Notes on Application of Finite Element Method to the Solution of the Poisson Equation

Consider the arbitrary two dimensional domain, Ω shown in Figure 1. The Poisson equation for this domain can be written

$$\nabla^2\phi = 1 \qquad (1)$$
$$\phi = 0 \quad \text{on } \partial\Omega \qquad (2)$$

where we have specified a constant right hand side, and dirichilet boundary conditions.

Applying the weighted residual method, we get:

$$\int_\Omega (\nabla^2 u - 1)\nu \mathrm{d}\Omega = 0 \qquad (3)$$

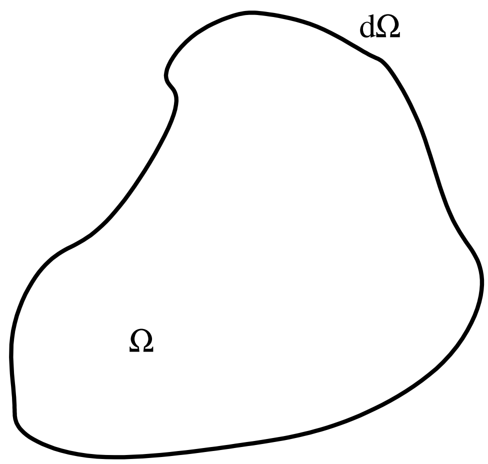


Figure 1: Domain for the Poisson Equation

Now, making use of the divergence theorem to reduce the order of the weighted residual equation we may express the integral as

$$\int_{\partial\Omega} \nu\nabla u \cdot n\partial\Omega - \int_\Omega \nabla u \cdot \nabla\nu \mathrm{d}\Omega - \int_\Omega \nu \mathrm{d}\Omega = 0 \qquad (4)$$

Equation 4 is called the symmetric weak form of the Poisson equation. The equation contains three separate integrals. A boundary integral involving the flux of u, the symmetric integral for the integral domain and the load integral. Examining Equation 4 we see that an amissable solution for u and ν only requires a C^0 continuous function. As long as the test function ν satisfies this requirement may used. So, for convenience we may also specify that ν vanishes at the boundary, thus the boundary integral also will vanish.

Due to the arbitrariness of the solution domain, we discretize the computational domain and evaluate the symmetric weak form as the sum of integrals over triangular patches. Figure 2 shows the discretized domain. Since we only require that the test and trial functions are C^0 continuous, we can represent u as a linear function of the nodal values as shown in Figure 2. Here we assume local linear trial and test functions of the form

$$u^j = \sum q_i^j N_i \qquad i = 1, 2, 3$$
$$\nu^j = \sum p_k^j N_k \qquad k = 1, 2, 3$$

Substituting these test and trial function into the second integral of the symmetric

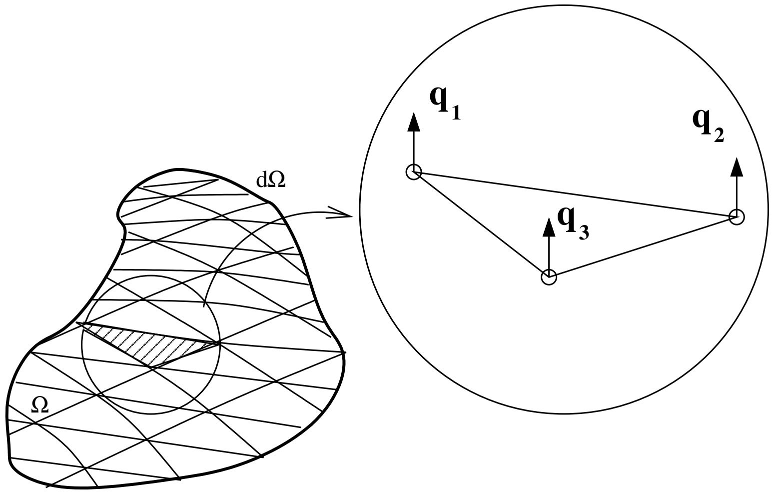


Figure 2: Domain Discretization

weak form we get

$$\begin{aligned}\int_\Omega \nabla u \cdot \nabla \nu \mathrm{d}\Omega &= \sum_{j=1}^{N} \int_\Omega \nabla(q_n^j N_n) \cdot \nabla(p_m^j N_m)\mathrm{d}\Omega \\ &= \sum_{j=1}^{N} \int_\Omega q_n^j N_{n,i} p_m^j N_{m,i} \mathrm{d}\Omega \\ &= \sum_{j=1}^{N} p_m^j K_{m,n}^j q_n^j\end{aligned}$$

where $K_{m,n}^j$, the element stiffness matrix, is given by

$$K_{m,n}^j = \int_{\Omega_j} N_{n,i} N_{m,i} \mathrm{d}\Omega_\mathrm{j} \quad (5)$$

Corrdinate Transformation and Shape Functions

Before we write the shape functions explicitly, we first define a local linear transformation under which the coordinates of each element are mapped into the ξ, η plane. Figure 3 shows graphically what this transformation would look like. To explicitly define this transformation, we represent x and y as linear functions of ξ and η in the form

$$\begin{aligned}x &= a + b\xi + c\eta \\ y &= d + e\xi + g\eta\end{aligned}$$

By evaluating these expressions at all three values of x, y corresponding to values of

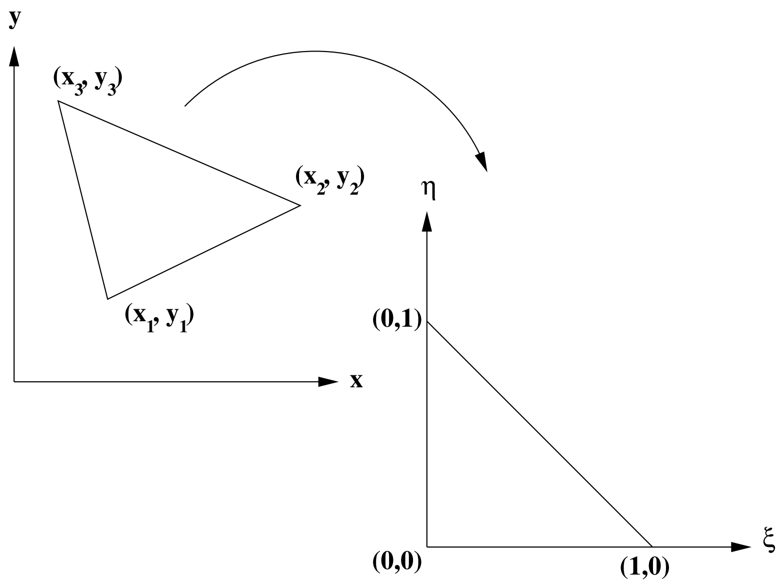


Figure 3: Transformation from Global to Local Coordinate System

ξ, η, we get a system of six equations for a, b, c, d, e, and g.

$$
\begin{aligned}
x_1 &= a \\
x_2 &= a + b \\
x_3 &= a + c \\
y_1 &= d \\
y_2 &= d + e \\
y_3 &= d + g
\end{aligned}
$$

Solving these equations we see the correct local tranformation is

$$
\begin{array}{ccccccc}
x & = & x_1 & + & (x_2 - x_1)\xi & + & (x_3 - x_1)\eta \\
y & = & y_1 & + & (y_2 - y_1)\xi & + & (y_3 - y_1)\eta
\end{array} \tag{6}
$$

Also, we may evaluate the Jacobian of the transformation to give us the relationship between the two coordinate systems as

$$
\mathbf{J} = \begin{bmatrix} \dfrac{\partial \mathrm{x}}{\partial \xi} & \dfrac{\partial \mathrm{x}}{\partial \eta} \\ \dfrac{\partial \mathrm{y}}{\partial \xi} & \dfrac{\partial \mathrm{y}}{\partial \eta} \end{bmatrix} = \begin{bmatrix} (x_2 - x_1) & (x_3 - x_1) \\ (y_2 - y_1) & (y_3 - y_1) \end{bmatrix} = \begin{bmatrix} \Delta \mathrm{x}_2 & \Delta \mathrm{x}_3 \\ \Delta \mathrm{y}_2 & \Delta \mathrm{y}_3 \end{bmatrix} \tag{7}
$$

where we have introduced the notation $(x_2 - x_1) = \Delta \mathrm{x}_2$ for a convenience.

To define the shape functions we note the desired boundary conditions for each shape function as shown in Table 1. Since the global requirement of continuity on the trial function is only C^0 continuity, we may assume a linear form for the local shape function $(N_i = c_1^i + c_2^i \xi + c_3^i \eta)$. Evaluating the boundary conditions we get a system of nine

	(x_1, y_1)	(x_2, y_2)	(x_3, y_3)
N_1	1	0	0
N_2	0	1	0
N_3	0	0	1

Table 1: Boundary Conditions for Shape Functions

equations with nine unknowns.

$$\begin{aligned}
N_1(1) &= 1 &&= c_1^1 \\
N_1(2) &= 0 &&= c_1^1 + c_2^1 \\
N_1(3) &= 0 &&= c_1^1 + c_3^1 \\
N_2(1) &= 0 &&= c_1^2 \\
N_2(2) &= 1 &&= c_1^2 + c_2^2 \\
N_2(3) &= 0 &&= c_1^2 + c_3^2 \\
N_3(1) &= 0 &&= c_1^3 \\
N_3(2) &= 0 &&= c_1^3 + c_2^3 \\
N_3(3) &= 1 &&= c_1^3 + c_3^3
\end{aligned}$$

Solving this system, we arrive at the shape functions in the element local coordinate system as

$$\begin{aligned}
N_1 &= 1 - \xi - \eta \\
N_2 &= \xi \\
N_3 &= \eta
\end{aligned} \tag{8}$$

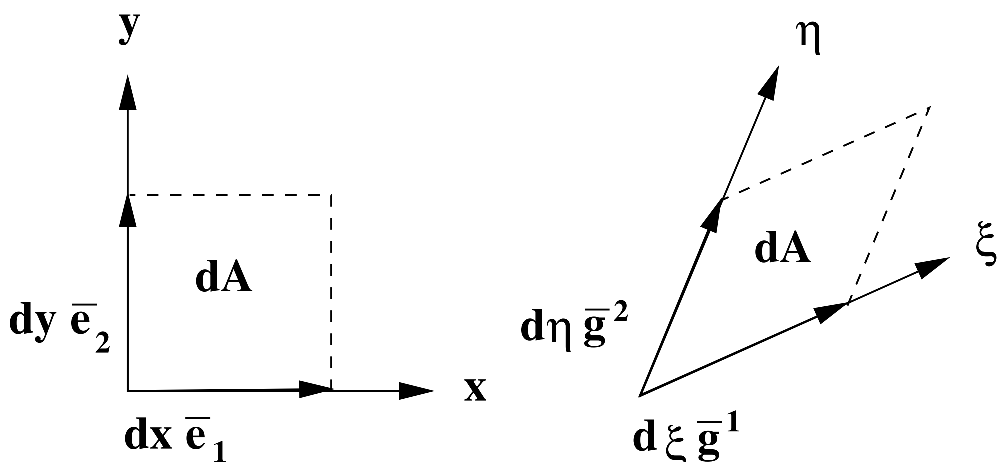
Coordinate System Transformations for Area Integrals

Because of the simplicity of the element local coordinate system, it is desirable to evaluate the stiffness matrix integral in this coordinate system. However, since the original equation is in terms of the standard Cartesian coordinate system, we must properly rewrite the integral in terms of the local coordinates. To do this, we must properly express a differential area element in this new system. In Figure 4(a) we see the differential area element represented in the Cartesian coordinate system. Noting that the area of the parallelagram formed between two non-parallel vectors can found by the norm of the cross product of the two vectors, we express the area element dA in terms of the unit base vectors $\bar{\mathbf{e}}_1$,$\bar{\mathbf{e}}_2$ and differential lengths dx, dy as:

$$\begin{aligned}
\mathrm{dA} &= \|\mathrm{dx}\,\bar{\mathbf{e}}_1 \times \mathrm{dy}\,\bar{\mathbf{e}}_2\| \\
&= \|\bar{\mathbf{e}}_1 \times \bar{\mathbf{e}}_2\| \mathrm{dx}\,\mathrm{dy} \\
&= \|\bar{\mathbf{e}}_3\| \mathrm{dx}\,\mathrm{dy} \\
&= \mathrm{dx}\,\mathrm{dy}
\end{aligned}$$

Similarly, for the element local coordinate system we may represent the differential area element in terms of the covariant base vectors $\bar{\mathbf{g}}^1$,$\bar{\mathbf{g}}^2$ and the differential lengths $\mathrm{d}\xi$, $\mathrm{d}\eta$ as:

$$\begin{aligned}
\mathrm{dA} &= \|\mathrm{d}\xi\,\bar{\mathbf{g}}^1 \times \mathrm{d}\eta\,\bar{\mathbf{g}}^2\| \\
&= \|\bar{\mathbf{g}}^1 \times \bar{\mathbf{g}}^2\| \mathrm{d}\xi\,\mathrm{d}\eta
\end{aligned}$$



(a) Cartesian Coordinate System (b) Element Local Coordinate System

Figure 4: Area Elements in Respective Coordinate Systems

To evaluate the cross product, $\overline{\mathbf{g}}^1 \times \overline{\mathbf{g}}^2$, we first express a vector $\overline{\mathbf{R}}$ in terms of both coordinate systems as:

$$\overline{\mathbf{R}} = \overline{\mathbf{g}}^1\,\xi + \overline{\mathbf{g}}^2\,\eta = \overline{\mathbf{e}}_1\,\mathrm{x} + \overline{\mathbf{e}}_2\,\mathrm{y}$$

By definition the covariant base vectors are:

$$\overline{\mathbf{g}}^1 = \frac{\partial \overline{\mathbf{R}}}{\partial \xi} = \frac{\partial(\overline{\mathbf{e}}_1\,\mathrm{x} + \overline{\mathbf{e}}_2\,\mathrm{y})}{\partial \xi} + \frac{\partial(\overline{\mathbf{e}}_1\,\mathrm{x} + \overline{\mathbf{e}}_2\,\mathrm{y})}{\partial \xi} = \overline{\mathbf{e}}_1\,\frac{\partial \mathrm{x}}{\partial \xi} + \overline{\mathbf{e}}_2\,\frac{\partial \mathrm{y}}{\partial \xi}$$
$$\overline{\mathbf{g}}^2 = \frac{\partial \overline{\mathbf{R}}}{\partial \eta} = \frac{\partial(\overline{\mathbf{e}}_1\,\mathrm{x} + \overline{\mathbf{e}}_2\,\mathrm{y})}{\partial \eta} + \frac{\partial(\overline{\mathbf{e}}_1\,\mathrm{x} + \overline{\mathbf{e}}_2\,\mathrm{y})}{\partial \eta} = \overline{\mathbf{e}}_1\,\frac{\partial \mathrm{x}}{\partial \eta} + \overline{\mathbf{e}}_2\,\frac{\partial \mathrm{y}}{\partial \eta}$$

Now that we have expressed the covariant base vectors in terms of the known cartesian base vectors we may evaluate the cross product. The result is

$$\|\overline{\mathbf{g}}^1 \times \overline{\mathbf{g}}^2\| = \left\| \begin{matrix} \frac{\partial \mathrm{x}}{\partial \xi} & \frac{\partial \mathrm{y}}{\partial \xi} \\ \frac{\partial \mathrm{x}}{\partial \eta} & \frac{\partial \mathrm{y}}{\partial \eta} \end{matrix} \right\| = \frac{\partial \mathrm{x}}{\partial \xi}\,\frac{\partial \mathrm{y}}{\partial \eta} - \frac{\partial \mathrm{x}}{\partial \eta}\,\frac{\partial \mathrm{y}}{\partial \xi} = \mathrm{J}$$

Recognizing the matrix resulting from the cross product $\overline{\mathbf{g}}^1 \times \overline{\mathbf{g}}^2$ as the Jacobian of the transformation, we have used the (J) to denote that the norm of the cross product is also the determinant of the Jacobian matrix. Completing the expression for the differential area element in the element local coordinate system, we have

$$\mathrm{dA} = \mathrm{J}\;\mathrm{d}\xi\;\mathrm{d}\eta$$

To summarize, we have shown that in order properly evaluate an are integral in the element local coordinate system, we should use Equation 9. This equation arose from reexpressing the differential area in the element local coordinate system in terms of the original cartesian coordinate system.

$$\iint f(\mathrm{x},\mathrm{y})\;\mathrm{dxdy} = \iint f(\xi,\eta)\mathrm{J}\;\mathrm{d}\xi\;\mathrm{d}\eta \tag{9}$$

Evaluating Local Stiffness Matrices

Now that we have defined the shape functions and a transformation from the global coordinate system to an element local system, we return to the local stiffness matrices. Examining Equation 5, and using the Jacobian matrix as in Equaiton 7, we can find the derivatives of the shape functions as

$$\left[\begin{array}{c}\frac{\partial N_m^j}{\partial x}\\ \frac{\partial N_m^j}{\partial y}\end{array}\right]\left[\begin{array}{cc}\frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta}\\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta}\end{array}\right]=\left[\begin{array}{c}\frac{\partial N_m^j}{\partial x}\\ \frac{\partial N_m^j}{\partial y}\end{array}\right][\mathbf{J}]=\left[\begin{array}{c}\frac{\partial N_m^j}{\partial \xi}\\ \frac{\partial N_m^j}{\partial \eta}\end{array}\right]$$

thus

$$\left[\begin{array}{c}\frac{\partial N_m^j}{\partial x}\\ \frac{\partial N_m^j}{\partial y}\end{array}\right]=\left[\begin{array}{c}\frac{\partial N_m^j}{\partial \xi}\\ \frac{\partial N_m^j}{\partial \eta}\end{array}\right][\mathbf{J}]^{-1}$$

or

$$\left\{\begin{array}{c}\frac{\partial N_n^j}{\partial x}\\ \frac{\partial N_n^j}{\partial y}\end{array}\right\}=[\mathbf{J}]^{-\mathrm{T}}\left\{\begin{array}{c}\frac{\partial N_n^j}{\partial \xi}\\ \frac{\partial N_n^j}{\partial \eta}\end{array}\right\}$$

where $j = 1, 2, ...N$ is the number of the element and $m = 1, 2, 3$ is the number of the corresponding element local shape function. Using the matrix notation developed above, we can now express Equation 5 as

$$K_{m,n}^j=\iint\left[\begin{array}{c}\frac{\partial N_m^j}{\partial \xi}\\ \frac{\partial N_m^j}{\partial \eta}\end{array}\right][\mathbf{J}]^{-1}[\mathbf{J}]^{-\mathrm{T}}\left\{\begin{array}{c}\frac{\partial N_n^j}{\partial \xi}\\ \frac{\partial N_n^j}{\partial \eta}\end{array}\right\}\mathrm{J}\ \mathrm{d}\xi\ \mathrm{d}\eta \qquad (10)$$

Using Equation 8 we can easily evaluate the local derivatives of the element shape functions. which are:

$$\begin{array}{llll}\frac{\partial N_1}{\partial \xi} = -1 & \frac{\partial N_1}{\partial \eta} = -1\\ \frac{\partial N_2}{\partial \xi} = 1 & \frac{\partial N_2}{\partial \eta} = 0\\ \frac{\partial N_3}{\partial \xi} = 0 & \frac{\partial N_3}{\partial \eta} = 1\end{array} \qquad (11)$$

To obtain expressions for $[\mathbf{J}]^{-1}$ and $[\mathbf{J}]^{-\mathrm{T}}$ we start with the Jacobian as in Equation 7 and perform the standard inversion and transpose for a two by two matrix. This yields

$$\begin{aligned}\mathbf{J} &= \left[\begin{array}{cc}\Delta \mathrm{x}_2 & \Delta \mathrm{x}_3\\ \Delta \mathrm{y}_2 & \Delta \mathrm{y}_3\end{array}\right]\\ \mathbf{J}^{-1} &= \frac{1}{\mathrm{J}}\left[\begin{array}{cc}\Delta \mathrm{y}_3 & -\Delta \mathrm{x}_3\\ -\Delta \mathrm{y}_2 & \Delta \mathrm{x}_2\end{array}\right]\\ \mathbf{J}^{-\mathrm{T}} &= \frac{1}{\mathrm{J}}\left[\begin{array}{cc}\Delta \mathrm{y}_3 & -\Delta \mathrm{y}_2\\ -\Delta \mathrm{x}_3 & \Delta \mathrm{x}_2\end{array}\right]\end{aligned}$$

where J is the determinant of the Jacobian given by

$$\mathrm{J} = \Delta \mathrm{x}_2 \Delta \mathrm{y}_3 - \Delta \mathrm{x}_3 \Delta \mathrm{y}_2.$$

Using these expressions, one can compute the product $\mathbf{J}^{-1}\mathbf{J}^{-\mathrm{T}}$ as

$$\begin{aligned}
\mathbf{J}^{-1}\mathbf{J}^{-\mathrm{T}} &= \frac{1}{\mathrm{J}^2}\begin{bmatrix} \Delta \mathrm{y}_3 & -\Delta \mathrm{x}_3 \\ -\Delta \mathrm{y}_2 & \Delta \mathrm{x}_2 \end{bmatrix}\begin{bmatrix} \Delta \mathrm{y}_3 & -\Delta \mathrm{y}_2 \\ -\Delta \mathrm{x}_3 & \Delta \mathrm{x}_2 \end{bmatrix} \\
&= \frac{1}{\mathrm{J}^2}\begin{bmatrix} \Delta \mathrm{x}_3^2 + \Delta \mathrm{y}_3^2 & -(\Delta \mathrm{x}_2 \Delta \mathrm{x}_3 + \Delta \mathrm{y}_2 \Delta \mathrm{y}_3) \\ -(\Delta \mathrm{x}_2 \Delta \mathrm{x}_3 + \Delta \mathrm{y}_2 \Delta \mathrm{y}_3) & \Delta \mathrm{x}_2^2 + \Delta \mathrm{y}_2^2 \end{bmatrix} \\
&= \frac{1}{\mathrm{J}^2}\begin{bmatrix} \lambda_3 & -\lambda_{23} \\ -\lambda_{23} & \lambda_2 \end{bmatrix}
\end{aligned} \tag{12}$$

where the notataion $\lambda_2 = \Delta \mathrm{x}_2^2 + \Delta \mathrm{y}_2^2$ has been introduced for convenience. The element stiffness matrix now becomes

$$K^j_{n,m} = \iint \frac{1}{\mathrm{J}} \begin{bmatrix} \dfrac{\partial N^j_n}{\partial \xi} \\ \dfrac{\partial N^j_n}{\partial \eta} \end{bmatrix} \begin{bmatrix} \lambda_3 & -\lambda_{23} \\ -\lambda_{23} & \lambda_2 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial N^j_m}{\partial \xi} \\ \dfrac{\partial N^j_m}{\partial \eta} \end{Bmatrix} \mathrm{d}\xi \, \mathrm{d}\eta$$

Notice that all the quantities involved in the integral are constant for a given element. Thus, evaluating the integral over a triangular element simply gives us 1/2 times the constant integrand. Using Equation 11 we see that the components of the element stiffness matrix as:

$$\begin{aligned}
\mathrm{J}\,K^j_{1,1} &= \lambda_2 + \lambda_3 - 2\lambda_{23} & \mathrm{J}\,K^j_{1,2} &= \lambda_{23} - \lambda_3 & \mathrm{J}\,K^j_{1,3} &= \lambda_{23} - \lambda_2 \\
\mathrm{J}\,K^j_{2,1} &= \lambda_{23} - \lambda_3 & \mathrm{J}\,K^j_{2,2} &= \lambda_3 & \mathrm{J}\,K^j_{2,3} &= -\lambda_{23} \\
\mathrm{J}\,K^j_{3,1} &= \lambda_{23} - \lambda_2 & \mathrm{J}\,K^j_{3,2} &= -\lambda_{23} & \mathrm{J}\,K^j_{3,3} &= \lambda_2
\end{aligned}$$

with

$$\mathrm{J} = \Delta \mathrm{x}_2 \Delta \mathrm{y}_3 - \Delta \mathrm{x}_3 \Delta \mathrm{y}_2$$

Assembly of the Global Stiffness Matrix and Load Vector

To assemble the global stiffness matrix we must place componenets of the local stiffness matrices in their appropriate locations in the gobal stiffness matrix. To accomplish this, we must first have a mapping from the local node numbering to the global node numbering. For the case of our simple triangular element, this means that for each local node ($il = 1, 2, 3$) we must have a global node number ($ig \in [1, ...N]$), where N is the total number of nodes. Once we have this mapping, all that remains is to cycle through each element and the local stiffness matrix componenets to the global stiffness matrix component as follows:

$$K(ig, jg) = K(ig, jg) + K^j_{local}(ie, je) \quad \begin{cases} \text{for } ie = 1, 2, 3 \\ \text{for } je = 1, 2, 3 \\ \text{for } j = 1, ...N \end{cases}$$

To assemble the load vector, we first evaluate the integral from Equation 4

$$\int_{\Omega} \nu \mathrm{d}\Omega = \sum_{j=1}^{N} \iint p_k^j N_k^j \mathrm{J}\, \mathrm{d}\xi\, \mathrm{d}\eta \quad \mathrm{k} = 1, 2, 3$$
$$= \lfloor p_k^j \rfloor \{Q_k^j\}$$

where

$$Q_k^j = \iint N_k^j \mathrm{J}\, \mathrm{d}\xi\, \mathrm{d}\eta = \frac{1}{6}\, \mathrm{J}$$

Note that integration of each shape function over a triangular element comes out to be the same constant for a given element. Now, assembly of a global load vector follow much the same process as for the stiffness matrix, where the following rule is used:

$$Q(ig) = Q(ig) + Q_{local}^j(ie) \quad \begin{cases} \text{for } ie = 1, 2, 3 \\ \text{for } j = 1, ...N \end{cases}$$

Solution of the FEM System of Equations

Returning to the Symmetric Weak form of the governing equation, we now may express the integrals in the global domain in terms of summations of the integrals in the element local domains. The result is

$$\lfloor p_n \rfloor \left[K_{n,m}\right] \{q_m\} = \lfloor p_k \rfloor \{Q_k\}$$

Here we have left of the element index j to signify these as global vectors and matrices. Since the coefficient vector p^j is arbitrary, we chose it to be a series of linearly independent vectors such that we end up with the following system of equations.

$$\left[K_{n,m}\right] \{q_m\} = \{Q_n\}$$

Application of the Boundary Conditions

The boundary condition is a Dirichilet type, specifically $\phi = 0$ on the boundary. To apply this boundary condition to our system of equations, we must simply modify the global load vector and stiffness matrix to fix the values of ϕ at the boundary nodes. For instance. Suppose global node n is a boundary node. Set all of the coefficients corresponding to the equation for node n in the global stiffness matrix equal to zero except for the diagonal which is set to one. For the load vecotor, we set the nth component also to zero. This is shown below, and must be repeated for each boundary node.

$$\begin{bmatrix} k_{1,1} & \dots & k_{1,n} & \dots & k_{1,N} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ k_{n,1} & \dots & k_{n,n} & \dots & k_{n,N} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ k_{N,1} & \dots & k_{N,n} & \dots & k_{N,N} \end{bmatrix} \left\{ \begin{array}{c} q_1 \\ \vdots \\ q_n \\ \vdots \\ q_N \end{array} \right\} = \left\{ \begin{array}{c} Q_1 \\ \vdots \\ Q_n \\ \vdots \\ Q_N \end{array} \right\}$$

$$\Downarrow$$

$$\begin{bmatrix} k_{1,1} & \dots & k_{1,n} & \dots & k_{1,N} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ k_{N,1} & \dots & k_{N,n} & \dots & k_{N,N} \end{bmatrix} \left\{ \begin{array}{c} q_1 \\ \vdots \\ q_n \\ \vdots \\ q_N \end{array} \right\} = \left\{ \begin{array}{c} Q_1 \\ \vdots \\ 0 \\ \vdots \\ Q_N \end{array} \right\}$$

We may now solve the modified system of equations for q, the nodal values of the solution variable ϕ

$$\{q_m\} = \left[K^*_{n,m}\right]^{-1} \{Q^*_n\}$$