Notes: http://bessie.che.uc.edu/tlb/teach/undergrad/chem381f97/math/node7.html

$$\text{soln} = \int_0^1 \int_0^1 \{\partial_\xi u, \partial_\eta u\}\, J^{-T} J^{-1} \begin{pmatrix} \partial_\xi \nu \\ \partial_\eta \nu \end{pmatrix} \|J\|\, d\xi\, d\eta$$

$$\text{local stiffness matrix}: K^j_{mn} = \int_0^1 \int_0^1 \{\partial_\xi N_m, \partial_\eta N_m\}\, J^{-T} J^{-1} \begin{pmatrix} \partial_\xi N_n \\ \partial_\eta N_n \end{pmatrix} \|J\|\, d\xi\, d\eta$$

Triangle Element

shape functions : $N_1 = 1 - \xi - \eta,\ N_2 = \xi,\ N_3 = \eta$

trial and test function : $u = \sum_{i=1}^{3} q_i N_i,\ v = \sum_{i=1}^{3} p_k N_k \to \frac{\partial u}{\partial \xi} = -q_1 + q_2,\ \frac{\partial v}{\partial \xi} = -p_1 + p_2,\ \frac{\partial u}{\partial \eta} = -q_1 + q_3,\ \frac{\partial v}{\partial \eta} = -p_1 + p_3$

$\vec{x} = \vec{x}_1 + (\vec{x}_2 - \vec{x}_1)\,\xi + (\vec{x}_3 - \vec{x}_1)\,\eta \to x = x_1 + (x_2 - x_1)\,\xi + (x_3 - x_1)\,\eta,\ y = y_1 + (y_2 - y_1)\,\xi + (y_3 - y_1)\,\eta$

$\Delta x_2 \equiv x_2 - x_1,\ \Delta y_2 \equiv y_2 - y_1,\ \Delta x_3 \equiv x_3 - x_1,\ \Delta y_3 \equiv y_3 - y_1$

Jacobian : $J = \begin{pmatrix} \partial_\xi x & \partial_\eta x \\ \partial_\xi y & \partial_\eta y \end{pmatrix} = \begin{pmatrix} \Delta x_2 & \Delta x_3 \\ \Delta y_2 & \Delta y_3 \end{pmatrix},\ \{dx, dy\} = J\,\{d\xi, d\eta\}$

eigenvalues : $\lambda_2 \equiv \Delta x_2^2 + \Delta y_2^2,\ \lambda_{23} \equiv \Delta x_2\, \Delta x_3 + \Delta y_2\, \Delta y_3,\ \lambda_3 \equiv \Delta x_3^2 + \Delta y_3^2$

$$K^j_{mn} = \frac{1}{\Delta x_2\, \Delta y_3 - \Delta x_3\, \Delta y_2} \begin{pmatrix} \lambda_2 + \lambda_3 - 2\,\lambda_{23} & \lambda_{23} - \lambda_3 & \lambda_{23} - \lambda_2 \\ \lambda_{23} - \lambda_3 & \lambda_3 & -\lambda_{23} \\ \lambda_{23} - \lambda_2 & -\lambda_{23} & \lambda_2 \end{pmatrix}$$

Quad Element

shape functions : $N_1 = (\eta - 1)\,(\xi - 1),\ N_2 = \xi\,(1 - \eta),\ N_3 = \eta\,(1 - \xi),\ N_4 = \xi\,\eta$

$\vec{x} = \vec{x}_1 + (\vec{x}_2 - \vec{x}_1)\,\xi + (\vec{x}_3 - \vec{x}_1)\,\eta + (\vec{x}_1 - \vec{x}_2 - \vec{x}_3 + \vec{x}_4)\,\xi\,\eta$

$\Delta x_2 \equiv x_2 - x_1,\ \Delta y_2 \equiv y_2 - y_1,\ \Delta x_3 \equiv x_3 - x_1,\ \Delta y_3 \equiv y_3 - y_1,\ \Delta x_a \equiv x_1 - x_2 - x_3 + x_4,\ \Delta y_a \equiv y_1 - y_2 - y_3 + y_4$

Jacobian : $J = \begin{pmatrix} \partial_\xi x & \partial_\eta x \\ \partial_\xi y & \partial_\eta y \end{pmatrix} = \begin{pmatrix} \Delta x_2 + \eta\, \Delta x_a & \Delta x_3 + \xi\, \Delta x_a \\ \Delta y_2 + \eta\, \Delta y_a & \Delta y_3 + \xi\, \Delta y_a \end{pmatrix}$

eigenvalues : $\lambda_2 \equiv \Delta x_2^2 + \Delta y_2^2,\ \lambda_{23} \equiv \Delta x_2\, \Delta x_3 + \Delta y_2\, \Delta y_3,\ \lambda_3 \equiv \Delta x_3^2 + \Delta y_3^2$

$K^j_{mn} = x$

Program

Calculate the x, y coordinates of the nodes :

```
a = b = 2.0; nx = 7; ny = 5;
nodes = Chop[Flatten[Table[{x, y}, {y, −b / 2, b / 2, b / (ny − 1)}, {x, −a / 2, a / 2, a / (nx − 1)}], 1]]
```

```
{{−1., −1.}, {−0.666667, −1.}, {−0.333333, −1.}, {0, −1.}, {0.333333, −1.}, {0.666667, −1.}, {1., −1.},
  {−1., −0.5}, {−0.666667, −0.5}, {−0.333333, −0.5}, {0, −0.5}, {0.333333, −0.5}, {0.666667, −0.5},
  {1., −0.5}, {−1., 0}, {−0.666667, 0}, {−0.333333, 0}, {0, 0}, {0.333333, 0}, {0.666667, 0}, {1., 0},
  {−1., 0.5}, {−0.666667, 0.5}, {−0.333333, 0.5}, {0, 0.5}, {0.333333, 0.5}, {0.666667, 0.5}, {1., 0.5},
  {−1., 1.}, {−0.666667, 1.}, {−0.333333, 1.}, {0, 1.}, {0.333333, 1.}, {0.666667, 1.}, {1., 1.}}
```

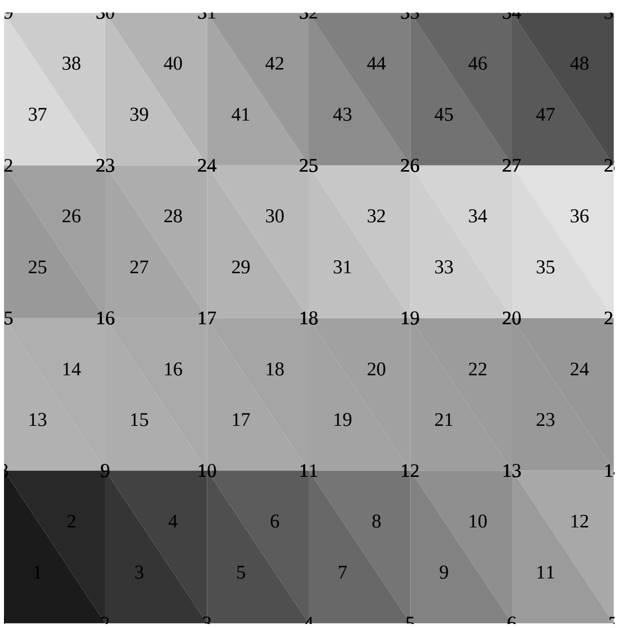
Calculate the indices of nodes of the triangle elements :

```
elements = Flatten[Table[{{i, i + 1, i + nx}, {i + 1, i + nx + 1, i + nx}} + (j − 1) nx, {j, 1, ny − 1}, {i, 1, nx − 1}], 2]
```

```
{{1, 2, 8}, {2, 9, 8}, {2, 3, 9}, {3, 10, 9}, {3, 4, 10}, {4, 11, 10}, {4, 5, 11}, {5, 12, 11}, {5, 6, 12}, {6, 13, 12}, {6, 7, 13},
  {7, 14, 13}, {8, 9, 15}, {9, 16, 15}, {9, 10, 16}, {10, 17, 16}, {10, 11, 17}, {11, 18, 17}, {11, 12, 18}, {12, 19, 18}, {12, 13, 19},
  {13, 20, 19}, {13, 14, 20}, {14, 21, 20}, {15, 16, 22}, {16, 23, 22}, {16, 17, 23}, {17, 24, 23}, {17, 18, 24}, {18, 25, 24},
  {18, 19, 25}, {19, 26, 25}, {19, 20, 26}, {20, 27, 26}, {20, 21, 27}, {21, 28, 27}, {22, 23, 29}, {23, 30, 29}, {23, 24, 30},
  {24, 31, 30}, {24, 25, 31}, {25, 32, 31}, {25, 26, 32}, {26, 33, 32}, {26, 27, 33}, {27, 34, 33}, {27, 28, 34}, {28, 35, 34}}
```

Here is a picture to visualize the triangle elements :

```
jmax = Length[elements]; $DefaultFont = {"Times−Roman", 9};
PlotColor[hue_] := Hue[2 (1 − Min[1, Max[0, hue]]) / 3];
Show[Graphics[Table[plist = Map[nodes[[#]] &, elements[[j]]]; {PlotColor[(j − 1) / (jmax − 1)], Polygon[plist],
        RGBColor[0, 0, 0], Text[j, Plus @@ plist / 3], Table[Text[elements[[j, i]], plist[[i]]], {i, 1, 3}]}, {j, 1, jmax, 1}]],
    AspectRatio → Automatic, PlotRange → {{−1, 1} a / 2, {−1, 1} b / 2}];
```



Calculate the local stiffness matrices :

my calculations (does this agree with Roy?) : $\lambda_2 = \Delta x_2^2 + \Delta y_2^2,\ \lambda_{23} = \Delta x_2\,\Delta y_2 + \Delta x_3\,\Delta y_3,\ \lambda_3 = \Delta x_3^2 + \Delta y_3^2,\ \lambda_a = \Delta x_3^2 - \Delta y_2^2$

$$K_{local} = \frac{1}{\Delta x_2\,\Delta y_3 - \Delta x_3\,\Delta y_2}$$

$$\begin{pmatrix} (\Delta x_2 - \Delta y_2)^2 + (\Delta x_3 - \Delta y_3)^2 & \Delta x_2\,\Delta y_2 - \Delta y_2^2 + \Delta x_3\,\Delta y_3 - \Delta y_3^2 & -\Delta x_2^2 - \Delta x_3^2 + \Delta x_2\,\Delta y_2 + \Delta x_3\,\Delta y_3 \\ \Delta x_2\,\Delta y_2 - \Delta y_2^2 + \Delta x_3\,\Delta y_3 - \Delta y_3^2 & \Delta y_2^2 + \Delta y_3^2 & -\Delta x_2\,\Delta y_2 - \Delta x_3\,\Delta y_3 \\ -\Delta x_2^2 - \Delta x_3^2 + \Delta x_2\,\Delta y_2 + \Delta x_3\,\Delta y_3 & -\Delta x_2\,\Delta y_2 - \Delta x_3\,\Delta y_3 & \Delta x_2^2 + \Delta x_3^2 \end{pmatrix}$$

$$= \frac{1}{\Delta x_2\,\Delta y_3 - \Delta x_3\,\Delta y_2} \begin{pmatrix} \lambda_2 + \lambda_3 - 2\,\lambda_{23} & \lambda_{23} - \lambda_3 + \lambda_a & \lambda_{23} - \lambda_2 - \lambda_a \\ \lambda_{23} - \lambda_3 + \lambda_a & \lambda_3 - \lambda_a & -\lambda_{23} \\ \lambda_{23} - \lambda_2 - \lambda_a & -\lambda_{23} & \lambda_2 + \lambda_a \end{pmatrix}$$

Klocal =
Map[Module[{}, Do[{Δx_i, Δy_i} = nodes[[#[[i]]]] − nodes[[#[[1]]]], {i, 2, 3}]; $\lambda_2 = \Delta x_2^2 + \Delta y_2^2$; $\lambda_{23} = \Delta x_2\,\Delta x_3 + \Delta y_2\,\Delta y_3$;
$\lambda_3 = \Delta x_3^2 + \Delta y_3^2$; d = $\Delta x_2\,\Delta y_3 - \Delta x_3\,\Delta y_2$; Chop[$\frac{1}{d} \begin{pmatrix} \lambda_2 + \lambda_3 - 2\,\lambda_{23} & \lambda_{23} - \lambda_3 & \lambda_{23} - \lambda_2 \\ \lambda_{23} - \lambda_3 & \lambda_3 & -\lambda_{23} \\ \lambda_{23} - \lambda_2 & -\lambda_{23} & \lambda_2 \end{pmatrix}$]] &, elements];

Map[
MatrixForm,
Klocal]

$$\left\{\begin{pmatrix} 2.16667 & -1.5 & -0.666667 \\ -1.5 & 1.5 & 0 \\ -0.666667 & 0 & 0.666667 \end{pmatrix}, \begin{pmatrix} 0.666667 & -0.666667 & 0 \\ -0.666667 & 2.16667 & -1.5 \\ 0 & -1.5 & 1.5 \end{pmatrix}, \begin{pmatrix} 2.16667 & -1.5 & -0.666667 \\ -1.5 & 1.5 & 0 \\ -0.666667 & 0 & 0.666667 \end{pmatrix},\right.$$

$$\begin{pmatrix} 0.666667 & -0.666667 & 0 \\ -0.666667 & 2.16667 & -1.5 \\ 0 & -1.5 & 1.5 \end{pmatrix}, \begin{pmatrix} 2.16667 & -1.5 & -0.666667 \\ -1.5 & 1.5 & 0 \\ -0.666667 & 0 & 0.666667 \end{pmatrix}, \begin{pmatrix} 0.666667 & -0.666667 & 0 \\ -0.666667 & 2.16667 & -1.5 \\ 0 & -1.5 & 1.5 \end{pmatrix},$$

$$\begin{pmatrix} 2.16667 & -1.5 & -0.666667 \\ -1.5 & 1.5 & 0 \\ -0.666667 & 0 & 0.666667 \end{pmatrix}, \begin{pmatrix} 0.666667 & -0.666667 & 0 \\ -0.666667 & 2.16667 & -1.5 \\ 0 & -1.5 & 1.5 \end{pmatrix}, \begin{pmatrix} 2.16667 & -1.5 & -0.666667 \\ -1.5 & 1.5 & 0 \\ -0.666667 & 0 & 0.666667 \end{pmatrix},$$

$$\begin{pmatrix} 0.666667 & -0.666667 & 0 \\ -0.666667 & 2.16667 & -1.5 \\ 0 & -1.5 & 1.5 \end{pmatrix}, \begin{pmatrix} 2.16667 & -1.5 & -0.666667 \\ -1.5 & 1.5 & 0 \\ -0.666667 & 0 & 0.666667 \end{pmatrix}, \begin{pmatrix} 0.666667 & -0.666667 & 0 \\ -0.666667 & 2.16667 & -1.5 \\ 0 & -1.5 & 1.5 \end{pmatrix},$$

$$\begin{pmatrix} 2.16667 & -1.5 & -0.666667 \\ -1.5 & 1.5 & 0 \\ -0.666667 & 0 & 0.666667 \end{pmatrix}, \begin{pmatrix} 0.666667 & -0.666667 & 0 \\ -0.666667 & 2.16667 & -1.5 \\ 0 & -1.5 & 1.5 \end{pmatrix}, \begin{pmatrix} 2.16667 & -1.5 & -0.666667 \\ -1.5 & 1.5 & 0 \\ -0.666667 & 0 & 0.666667 \end{pmatrix},$$

$$\begin{pmatrix} 0.666667 & -0.666667 & 0 \\ -0.666667 & 2.16667 & -1.5 \\ 0 & -1.5 & 1.5 \end{pmatrix}, \begin{pmatrix} 2.16667 & -1.5 & -0.666667 \\ -1.5 & 1.5 & 0 \\ -0.666667 & 0 & 0.666667 \end{pmatrix}, \begin{pmatrix} 0.666667 & -0.666667 & 0 \\ -0.666667 & 2.16667 & -1.5 \\ 0 & -1.5 & 1.5 \end{pmatrix},$$

$$\begin{pmatrix} 2.16667 & -1.5 & -0.666667 \\ -1.5 & 1.5 & 0 \\ -0.666667 & 0 & 0.666667 \end{pmatrix}, \begin{pmatrix} 0.666667 & -0.666667 & 0 \\ -0.666667 & 2.16667 & -1.5 \\ 0 & -1.5 & 1.5 \end{pmatrix}, \begin{pmatrix} 2.16667 & -1.5 & -0.666667 \\ -1.5 & 1.5 & 0 \\ -0.666667 & 0 & 0.666667 \end{pmatrix},$$

$$\begin{pmatrix} 0.666667 & -0.666667 & 0 \\ -0.666667 & 2.16667 & -1.5 \\ 0 & -1.5 & 1.5 \end{pmatrix}, \begin{pmatrix} 2.16667 & -1.5 & -0.666667 \\ -1.5 & 1.5 & 0 \\ -0.666667 & 0 & 0.666667 \end{pmatrix}, \begin{pmatrix} 0.666667 & -0.666667 & 0 \\ -0.666667 & 2.16667 & -1.5 \\ 0 & -1.5 & 1.5 \end{pmatrix},$$

$$\begin{pmatrix} 2.16667 & -1.5 & -0.666667 \\ -1.5 & 1.5 & 0 \\ -0.666667 & 0 & 0.666667 \end{pmatrix}, \begin{pmatrix} 0.666667 & -0.666667 & 0 \\ -0.666667 & 2.16667 & -1.5 \\ 0 & -1.5 & 1.5 \end{pmatrix}, \begin{pmatrix} 2.16667 & -1.5 & -0.666667 \\ -1.5 & 1.5 & 0 \\ -0.666667 & 0 & 0.666667 \end{pmatrix},$$

$$\begin{pmatrix} 0.666667 & -0.666667 & 0 \\ -0.666667 & 2.16667 & -1.5 \\ 0 & -1.5 & 1.5 \end{pmatrix}, \begin{pmatrix} 2.16667 & -1.5 & -0.666667 \\ -1.5 & 1.5 & 0 \\ -0.666667 & 0 & 0.666667 \end{pmatrix}, \begin{pmatrix} 0.666667 & -0.666667 & 0 \\ -0.666667 & 2.16667 & -1.5 \\ 0 & -1.5 & 1.5 \end{pmatrix},$$

$$\begin{pmatrix} 2.16667 & -1.5 & -0.666667 \\ -1.5 & 1.5 & 0 \\ -0.666667 & 0 & 0.666667 \end{pmatrix}, \begin{pmatrix} 0.666667 & -0.666667 & 0 \\ -0.666667 & 2.16667 & -1.5 \\ 0 & -1.5 & 1.5 \end{pmatrix}, \begin{pmatrix} 2.16667 & -1.5 & -0.666667 \\ -1.5 & 1.5 & 0 \\ -0.666667 & 0 & 0.666667 \end{pmatrix},$$

$$\begin{pmatrix} 0.666667 & -0.666667 & 0 \\ -0.666667 & 2.16667 & -1.5 \\ 0 & -1.5 & 1.5 \end{pmatrix}, \begin{pmatrix} 2.16667 & -1.5 & -0.666667 \\ -1.5 & 1.5 & 0 \\ -0.666667 & 0 & 0.666667 \end{pmatrix}, \begin{pmatrix} 0.666667 & -0.666667 & 0 \\ -0.666667 & 2.16667 & -1.5 \\ 0 & -1.5 & 1.5 \end{pmatrix},$$

$$\begin{pmatrix} 2.16667 & -1.5 & -0.666667 \\ -1.5 & 1.5 & 0 \\ -0.666667 & 0 & 0.666667 \end{pmatrix}, \begin{pmatrix} 0.666667 & -0.666667 & 0 \\ -0.666667 & 2.16667 & -1.5 \\ 0 & -1.5 & 1.5 \end{pmatrix}, \begin{pmatrix} 2.16667 & -1.5 & -0.666667 \\ -1.5 & 1.5 & 0 \\ -0.666667 & 0 & 0.666667 \end{pmatrix},$$

$$\begin{pmatrix} 0.666667 & -0.666667 & 0 \\ -0.666667 & 2.16667 & -1.5 \\ 0 & -1.5 & 1.5 \end{pmatrix}, \begin{pmatrix} 2.16667 & -1.5 & -0.666667 \\ -1.5 & 1.5 & 0 \\ -0.666667 & 0 & 0.666667 \end{pmatrix}, \begin{pmatrix} 0.666667 & -0.666667 & 0 \\ -0.666667 & 2.16667 & -1.5 \\ 0 & -1.5 & 1.5 \end{pmatrix},$$

$$\begin{pmatrix} 2.16667 & -1.5 & -0.666667 \\ -1.5 & 1.5 & 0 \\ -0.666667 & 0 & 0.666667 \end{pmatrix}, \begin{pmatrix} 0.666667 & -0.666667 & 0 \\ -0.666667 & 2.16667 & -1.5 \\ 0 & -1.5 & 1.5 \end{pmatrix}, \begin{pmatrix} 2.16667 & -1.5 & -0.666667 \\ -1.5 & 1.5 & 0 \\ -0.666667 & 0 & 0.666667 \end{pmatrix},$$

$$\begin{pmatrix} 0.666667 & -0.666667 & 0 \\ -0.666667 & 2.16667 & -1.5 \\ 0 & -1.5 & 1.5 \end{pmatrix}, \begin{pmatrix} 2.16667 & -1.5 & -0.666667 \\ -1.5 & 1.5 & 0 \\ -0.666667 & 0 & 0.666667 \end{pmatrix}, \begin{pmatrix} 0.666667 & -0.666667 & 0 \\ -0.666667 & 2.16667 & -1.5 \\ 0 & -1.5 & 1.5 \end{pmatrix}\}$$

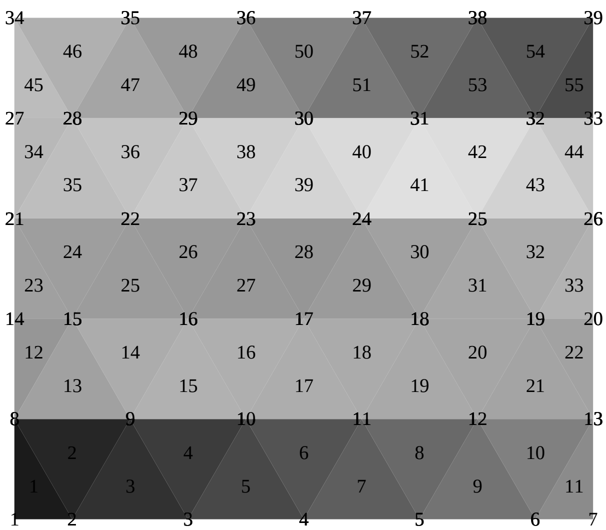
Assemble global stiffness matrix :

```
nxy = Length[nodes]; Kglobal = Table[0, {nxy}, {nxy}];
Do[element = elements⟦j⟧;
  Do[Kglobal⟦element⟦m⟧, element⟦n⟧⟧ += Klocal⟦j, m, n⟧, {m, 1, 3}, {n, 1, 3}], {j, 1, jmax}];
Kglobal // MatrixForm
```

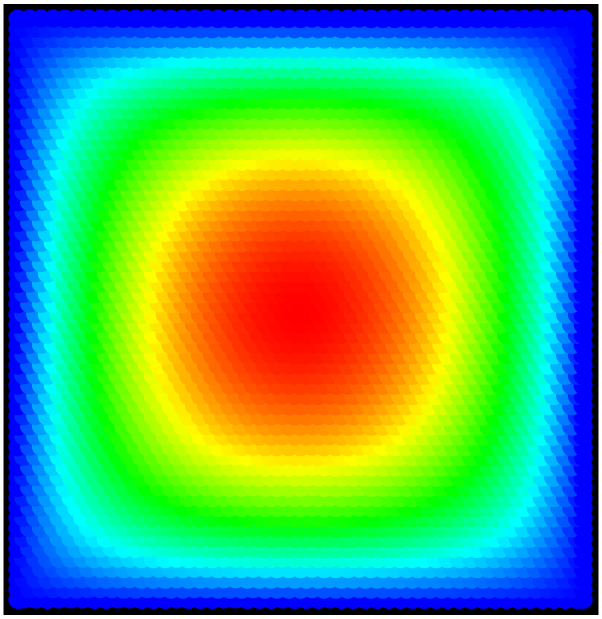
2.16667	−1.5	0	0	0	0	0	−0.666667	0	0	0
−1.5	4.33333	−1.5	0	0	0	0	0	−1.33333	0	0
0	−1.5	4.33333	−1.5	0	0	0	0	0	−1.33333	0
0	0	−1.5	4.33333	−1.5	0	0	0	0	0	−1.33
0	0	0	−1.5	4.33333	−1.5	0	0	0	0	0
0	0	0	0	−1.5	4.33333	−1.5	0	0	0	0
0	0	0	0	0	−1.5	2.16667	0	0	0	0
−0.666667	0	0	0	0	0	0	4.33333	−3.	0	0
0	−1.33333	0	0	0	0	0	−3.	8.66667	−3.	0
0	0	−1.33333	0	0	0	0	0	−3.	8.66667	−3
0	0	0	−1.33333	0	0	0	0	0	−3.	8.66
0	0	0	0	−1.33333	0	0	0	0	0	−3
0	0	0	0	0	−1.33333	0	0	0	0	0
0	0	0	0	0	0	−0.666667	0	0	0	0
0	0	0	0	0	0	0	−0.666667	0	0	0
0	0	0	0	0	0	0	0	−1.33333	0	0
0	0	0	0	0	0	0	0	0	−1.33333	0
0	0	0	0	0	0	0	0	0	0	−1.33
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

■ Honey Comb

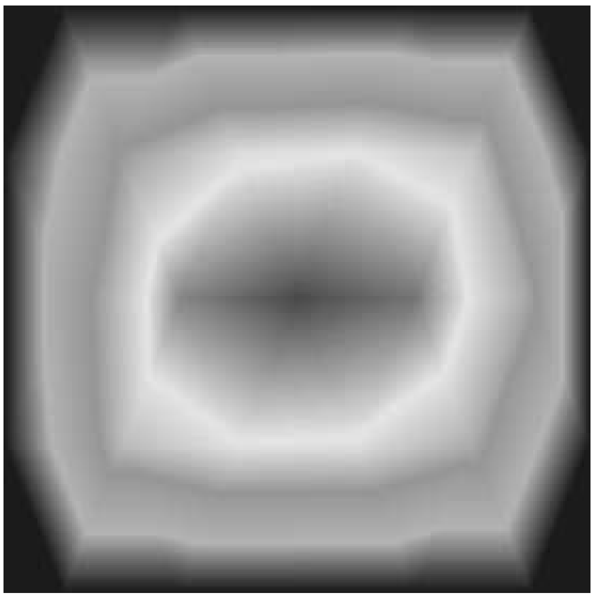
```
nx = 50; ny = Round[2 nx Tan[π / 6]]; dx = 1.0 / (nx − 2);
nodes = Flatten[Table[Table[{Min[1, Max[−1, x]], dx (j − 0.5 (ny + 1)) / Tan[π / 6]},
        {x, −(1 + dx Mod[j, 2]), 1 + dx, 2 dx}], {j, 1, ny}], 1];
elements = Flatten[Table[Table[i + (j − 1) nx − Floor[j / 2] + Join[If[Mod[j, 2] == 1 || i > 1, {{0, 1, nx}}, {}],
           If[Mod[j, 2] == 0 || i < nx − 1, {{1, nx + 1, nx}}, {}]], {i, 1, nx − 1}], {j, 1, ny − 1}], 2];
jmax = Length[elements]; PlotColor[hue_] := Hue[2 (1 − Min[1, Max[0, hue]]) / 3];

$DefaultFont = {"Times−Roman", 9};
Show[Graphics[Table[plist = Map[nodes[[#]] &, elements[[j]]];
      {PlotColor[(j − 1) / (jmax − 1)], Polygon[plist], RGBColor[0, 0, 0], Text[j, Plus @@ plist / 3],
        Table[Text[elements[[j, i]], plist[[i]]], {i, 1, 3}]}, {j, 1, jmax, 1}]], AspectRatio → Automatic, PlotRange → All];
```



```
Klocal =
  Map[Module[{}, Do[{Δx_i, Δy_i} = nodes[[#[[i]]]] − nodes[[#[[1]]]], {i, 2, 3}]; λ2 = Δx2^2 + Δy2^2; λ23 = Δx2 Δx3 + Δy2 Δy3;
      λ3 = Δx3^2 + Δy3^2; d = Δx2 Δy3 − Δx3 Δy2; Chop[1/d {{λ2 + λ3 − 2 λ23, λ23 − λ3, λ23 − λ2}, {λ23 − λ3, λ3, −λ23}, {λ23 − λ2, −λ23, λ2}}]] &, elements];
nxy = Length[nodes]; Kglobal = Table[0, {nxy}, {nxy}];
Do[element = elements[[j]]; Do[Kglobal[[element[[m]], element[[n]]]] += Klocal[[j, m, n]], {m, 1, 3}, {n, 1, 3}], {j, 1, jmax}];
Q = Table[0, {nxy}]; Map[Module[{}, Do[{Δx_i, Δy_i} = nodes[[#[[i]]]] − nodes[[#[[1]]]], {i, 2, 3}];
    Do[Q[[#[[n]]]] += (Δx2 Δy3 − Δx3 Δy2) / 6, {n, 1, 3}]] &, elements];
ymax = Max[Map[#[[2]] &, nodes]];
Do[{x, y} = nodes[[i]];
  If[Abs[x] == 1 || Abs[y] == ymax, Kglobal[[i]] = Table[If[j == i, 1, 0], {j, 1, nxy}]; Q[[i]] = 0], {i, 1, nxy}];
q = LinearSolve[Kglobal, Q]; q1 = Min[q]; q2 = Max[q];
Show[Graphics[{PointSize[1.5 dx], Table[{PlotColor[(q[[i]] − q1) / (q2 − q1)], Point[nodes[[i]]]}, {i, 1, nxy}]},
    AspectRatio → Automatic, Background → RGBColor[0, 0, 0]]];
```



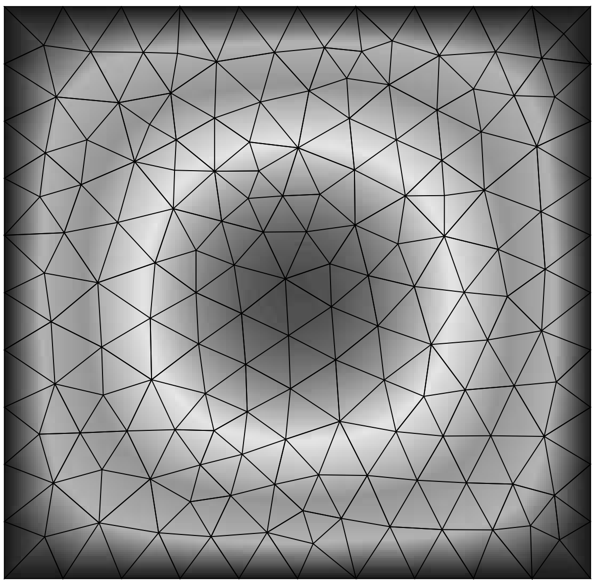
```
(* Interpolated plot, runtime: 2.3 minutes *)
Interpolate[x_, y_] :=
  Module[{j = 1, ξ = 2, η = 0}, While[! (0 ≤ ξ ≤ 1 && 0 ≤ η ≤ 1 && ξ + η ≤ 1), element = elements[[j]];
    {x1, y1} = nodes[[element[[1]]]]; Do[{Δx_i, Δy_i} = nodes[[element[[i]]]] − {x1, y1}, {i, 2, 3}]; d = Δx_2 Δy_3 − Δx_3 Δy_2;
    {ξ, η} = ((x − x1) {Δy_3, −Δy_2} + (y − y1) {−Δx_3, Δx_2}) / d; j++]; q[[element]].{1 − ξ − η, ξ, η}];
ymax = 0.5 dx (ny − 1) / Tan[π / 6];
DensityPlot[Interpolate[x, y], {x, −1, 1}, {y, −ymax, 0.999 ymax},
  PlotPoints → 275, Mesh → False, Frame → False, ColorFunction → PlotColor];
```



■ Unstructured Grid

```
nodes =
  Transpose[Import["C:/School/Engineering/ME207 − Computer Modeling/Unstructured Grid/mesh_p.txt", "Table"]];
elements = Transpose[Round[Delete[
    Import["C:/School/Engineering/ME207 − Computer Modeling/Unstructured Grid/mesh_t.txt", "Table"], 4]]];
jmax = Length[elements]; xlist = Map[#[[1]] &, nodes]; ylist = Map[#[[2]] &, nodes];
{x1, x2} = {Min[xlist], Max[xlist]}; {y1, y2} = {Min[ylist], Max[ylist]};
nxy = Length[nodes]; Q = Table[0, {nxy}];
Klocal = Map[Module[{}, Do[{Δx_i, Δy_i} = nodes[[#[[i]]]] − nodes[[#[[1]]]], {i, 2, 3}]; λ2 = Δx_2^2 + Δy_2^2;
    λ23 = Δx_2 Δx_3 + Δy_2 Δy_3; λ3 = Δx_3^2 + Δy_3^2; d = Δx_2 Δy_3 − Δx_3 Δy_2; Do[Q[[#[[n]]]] += d / 6, {n, 1, 3}];
    Chop[1/d {{λ2 + λ3 − 2 λ23, λ23 − λ3, λ23 − λ2},
              {λ23 − λ3, λ3, −λ23},
              {λ23 − λ2, −λ23, λ2}}]] &, elements]; nxy = Length[nodes];
Kglobal = Table[0, {nxy}, {nxy}];
Do[element = elements[[j]];
  Do[Kglobal[[element[[m]], element[[n]]]] += Klocal[[j, m, n]], {m, 1, 3}, {n, 1, 3}], {j, 1, jmax}];
Do[{x, y} = nodes[[i]]; If[x == x1 || x == x2 || y == y1 || y == y2, Kglobal[[i]] = Table[If[j == i, 1, 0], {j, 1, nxy}]; Q[[i]] = 0],
  {i, 1, nxy}]; q = Chop[LinearSolve[Kglobal, Q]];
```

```
(* 17 minutes *)
Interpolate[x_, y_] :=
  Module[{j = 1, ξ = 2, η = 0, x1, y1}, While[! (0 ≤ ξ ≤ 1 && 0 ≤ η ≤ 1 && ξ + η ≤ 1), element = elements〚j〛;
    {x1, y1} = nodes〚element〚1〛〛; Do[{Δxi, Δyi} = nodes〚element〚i〛〛 − {x1, y1}, {i, 2, 3}]; d = Δx2 Δy3 − Δx3 Δy2;
    {ξ, η} = ((x − x1) {Δy3, −Δy2} + (y − y1) {−Δx3, Δx2}) / d; j++]; q〚element〛.{1 − ξ − η, ξ, η}];
DensityPlot[Interpolate[x, y], {x, x1, x2}, {y, y1, y2}, PlotPoints → 275, AspectRatio → Automatic,
  Mesh → False, Frame → False, ColorFunction → (Hue[2 (1 − #) / 3] &),
  Epilog → Table[{Line[Map[nodes〚#〛 &, elements〚j〛]]}, {j, 1, jmax}]];
```



```
n = 275; image = Table[0, {n}, {n}];
Interpolate[x_, y_] := Module[{x1, y1}, {x1, y1} = plist⟦1⟧; Do[{Δxi, Δyi} = plist⟦i⟧ − {x1, y1}, {i, 2, 3}];
    {ξ, η} = ((x − x1) {Δy3, −Δy2} + (y − y1) {−Δx3, Δx2}) / (Δx2 Δy3 − Δx3 Δy2); q⟦element⟧.{1 − ξ − η, ξ, η}];
Do[element = elements⟦j⟧; plist = nodes⟦element⟧; xlist = Map[#⟦1⟧ &, plist]; ylist = Map[#⟦2⟧ &, plist];
  xIntersect[{{x1_, y1_}, {x2_, y2_}}] := x1 + If[y1 == y2, 0, (y − y1) (x2 − x1) / (y2 − y1)];
  Do[y = y1 + (y2 − y1) (i − 1) / (n − 1); jlist =
     Floor[n (Select[Map[xIntersect, Partition[plist, 2, 1, 1]], (Min[xlist] ≤ # ≤ Max[xlist]) &] − x1) / (x2 − x1)] + 1;
   If[jlist =!= {}, Do[x = x1 + (x2 − x1) (jj − 1) / (n − 1);
      image⟦Max[1, Min[n, i]], Max[1, Min[n, jj]]⟧ = Interpolate[x, y], {jj, Min[jlist], Max[jlist]}]],
   {i, Floor[n (Min[ylist] − y1) / (y2 − y1)] + 1, Floor[n (Max[ylist] − y1) / (y2 − y1)] + 1}], {j, 1, jmax}];
ListDensityPlot[image, AspectRatio → Automatic, Mesh → False, Frame → False, ColorFunction → (Hue[2 (1 − #) / 3] &),
  Epilog → Table[{Line[Map[n (nodes⟦#⟧ + 1) / 2 &, elements⟦j⟧]]}, {j, 1, jmax}]];
```