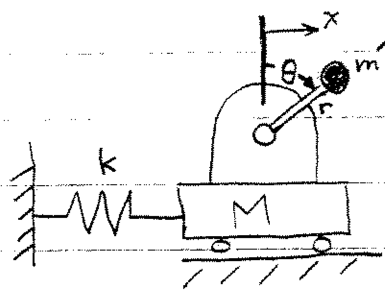
Equation (5) shows that as $\zeta \to 0$ $|G(j\omega_r)| \to \infty$ and (4) shows $\omega_r \to \omega_n$.
Physically this means that an <u>undamped</u> system will blow up (break) when forced at its natural frequency!

A Bode Plot of $20 \log|G(j\omega)|$ vs. $\frac{\omega}{\omega_n}$ is shown for typical ζ on the next page.



20
10
0
-10
db
0°
ϕ
-90°
-180°
$\zeta = 0.1$
$\zeta = 0.2$
$\zeta = 0.3$
$\zeta = 0.5$
$\zeta = 0.7$
$\zeta = 1.0$
Asymptotes
0.1
0.2
0.4
0.6
0.8
1
2
4
6
8
10
$\frac{\omega}{\omega_n}$

A common vibration problem is due to rotating machinery with unbalanced loads. Consider the model system

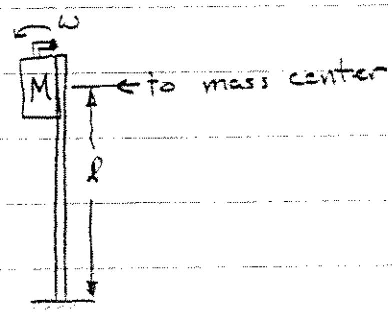


where m is an unbalanced mass rotating with constant angular velocity $\dot{\theta} = \omega$. The force that m applies on M is radially outward of magnitude $F = mr\omega^2$. The component of this force in the x direction is $mr\omega^2 \sin\theta$. For constant $\dot{\theta}$, $\theta = \omega t$, so the forcing function is

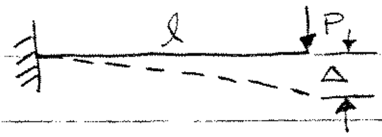
$$f = mr\omega^2 \sin\omega t .$$

This sine wave produces a frequency response of the machine as just described, but note that the amplitude of the input wave is a function of ω^2, so it increases with motor velocity.

The spring k could represent structural rigidity. In experiment 3, we have a motor with an unbalanced load mounted on a cantilevered beam.

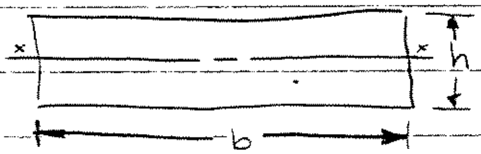


The spring constant of the structure can be found in any strength of materials book from the load-deflection relation:



$$\Delta = \frac{P\ell^3}{3EI}$$

Where P is an applied load, Δ is the deflection at the beam tip, E is the modulus of elasticity, and I is the area moment of inertia of the beam cross-section. For our beam



$$I_{xx} = \frac{1}{12} b h^3$$

The spring constant is $k = \frac{F}{x} = \frac{P}{\Delta} = \frac{3EI}{\ell^3}$

Because the beam itself has mass, M should be increased. An approximate amount to increase M by is $.236\,m$, where m is the total mass of the beam. This number can be found using energy methods of vibration analysis. The estimated natural frequency is

$$\omega_n^2 \cong \frac{k}{M_{tot}} = \frac{3EI}{\ell^3(M + .236m)}.$$

This is only an approximation, but a good one. There are also higher harmonics, or other natural frequencies of the beam as you will see in experiment 7.

Second Order Control Systems

Although the ~~above~~ discussion on page 29 on motor velocity control indicates that $\omega(t) \rightarrow \omega_d(t)$ with time constant τ, it actually does not work so well because friction has been ignored. If you touch the rotating shaft in experiment 4 you will introduce an error. One common method used to eliminate the error is to add integral control action as follows.

The motor dynamics with friction are

$$T - T_f = J\dot{\omega},$$

where T_f is the torque due to friction. The form of a proportional controller is

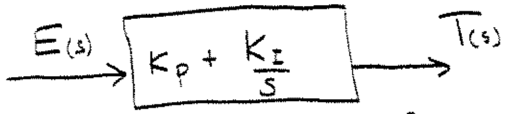
$$T = -k_p e,$$

where $e = (\omega - \omega_d)$, the error of the actual angular velocity from the desired value ω_d. The form of a proportional plus integral control is

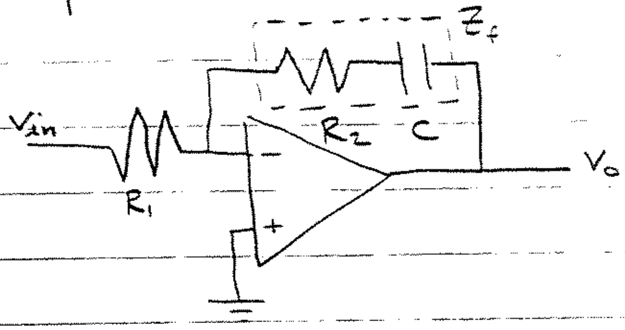
$$T = -k_p e - k_I \int_0^t e\,dt, \quad \text{or} \quad T(s) = -E(s)\left(K_p + \frac{K_I}{s}\right)$$

where K_I is the integral gain. Now, if $e(t) \neq 0$ say it is some constant ϵ, then $\int e\,dt$ increases with time and eventually the torque becomes high enough to overcome friction.

The block diagram of a P-I compensator is



A PI control circuit can be implemented with op. amps as



$$V_o = -\frac{Z_f}{Z_{in}} V_{in} \qquad Z_{in} = R_1$$

$$Z_f = R_2 + \frac{1}{Cs}$$ (two impedances in series)

$$V_o = -\frac{1}{R_1}\left(R_2 + \frac{1}{Cs}\right) V_{in}$$

$$T \Rightarrow V_o = -\left[\frac{R_2}{R_1} + \frac{1}{R_1 C s}\right] V_{in} \Rightarrow E$$

$$K_p = \frac{R_2}{R_1} \qquad K_I = \frac{1}{R_1 C}$$

To show how the whole system works mathematically, look at the closed loop system. For $\omega_d = \text{constant}$, $\dot{e} = \dot{\omega}$, hence

$$T - T_f = J\dot{e}$$

$$-k_p e - k_I \int e\,dt - T_f = J\dot{e}$$

or,

$$J\dot{e} + k_p e + k_I \int e\,dt = -T_f .$$

Now take the derivative of this equation with respect to time and assume $T_f = \text{constant}$ to give

$$J\ddot{e} + k_p\dot{e} + k_I e = 0 .$$

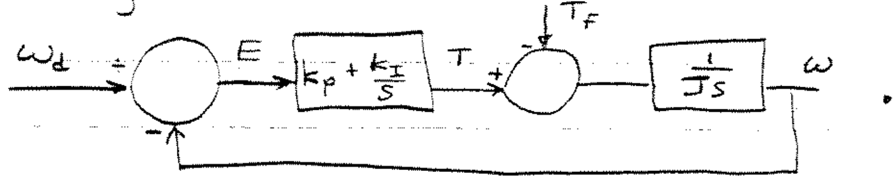
This is an unforced second order system that will converge to $e(\infty) = 0$ as long as the roots of the characteristic equation are in the left half s plane. This means that $k_p > 0$ and $k_I > 0$. The response can be found from

$$E(s)\left(Js^2 + k_p s + k_I\right) = \underbrace{IC(s)}_{\text{initial conditions}}$$

$$E(s) = \frac{IC(s)}{Js^2 + k_p s + k_I} = \frac{IC/J}{s^2 + k_p/J\, s + k_I/J}$$

so $\omega_n^2 = \frac{k_I}{J}$ and $2\zeta\omega_n = \frac{k_p}{J}$, and these numbers can be set to be any values just by adjusting two gains, k_p and k_I.

The block diagram is



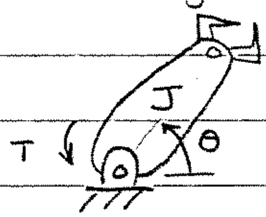
You should confirm that this matches the above differential equations.

Notes for Lab 6

A PD motor position control system.

By far, the most common control system used in industry is a motor position controller. Examples are robot arm controllers, radar tracking systems, numerically controlled milling machines and lathes, remote controlled toys, ...

Consider a one jointed robot arm,



First suppose there is no friction and no gravity. We have the ability to apply any torque T we want to the arm from a DC motor. How can we design a controller which will quickly position the arm to any desired angle θ_d ?

We could first try a proportional controller as in experiment 2.

Dynamic Model $\quad T = J\ddot{\theta} \quad$ or $\quad T = s^2 J \,\Theta(s)$

$$T \longrightarrow \boxed{\frac{1}{Js^2}} \longrightarrow \theta$$

Controller

$T = -k_p(\theta - \theta_d)$ ⟶ This is like a torsional spring

↑ need to measure, say with a potentiometer.

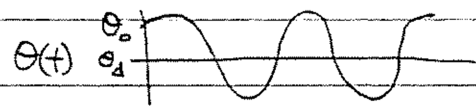
Closed loop system

$$J\ddot{\theta} = -k_p(\theta - \theta_d)$$

Define $e = \theta - \theta_d$. If θ_d = constant, $\ddot{\theta} = \ddot{e}$ and the closed loop system is

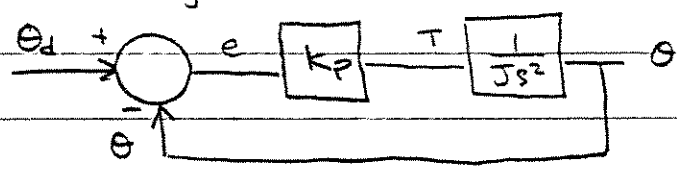
$$J\ddot{e} + k_p e = 0.$$

This is the same form as an undamped mass spring system (i.e. $m\ddot{x} + kx = 0$). The system is marginally stable, so the

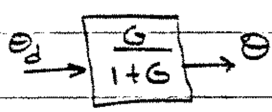


robot arm would oscillate forever about the unstretched "spring" position $\theta = \theta_d$.

In terms of block diagrams:



The closed loop transfer function is:



$$\frac{G}{1+G} = \frac{\frac{k_p}{Js^2}}{1 + \frac{k_p}{Js^2}} = \frac{k_p}{Js^2 + k_p},$$

hence $\zeta = 0$ and there are 2 roots on the imaginary axis.

To fix the problem, we need to somehow add damping to the system. If a tachometer signal is available, the following control law works

$$T = -k_p(\theta - \theta_d) - k_v \dot{\theta}$$

$\uparrow$ from tachometer

The closed loop system is

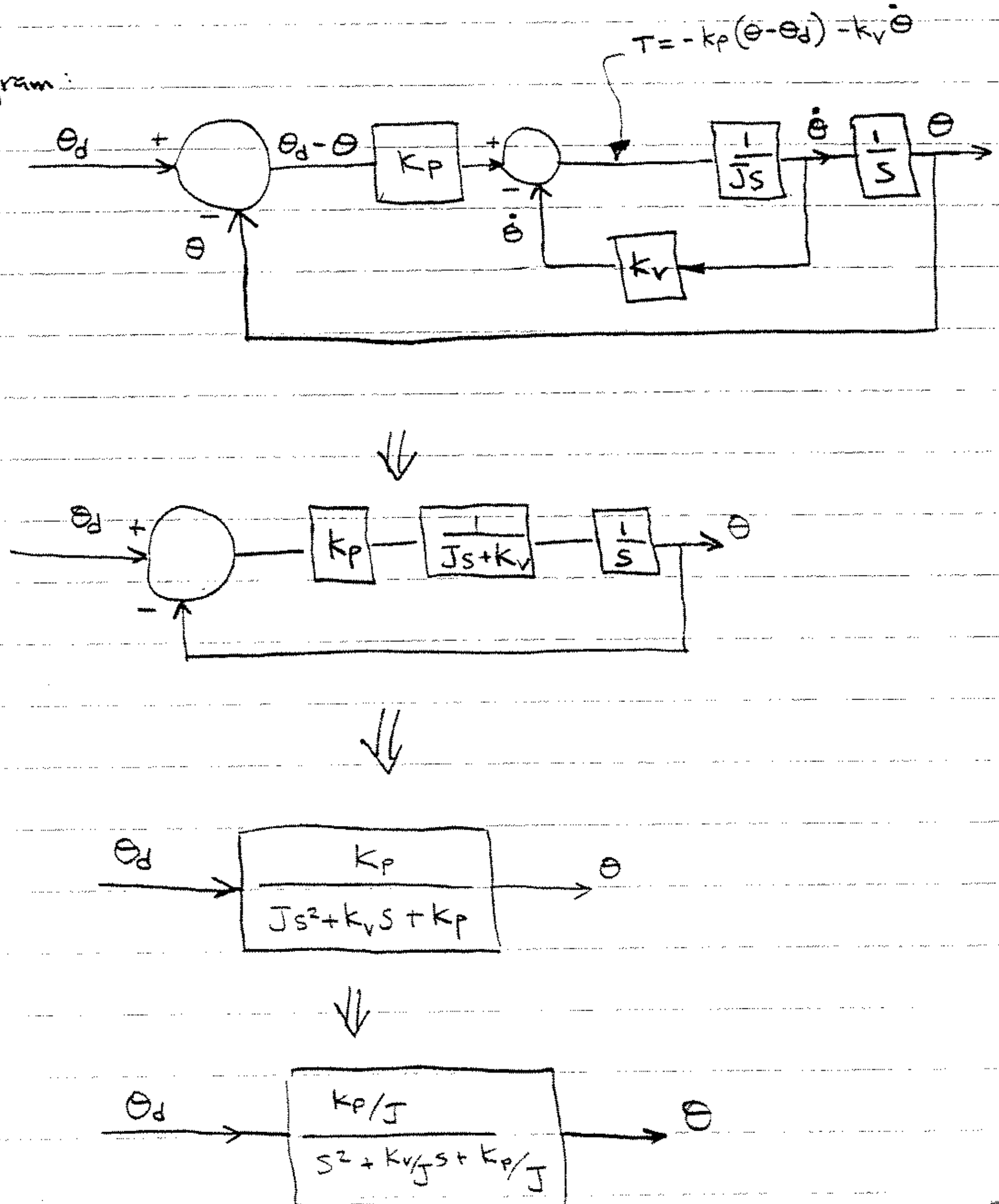
$$J\ddot{\theta} = -k_p(\theta - \theta_d) - k_v\dot{\theta}$$

or

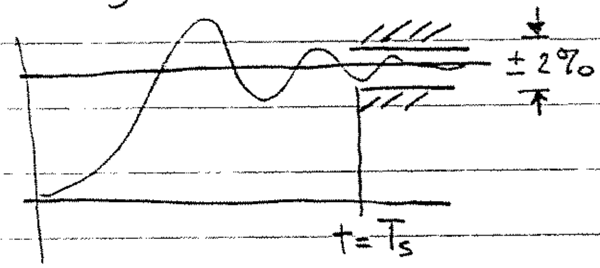
$$J\ddot{e} + k_v\dot{e} + k_p e = 0,$$

so we now have the required damping term. The system is stable for any $k_v > 0$ and $k_p > 0$.

Block Diagram:



Typically control system designers care very much about performance. That is, they want $e(t) = \theta - \theta_d$ to go to zero quickly. We said in the last lecture that the time constant for a second order system is $\tau = \frac{1}{\zeta\omega_n}$. It can be shown that if you wait $t = 4\tau$ seconds after an input is applied, the output will be within 2% of its final value. This is called the <u>settling time</u> $T_s = 4/\zeta\omega_n$.



So a motor control design specification may call for a settling time of .1 second with a ζ of .7. It is easy to do this electrically using the gains k_p and k_v. Since

$$(1)\quad 2\zeta\omega_n = \frac{k_v}{J} \quad \text{and} \quad (2)\quad \omega_n^2 = \frac{k_p}{J},$$

$$T_s = \frac{4}{\zeta\omega_n} \overset{(1)}{=} \frac{8J}{k_v} \Rightarrow \underline{\underline{k_v = \frac{8J}{T_s}}}$$

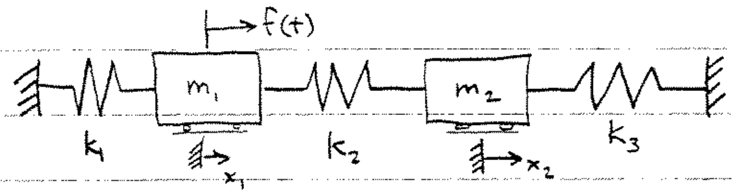
$$(2)\quad k_p = J\omega_n^2 \overset{(1)}{=} J\left(\frac{k_v}{2\zeta J}\right)^2 = \underline{\underline{\left(\frac{k_v}{2\zeta}\right)^2\frac{1}{J}}} \;.$$

Notes for Lab 7

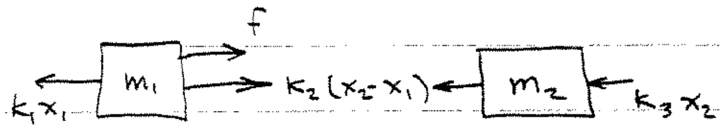
Systems with Two Modes of Vibration

No structure can ever be modelled exactly using just the simple mass-spring oscillator. In reality there are always higher frequency <u>modes</u> of <u>vibration</u> present in any structure. Even in Experiment 3, you probably noticed oscillations at higher frequencies than the fundamental (lowest one) present in the accelerometer output signal.

Consider a model for Experiment 5,



f is a force applied to m_1, and x_1, x_2 are the positions of m_1 and m_2 relative to a <u>fixed</u> reference frame. Draw a free body diagram to obtain the equations of motion. Assume $x_2 > x_1 > 0$:



$$m_1 \ddot{x}_1 = f + k_2(x_2 - x_1) - k_1 x_1$$
$$m_2 \ddot{x}_2 = -k_2(x_2 - x_1) - k_3 x_2$$

or,

$$m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = f$$
$$m_2 \ddot{x}_2 - k_2 x_1 + (k_2 + k_3) x_2 = 0$$

or in matrix form,

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} . \quad (2)$$

Symbolically we often write these as

$$(1) \qquad M\ddot{\underline{x}} + K\underline{x} = \underline{f}, \text{ where}$$

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad K = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix} \quad \underline{f} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

Equation (1) is a very general form that works for systems with any number of masses, $m_3, m_4 \cdots$ etc. It can be shown using energy methods that M and K are always symmetric, which means $M = M^T$, $K = K^T$ (T denotes transpose).

To solve for the motion $\underline{x}(t)$ given $f(t)$ we can still use Laplace transforms on (1)

$$\underline{F} = \begin{pmatrix} F(s) \\ 0 \end{pmatrix}$$

$$M s^2 \underline{X} + K\underline{X} = \underline{F} + \underline{I}(s), \quad \underline{X} = \begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix}, \text{ and}$$

$\underline{I}(s)$ is a vector arising from the initial conditions.
Solving for $\underline{X}(s)$ we have

$$\left[Ms^2 + K\right]\underline{X} = \underline{F} + \underline{I}(s)$$

$$(2) \qquad \underline{X} = \left[Ms^2 + K\right]^{-1}\left(\underline{F} + \underline{I}\right)$$

In this case it is easy to invert a 2×2 matrix because

$$\text{if} \quad A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \quad A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} c & -b \\ -b & a \end{bmatrix}; \quad \det(A) = ac - b^2$$

So $\quad Ms^2+K = \begin{bmatrix} m_1s^2+k_1+k_2 & -k_2 \\ -k_2 & m_2s^2+k_2+k_3 \end{bmatrix}$ and the inverse formula gives

$$[Ms^2+K]^{-1} = \frac{1}{(m_1s^2+k_1+k_2)(m_2s^2+k_2+k_3)-k_2^2}\begin{bmatrix} m_2s^2+k_2+k_3 & k_2 \\ k_2 & m_1s^2+k_1+k_2 \end{bmatrix}.$$

Plugging this into (2) and looking at the first component of X gives:

$$(3) \qquad X_1(s) = \frac{F(s)\,(m_2s^2+k_2+k_3)}{(m_1s^2+k_1+k_2)(m_2s^2+k_2+k_3)-k_2^2} \qquad (\text{with } I(s)=0).$$

You could also relate $X_1(s)$ to $F(s)$ by solving 2 equations and 2 unknowns without explicitly inverting $[Ms^2+K]$.

Now suppose $f(t)=\sin\omega t$ and we want $x_1(t)$. From (3) the transfer function X_1/F is known. Hence

$$\frac{X_1}{F}(j\omega) = \frac{(-m_2\omega^2+k_2+k_3)}{(-m_1\omega^2+k_1+k_2)(-m_2\omega^2+k_2+k_3)-k_2^2}.$$

or, expanding the denominator,

$$(4) \qquad \frac{X_1}{F}(j\omega) = \frac{(-m_2\omega^2+k_2+k_3)}{m_1m_2\omega^4-\omega^2\left(m_2(k_1+k_2)+m_1(k_2+k_3)\right)+(k_1k_2+k_3k_1+k_2k}$$

Note that the denominator is quadratic in ω^2, so the roots (poles) can be found with the quadratic formula.

In some cases you might want to stop $x_1(t)$ from vibrating even though a oscillating force is applied to m_1 with some frequency $\omega = \omega_0$.
Equation (4) shows that

$$\left| \frac{X_1}{F}(j\omega_0) \right| = 0 \qquad \text{when}$$

we choose $\underline{\dfrac{k_2+k_3}{m_2} = \omega_0^2}$ to make the

numerator of (4) zero. If you choose m_2, k_2, k_3 according to the above formula, $x_1(t) = 0$, when $\omega = \omega_0$. That is, mass m_1 <u>will not move</u> if $f = \sin \omega_0 t$. This is a common method for <u>vibration</u> <u>isolation</u>. You will find ω_0 for a given k_2, k_3, m_2 in Experiment 5.

Equation (4) also shows that

$$\left| \frac{X_1}{F}(j\omega) \right| \rightarrow \infty \qquad \text{when}$$

ω approaches a root of the denominator. This occurs at two distinct frequencies, say ω_1 and ω_2. Usually,

$$\omega_1 < \omega_0 < \omega_2 .$$

You will determine all three frequencies experimentally in Exp 5.

Modes of Vibration

More insight is gained regarding the motion of the system if we consider the case of free vibrations, i.e. $f(t) \equiv 0$.

Assume that $\underline{x}(t)$ has the form $\underline{x}_0 \sin(\omega t + \phi)$, so that

(5) $$\underline{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} x_{10} \sin(\omega t + \phi) \\ x_{20} \sin(\omega t + \phi) \end{pmatrix} = \underline{x}_0 (\sin \omega t + \phi).$$

The vector $\underline{x}_0 = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}$ is the amplitude of vibration for each mass. Now substitute (5) into

$$M\ddot{\underline{x}} + Kx = 0$$

to obtain

$$M - \omega^2 \underline{x}_0 \sin(\omega t + \phi) + K \underline{x}_0 \sin(\omega t + \phi) = 0$$

or

(6) $$\left[-\omega^2 M + K\right] \underline{x}_0 = 0 \qquad (\text{note } \omega^2 \text{ is a scalar}).$$

Now multiply (6) by $M^{-1} = \begin{bmatrix} \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}$ to obtain

(7) $$\left[-\omega^2 I + M^{-1} K\right] \underline{x}_0 = 0$$

Now, (7) has the general form $\left[A - \lambda I\right] \underline{x}_0 = 0$ which is called an <u>eigenvalue problem</u>. The question is, are there any values of $\underline{x}_0 \neq 0$ that satisfy (7)? For $\underline{x}_0 \neq 0$ to exist $[-\omega^2 I + M^{-1} K]$ must be <u>singular</u>, i.e. not invertible. If it were invertible, the only solution to (7) would be $\underline{x}_0 = 0$.

$$M^{-1}K = \begin{bmatrix} \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} \begin{bmatrix} k_1+k_2 & k_2 \\ -k_2 & k_2+k_3 \end{bmatrix} = \begin{bmatrix} (k_1+k_2)/m_1 & -k_2/m_1 \\ -k_2/m_2 & k_2+k_3/m_2 \end{bmatrix}$$

$$(8) \quad \left[-\omega^2 I + M^{-1}K\right] = \begin{bmatrix} (k_1+k_2)/m_1 - \omega^2 & -k_2/m_1 \\ -k_2/m_2 & (k_2+k_3)/m_2 - \omega^2 \end{bmatrix}$$

For $\left[-\omega^2 I + M^{-1}K\right]$ to be singular,

$$\det\left(-\omega^2 I + M^{-1}K\right) = 0 \quad \text{or}$$

$$\left(\frac{k_1+k_2}{m_1} - \omega^2\right)\left(\frac{k_2+k_3}{m_2} - \omega^2\right) - \frac{k_2^2}{m_1 m_2} = 0 \quad \text{or}$$

$$(9) \quad (k_1+k_2 - m_1\omega^2)(k_2+k_3 - m_2\omega^2) - k_2^2 = 0 .$$

The value of ω^2 that satisfies this equation is also a pole of the transfer function in (4). It is called an <u>eigenvalue</u> of $M^{-1}\mathbf{K}$. Suppose we solve for the two values of ω^2 that satisfy (9) and denote them as ω_1^2 and ω_2^2, as before, so that

$$\det\left(\left[-\omega_1^2 I + M^{-1}K\right]\right) = \det\left(\left[-\omega_2^2 I + M^{-1}K\right]\right) = 0.$$

At these frequencies, it is possible to have nonzero amplitudes of vibration, so the amplitudes $\underline{x}_0 = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}$

that satisfy (7) for $\omega = \omega_1$ or $\omega = \omega_2$ can be found. Note that there is no solution (other than $x_0 = 0$) possible for $\underline{x}_0$ at any frequency other than $\omega = \omega_1$ or $\omega = \omega_2$. Why? The form of (7) is

$$A \underline{x}_0 = 0, \quad \text{and} \quad \det(A) = 0 \quad \text{for}$$

$\omega = \omega_1, \omega_2$. Hence, the rows of A are <u>linearly dependent</u> on one another at $\omega = \omega_1$ or $\omega = \omega_2$. Say

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then let } \underline{x}_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix},$$

$$A \underline{x}_0 = \begin{pmatrix} a x_{10} + b x_{20} \\ c x_{10} + d x_{20} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad \left(A = \left[-\omega^2 I + M^{-1} K\right] \right. (8$$

The 2 equations contain the same information. We see that the top equation shows

$$(10) \qquad x_{10} = -\frac{b}{a} x_{20},$$

where $b = -k_2/m_1$ and $a = \left[(k_1 + k_2)/m_1 - \omega^2\right]$ for $\omega = \omega_1$ or $\omega = \omega_2$. Equation (10) is very helpful for visualizing the motion. That is, the amplitudes must be in exactly the ratio $\frac{x_{10}}{x_{20}} = -\frac{b}{a}$ in order for free vibration to be possible at all.

For example, consider the case when $m_1 = m_2 = k_1 = k_2 = k_3 = 1$.

Eq(8) gives
$$[-\omega^2 I - M^{-1}K] = \begin{bmatrix} 2-\omega^2 & -1 \\ -1 & 2-\omega^2 \end{bmatrix} \equiv A .$$

The characteristic equation (9) is $|A| = 0$:

$$(2-\omega^2)(2-\omega^2) - 1 = 0 \quad \text{or}$$

$$\omega^4 - 4\omega^2 + 3 = 0 .$$ (Also see the denominator of (4))

The roots are
$$\omega^2 = \frac{4 \pm \sqrt{16-12}}{2} = 2 \pm 1 = \underline{\underline{1 \text{ or } 3.}}$$

The amplitudes that correspond to the root $\omega_1^2 = 1$, with $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, satisfy (10)
$$x_{10} = -\left[\frac{-1}{1}\right] x_{20} = x_{20}$$

Hence $\underline{x}_0$ has the form

$$\underline{x}_0^1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \alpha ,$$

where α is a scalar amplitude for both masses and the superscript means the first mode. Hence, in the first mode of vibration for the system, both masses move to the left and right together in exactly the seme phase.

The amplitudes that correspond to the root $\omega_2^2 = 3$ satisfy

$$x_{10} = -\left[\frac{-1}{(2-3)}\right] x_{22} = -x_{20} .$$

For the second mode then $\underline{x}_0$ has the form

$$\underline{x}_0^2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \alpha .$$

directions, or 180° out of phase. They do this at a higher frequency. Note that the center spring deforms in this mode whereas it remains undeformed in the first mode, and this requires more energy during the vibration.

A general free vibration can be expressed as

$$x(t) = \underline{x}_0^1 \sin(\omega_1 t + \phi_1) + \underline{x}_0^2 \sin(\omega_2 t + \phi_2)$$

because this solution satisfies (1) for both $\underline{x}_0^1$ and $\underline{x}_0^2$. Hence a general motion is not usually all in one mode or the other, but a linear combination of the two.

Finally, note that the vibration isolation frequency for m_1 in this example is given by

$$\omega_0^2 = \frac{k_2 + k_3}{m_2} = 2$$

so

$$\omega_1^2 = 1 < \omega_0^2 = 2 < \omega_2^2 = 3 .$$