26.2 Spherical Thin Lenses

In comparison to aspherics, spherical surfaces (actually, segments of spheres) are easy to fabricate. Unlike the hyperboloid, ellipsoid, or paraboloid, the sphere has no single central symmetry axis. All its diameters are alike, and the sphere can be generated by simply randomly rocking a rough-cut glass disk against an approximately spherical grinding tool. The two will wear each other away until both are perfectly spherical. The vast majority of lenses in use today have surfaces that are segments of spheres—despite the fact that those spherical surfaces are not ideal and will result in imaging errors known as *aberrations*. By using several components made of different materials to form compound lenses, these errors can be controlled so well that image quality need only be limited by diffraction.

As has already been seen, we cannot expect perfect imagery from spherical surfaces; hence, it will be necessary to place limitations on the way a spherical lens can be used so that it behaves appropriately. Thus, we only allow the lens to receive rays that strike it *not far from the central axis and enter only at shallow angles*—rays of this kind are said to be **paraxial**. No matter how the rays are drawn (and they will often be depicted making large angles for the sake of clarity), the rays are nonetheless paraxial. To simplify matters further, we will only deal with **thin lenses**; that is, *lenses for which the radii of curvature of the surfaces are large compared to the thickness*. Such lenses are quite common—most telescope and eyeglass lenses are thin.

The distances of objects and images are usually measured from the lens and can be on either side of it. It is important to associate a specific sign with each such distance so that it can be properly manipulated algebraically. ***As a rule, light enters from the left***, and Fig. 26.5 shows how a typical ray is twice bent toward the central axis as it traverses a convex spherical lens. In Fig. 26.6a, we see two rays leaving the axial point source S and converging to the corresponding image point P. This is the basic geometry for which several possible sign conventions exist. We will take an **object distance** s_o to the *left* of the lens as positive and an **image distance** s_i to the *right* of the lens as positive. Furthermore, the radius of curvature of a lens surface is positive when its center point C is to the right of the surface. Here R_1, the radius of the first surface encountered, is positive while R_2, whose center C_2 is to the left, is negative. ***All interfaces that bulge toward the left have positive radii, and all interfaces that bulge right have negative radii.***

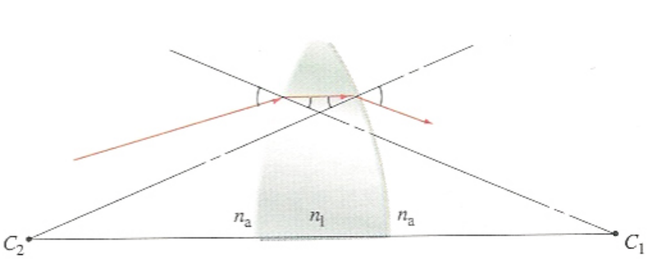
To derive an equation for the operation of a spherical lens, note that the optical path lengths traversed by all rays from S to P are equal. All portions of the diverging wavefront take the same time to reach P, and all arrive in-phase. Let's examine two such paths, one along the central axis and one higher up, some arbitrary distance y. The O.P.L. from S to A to G to P must be the same as that from S to H to J to P. The wavefronts just in front of and behind the lens, Σ and Σ', have radii $\overline{SH} = \overline{SA}$ and $\overline{GP} = \overline{JP}$, respectively, and so if the O.P.L. from A to G equals that from H to J, the overall optical path lengths along the two routes will be equal. The paraxial limitation requires that y be small and since the lens is thin, we can approximate the path from A to G as a straight line (Fig. 26.6b). For simplicity, assume the lens has an index n_1 and it is immersed in air with an index of one. Thus, S will be imaged at P provided that

$$n_1\overline{HI} + n_1\overline{IJ} = \overline{AB} + \overline{BC} + n_1\overline{CD} + n_1\overline{DE} + \overline{EF} + \overline{FG} \tag{26.3}$$

for all allowed values of y.

Since all the surfaces are spheres, the curves in the figure are circles, and there is a nice way to represent each of these little distances. Figure 26.6c shows a chord cutting a diameter ($2R$) of a circle perpendicularly. The piece σ is called the *saggita* (from the Latin for *arrow*) because it looks like an arrow resting on a bow. There is a theorem (proved in Problem 34) stating that for two such intersecting lines, the product of the two segments of one equals the

Figure 26.5 The radius drawn from C_1 is normal to the first surface, and as the ray enters the lens, it bends down *toward* that normal. The radius from C_2 is normal to the second surface, and as the ray emerges, since $n_1 > n_a$, the ray bends down *away* from that normal.



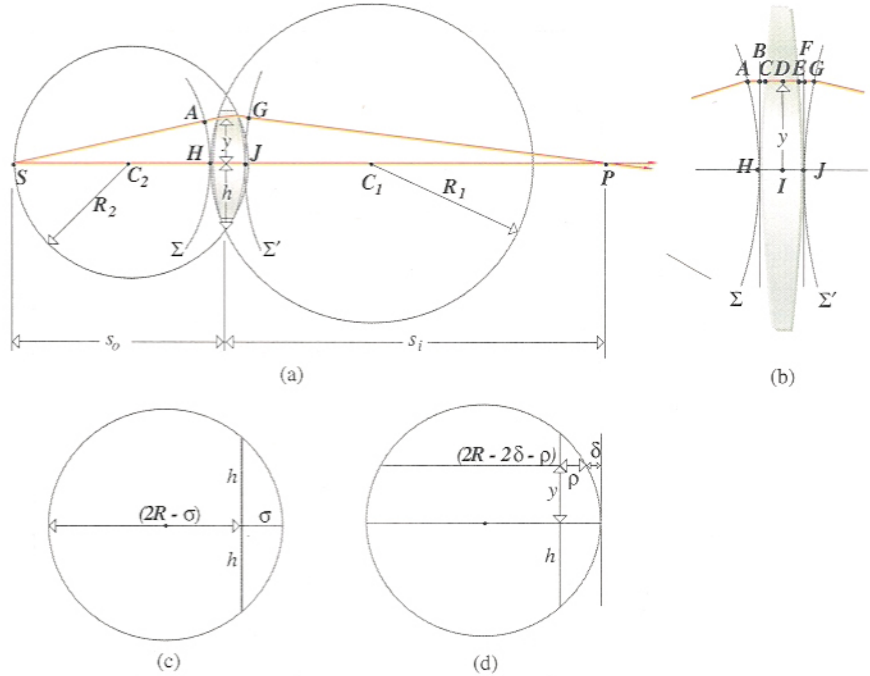


Figure 26.6 (a) The geometry of the thin lens. (b), (c), and (d) are associated with the approximations needed to determine the optical path lengths for rays from S to P.

product of the segments of the other:

$$\sigma(2R - \sigma) = (h)(h)$$

Hence
$$2R\sigma - \sigma^2 = h^2$$

But we are only interested in cases where $R \gg \sigma$ and there σ^2 is negligible compared to $2R\sigma$; hence, $2R\sigma \approx h^2$ and

$$\sigma \approx \frac{h^2}{2R} \tag{26.4}$$

Figure 26.6d shows another chord, the segment ρ resembling a saggita, and the length δ extending beyond the circle. In the same way as above, it follows (Problem 35) that

$$\rho \approx \frac{(h^2 - y^2)}{2R} \quad \text{and} \quad \delta \approx \frac{y^2}{2R} \tag{26.5}$$

These three approximations correspond to the segments in Fig. 26.6b where the surfaces of the lens form circles of radii R_1 and $-R_2$ and the radii of the wavefronts Σ and Σ' are represented by s_o and s_i, respectively. This last approximation is equivalent to saying that the lens is so thin we can measure the object and image distances from *either* its center or from its faces. Pressing on, we have

$$\overline{AB} = \delta_0 \approx y^2/2s_0 \qquad \overline{DE} = \rho_2 \approx (h^2 - y^2)/-2R_2$$
$$\overline{BC} = \delta_1 \approx y^2/2R_1 \qquad \overline{EF} = \delta_2 \approx y^2/-2R_2$$
$$\overline{CD} = \rho_1 \approx (h^2 - y^2)/2R_1 \qquad \overline{FG} = \delta_i \approx y^2/2s_i$$
$$\overline{HI} = \sigma_1 \approx h^2/2R_1 \qquad \overline{IJ} = \sigma_2 \approx h^2/-2R_2$$

Now, substituting all of these into Eq. (26.3) and shifting things around, we see that

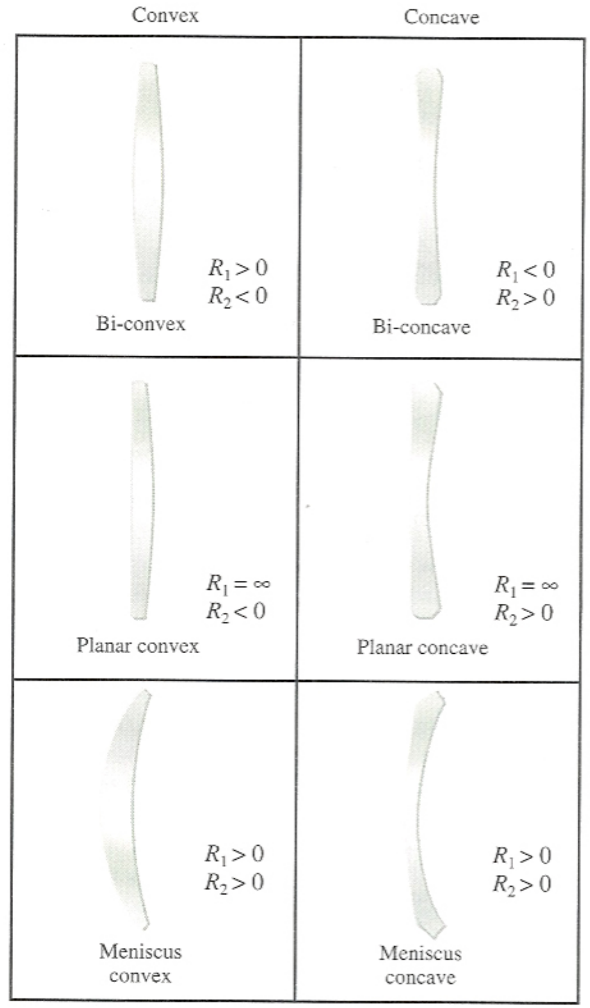


Figure 26.7 Cross sections of various centered spherical simple lenses. Take the surface on the left as number one, since it's encountered first by light coming from the left.

every term with h^2 vanishes; the $y^2/2$ factors out and cancels, leaving

$$\frac{1}{s_o} + \frac{1}{s_i} = (n_l - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{26.6}$$

This is the **Thin-Lens Equation**, often referred to as the **Lensmaker's Formula** because the left side (which treats what is going on external to the lens) is given in terms of the physical variables that would have to be selected to fabricate the lens (Fig. 26.7).

It can be shown (Problem 37) that, had the lens been immersed in a medium of index n_m rather than air, the Lensmaker's Formula would again result, but instead of the first term on the right being n_l, or equivalently $n_l/1$, it would now be n_l/n_m.