MAE 106 Laboratory Exercise #4
Solution

Q1: For a beam with a length of 8 inches:

$$
\begin{aligned}
I &= \frac{1}{12}bh^3 \\
&= \frac{(2in)(.125in)^3}{12} \\
&= 3.26 \times 10^{-4} in^4 \\
k &= \frac{3EI}{l^3} \\
&= \frac{(3)(30 \times 10^6 \frac{lb}{in^2})(3.26 \times 10^{-4} in^4)}{(8in)^3} \\
&= 57.2\frac{lb}{in} = 686.6\frac{lb}{ft} \\
\rho &= .283\frac{lb}{in^3} \\
m &= (.283)(2)(.125)(8) \\
&= .566lb \\
M_{total} &= 2.25lb + .283(.566lb) \\
&= 2.41lb \\
&= \frac{2.41}{32.2}\frac{lb \cdot s^2}{ft} \\
&= 0.075\frac{lb \cdot s^2}{ft} \\
\omega_n &= \sqrt{\frac{k}{m}} \\
&= \sqrt{\frac{686.6\frac{lb}{ft}}{0.075\frac{lb \cdot s^2}{ft}}} \\
&= 95.7\frac{rad}{sec} = 15.2Hz
\end{aligned}
$$

These computations are slightly easier when using SI units. When you use english units, you must be very careful to make sure that you get the correct units (namely $\frac{1}{sec}$) for ω_n. The most common problem is to forget to convert from pounds (which is a unit of force) to a suitable unit of mass or to convert from pounds when it is not necessary. It is also not a good idea to call the unit of mass a "slug", instead call it a $\frac{lb \cdot s^2}{ft}$ so that you can be assured of cancelling the correct units as I did in the above problem.

Q2: Typical accelerometer voltages are in the 100 to 300 mV range. This measurement, plus the assumption that the relationship between output voltage and acceleration is linear allows us to compute acceleration. For example, if the accelerometer read 250 mV at 1 gravity, then we use the following formula to compute acceleration:

$$a = 32.2\frac{ft}{sec^2}\left(\frac{V}{.25V}\right)$$

where V is the output reading (in Volts) of the accelerometer.

Q3: Typical damped natural frequencies are in the 7 to 20 Hz range.

Q4: Run at least two tests and find a_0, a_n and n for each test. Example:

Test 1: $a_0 = 306$ mV, $a_n = 140.6$ mV, $n = 15$

Test 2: $a_0 = 222$ mV, $a_n = 97$ mV, $n = 15$

Use this data and the formula:

$$\delta = ln\left(\frac{a_0}{a_n}\right)$$

$$\xi = \frac{\delta}{\sqrt{\delta^2 + 4\pi^2 n^2}} \approx \frac{\delta}{2\pi n}$$

to find exact and approximate ξ values for each test run. Then average these values to obtain the best estimate.

A sample solution:

$$\xi = \frac{.78}{.78^2 + 4\pi^2(15)^2} = .0083$$

Q5: This is different for each motor, but typical values are in the range of 450 to 500 revolutions per minute for a 2 V input signal and 950 to 1000 revolutions per minute for a 4 V input signal. We can combine these data points in order to develop a formula that relates the speed of the motor to the input voltage.

Example data: 495 rpm at 2 V, 968 rpm at 4 V. Converting rpm to radians per second:

$$495\frac{revolutions}{min}\left(\frac{1min}{60sec}\right)\left(\frac{2\pi\ radians}{revolution}\right) = 51\frac{rad}{sec}$$

$$968\frac{revolutions}{min}\left(\frac{1min}{60sec}\right)\left(\frac{2\pi\ radians}{revolution}\right) = 101\frac{rad}{sec}$$

From this data, we see that the gain is approximately $25\frac{rad}{sec \times Volt}$. We can multiply this gain by the input voltage to obtain the frequency of the forcing function in radians per second.

Q6: We asked you to record the input voltage at resonance. Typical values are in the 1.8 to 2 Volt range for the input voltage. You can now convert this using the results of Q5 to obtain the frequency of the forcing function at resonance. You can also get the frequency by examining the frequency of the accelerometer output trace on the scope.

The amplitude of the accelerometer output, combined with the formula in **Q2** will yield the maximum accelerations. If you measured the peak to peak voltage of the accelerometer output, you need to divide by 2 before using the formula in **Q2**.

The estimated tip motion could be anywhere from 1 inch (very small) to 4 or more inches depending upon how close you got to the actual resonance value. This estimated tip motion was denoted A_{res} in the lab.

Q7: Exciting the system at a frequency well past resonance should result in very small tip motions. A typical value is $\frac{1}{16}$ inch. This tip motion was denoted A_{high} in the lab.

Q8: Combine the estimated tip motions from **Q6** and **Q7** to find an estimate of ξ using

$$\xi = \frac{A_{high}}{2A_{res}}$$

A typical result might be

$$\xi = \frac{0.0625in}{2(2in)} = 0.0156$$

Q9 The transfer function is given by

$$
\begin{aligned}
G(s) &= \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \\
G(j\omega) &= \frac{\omega_n^2}{-\omega^2 + 2\xi\omega_n j\omega + \omega_n^2} \\
|G(j\omega)| &= \frac{\omega_n^2}{\sqrt{((\omega_n^2 - \omega^2)^2 + (2\xi\omega_n\omega)^2)}}
\end{aligned}
$$

Resonance occurs when the output is the maximum. This occurs when the denomenator of $|G(j\omega)|$ is minimum:

$$
\frac{d}{d\omega}\left((\omega_n^2 - \omega^2)^2 + (2\xi\omega_n\omega)^2\right) = -4(\omega_n^2 - \omega^2)\omega + 8\xi^2\omega_n^2\omega = 0
$$

Solving this equation for ω yields the resonance frequency:

$$
\omega_r = \omega_n\sqrt{1 - 2\xi^2}
$$

ξ must be less than .707 to insure that the square root does not yield an imaginary number.

Q10

$$
\begin{aligned}
G(s) &= \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \\
G(j\omega_r) &= \frac{\omega_n^2}{-\omega_r^2 + 2\xi\omega_n j\omega_r + \omega_n^2} \\
G(j\omega_r) &= \frac{\omega_n^2}{-\omega_n^2(1 - 2\xi^2) + 2\xi\omega_n^2 j\sqrt{1 - 2\xi^2} + \omega_n^2} \\
&= \frac{1}{2\xi j\sqrt{1 - 2\xi^2} + 2\xi^2} \\
|G(j\omega_r)| &= \frac{1}{\sqrt{4\xi^4 + 4\xi^2(1 - 2\xi^2)}} \\
&= \frac{1}{\sqrt{4\xi^2 - 4\xi^4}} \\
&= \frac{1}{2\xi\sqrt{1 - \xi^2}}
\end{aligned}
$$

Q11 It is fairly obvious that as $\xi \to 0$, $\omega_r \to \omega_n$ using the result of **Q9**.
From above we have

$$
G(j\omega) = \frac{\omega_n^2}{-\omega^2 + 2\xi\omega_n j\omega + \omega_n^2}
$$

Now, as $\omega \to \omega_n$, we have:

$$
\begin{aligned}
G(j\omega_n) &= \frac{\omega_n^2}{-\omega_n^2 + 2\xi\omega_n^2 j + \omega_n^2} \\
&= \frac{1}{2\xi j} \\
|G(j\omega_n)| &= \frac{1}{2\xi}
\end{aligned}
$$

Q12 As $\omega \to \infty$,

$$\begin{aligned} G(s) &= \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \\ G(j\omega) &= \frac{\omega_n^2}{-\omega^2 + 2\xi\omega_n j\omega + \omega_n^2} \\ \omega^2 G(j\omega) &= \frac{\omega^2\omega_n^2}{(\omega_n^2 - \omega^2) + 2\xi\omega_n j\omega} \\ &= \frac{\omega_n^2}{(\frac{\omega_n^2}{\omega^2} - 1) + \frac{2\xi\omega_n j}{\omega}} \end{aligned}$$

As $\omega \to \infty$, any term with ω in the denomenator becomes zero, so

$$\begin{aligned} G(j\omega) &= \frac{\omega_n^2}{(0-1)+0} \\ &= -\omega_n^2 \\ |G(j\omega)| &= \omega_n^2 \end{aligned}$$

Q13 Normally the **Q6** values are the most accurate, since **Q1** contained several assumptions about the mass distribution. **Q3** values can also be as accurate as the **Q6** values depending on how well you read the scope. I accepted any reasonable answers since several groups had problems or bad data associated with some tests.

Q14 For this problem, the values from **Q4** are most accurate. The values from **Q8** involved an "eyeball" measurement of the tip motion which is nearly impossible to obtain with great precision.

Q15 Use the formula:

$$c = 2\xi M\omega_n$$

where M is the total mass of the system.

Example solution:

- $M = 0.075\frac{lb \cdot s^2}{ft}$ from **Q2**
- $\xi = .0083$ from **Q4**
- ω_n from your best run (either **Q3** or **Q6**)

$$c = 2(.0083)(0.075\frac{lb \cdot s^2}{ft})(98\frac{rad}{sec} = .122\frac{lb \cdot sec}{ft}$$

Note that the units on c match the notes.

Q16 From **Q1**, we know that for a beam of length l_1:

$$\omega_1 = \sqrt{\frac{3EI}{M_1 l_1^3}}$$

For another beam of length l_2 we have

$$\omega_2 = \sqrt{\frac{3EI}{M_2 l_2^3}}$$

Combining these two equations gives:

$$\frac{\omega_2^2}{\omega_1^2} = \frac{3EI(M_1 l_1^3)}{3EI(M_2 l_2^3)}$$

For a beam of similar properties, we know that E and I are the same for each beam. If we also assume that $M_1 \approx M_2$ we have

$$\frac{\omega_2^2}{\omega_1^2} = \frac{l_1^3}{l_2^3}$$

We can now plug in the length of our beam from **Q1** and the natural frequency obtained from **Q13** to find the natural frequency of a beam with a length of 15 inches.

$$\omega_2^2 = (98\frac{rad}{sec})\left(\frac{8in}{15in}\right)^3$$