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MAE 207, Methods of Computer Modeling in Engineering and the Sciences
Isotropic Rectangular Beam Torsion

beam: <http://scienceworld.wolfram.com/physics/Beam.html>

■ **Finite Element Method (FEM) with Triangular Elements**

■ **Roy's notes**

Poisson equation: $\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = f \rightarrow (\nabla u \cdot \nabla v)_{\partial \Omega} - \int_{\Omega} (\nabla u \cdot \nabla v - f v) d\Omega = 0$

coordinates within triangle: ξ_1, ξ_2

linearized solution: $\phi \approx \alpha_1 + \alpha_2 \xi_1 + \alpha_3 \xi_2$

node 1: $(\xi_1)_{\text{node1}} = \xi_{11} \stackrel{?}{=} 0, (\xi_2)_{\text{node1}} = \xi_{21} \stackrel{?}{=} 0 \rightarrow \mathbf{q}_1 \equiv \phi_{\text{node1}} = \alpha_1 + \alpha_2 \xi_{11} + \alpha_3 \xi_{21} \stackrel{?}{=} \alpha_1$

node 2: $(\xi_1)_{\text{node2}} = \xi_{12} \stackrel{?}{=} 1, (\xi_2)_{\text{node2}} = \xi_{22} \stackrel{?}{=} 0 \rightarrow \mathbf{q}_2 \equiv \phi_{\text{node2}} = \alpha_1 + \alpha_2 \xi_{12} + \alpha_3 \xi_{22} \stackrel{?}{=} \alpha_1 + \alpha_2$

node 3: $(\xi_1)_{\text{node3}} = \xi_{13} \stackrel{?}{=} 0, (\xi_2)_{\text{node3}} = \xi_{23} \stackrel{?}{=} 1 \rightarrow \mathbf{q}_3 \equiv \phi_{\text{node3}} = \alpha_1 + \alpha_2 \xi_{13} + \alpha_3 \xi_{23} \stackrel{?}{=} \alpha_1 + \alpha_3$

$u = \mathbf{q}_1 N_1 + \mathbf{q}_2 N_2 + \mathbf{q}_3 N_3 \stackrel{?}{=} \phi$

$$\nabla N_i \cdot \nabla N_k = \begin{pmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial \xi} * & \frac{\partial}{\partial \xi} * \\ \frac{\partial}{\partial \xi} * & \frac{\partial}{\partial \xi} * \\ \frac{\partial}{\partial \xi} * & \frac{\partial}{\partial \xi} * \end{pmatrix}$$

what is N?

■ **class notes: 5/1/06**

$\text{Solve}[\{0 = b x_1 + c x_2 + d x_3, 1 = b x_1 + c x_2 + d x_3, 0 = b x_1 + c x_2 + d x_3\}, \{x_1, x_2, x_3\}]$

■ **Analytical Solution, Nasser's notes**

Pandlet

$\nabla^2 \phi = -2 G K$, angle at twist: $\frac{d\alpha}{dt}, \frac{T}{G J}$

$$\phi^{(\alpha, \eta)} = \frac{32 G a^3}{\pi^3} \sum_{n \text{ odd}}^{\infty} \frac{1}{n^3} (-1)^{\frac{n-1}{2}} \left(1 - \frac{\text{Cosh}[b]}{\text{Cosh}[b]} \right)$$

Fig1

twist rate: $\frac{d\alpha}{dz} = k z$, rectangular beam dimensions: $a \times b$, applied torque: T

modulus of rigidity: $G = \frac{\tau}{\gamma} = \frac{E}{2(1+\nu)}$, Young's modulus: E,

Poisson's ratio: ν , shearing stress: τ , shearing strain: γ

$$\text{torsion constant : } J = \frac{16}{3} a^3 b \left(1 - \frac{196 a}{b \pi^5} \sum_{n=1,3,5 \dots}^{\infty} \frac{1}{n^5} \text{Tanh} \left[\frac{n \pi b}{2 a} \right] \right) = 2.18498$$

$$\text{Prandtl stress function : } \Phi, \text{ Poisson equation : } \nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = -2 G K, \Phi_{\text{boundary}} = 0$$

$$\text{Timoshenko : } \Phi = \frac{32 G k a}{\pi^3} \sum_{n=1,3,5 \dots}^{\infty} \frac{1}{n^3} (-1)^{\frac{n-1}{2}} \left(1 - \frac{\text{Cosh} \left[\frac{n \pi y}{2 a} \right]}{\text{Cosh} \left[\frac{n \pi b}{2 a} \right]} \right) \text{Cos} \left[\frac{n \pi x}{2 a} \right],$$

$$\text{linear twist : } k = \frac{T}{G J}$$

$$\text{shear stress tensor field : } \tau_{yz} = -\frac{\partial \Phi}{\partial x}, \tau_{xz} = \frac{\partial \Phi}{\partial y}, \tau_{yx} = 0, \sigma_x = \sigma_y = \sigma_z = 0$$

$$\text{principle stresses : } \{\sigma_1, \sigma_2, \sigma_3\} = \text{Eigenvalues} \left[\begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix} \right]$$

$$\text{VonMises stress : } \sigma_v = \sqrt{((\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2) / 2} = (3 (\tau_{xz}^2 + \tau_{yz}^2))^{1/2}$$

strain tensor field (material constitutive relation)

$$\epsilon_x = \frac{1}{E} (\sigma_x - \nu (\sigma_y + \sigma_z)) = 0, \epsilon_y = \frac{1}{E} (\sigma_y - \nu (\sigma_x + \sigma_z)) = 0, \epsilon_z = \frac{1}{E} (\sigma_z - \nu (\sigma_x + \sigma_y)) = 0$$

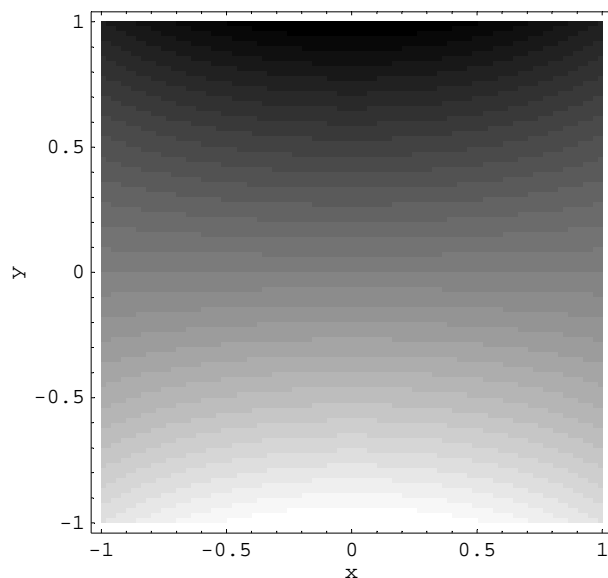
$$\gamma_{xy} = \frac{\tau_{xy}}{G} = 0, \gamma_{yz} = \frac{\tau_{yz}}{G}, \gamma_{xz} = \frac{\tau_{xz}}{G}$$

$$\text{angle of twist : } \alpha = k z$$

$$a = b = 2; T = G = 1; J = 2.1849801331564294; k = \frac{T}{G J};$$

$$\Phi = \frac{32 G k a}{\pi^3} \text{Sum} \left[\frac{1}{n^3} (-1)^{\frac{n-1}{2}} \left(1 - \frac{\text{Cosh} \left[\frac{n \pi y}{2 a} \right]}{\text{Cosh} \left[\frac{n \pi b}{2 a} \right]} \right) \text{Cos} \left[\frac{n \pi x}{2 a} \right], \{n, 1, 10, 2\} \right]; \tau_{yz} = -\partial_x \Phi; \tau_{xz} = \partial_y \Phi;$$

DensityPlot[τ_{xz} , {x, -a/2, a/2}, {y, -a/2, a/2},
PlotPoints → 100, Mesh → False, FrameLabel → {"x", "y"}];



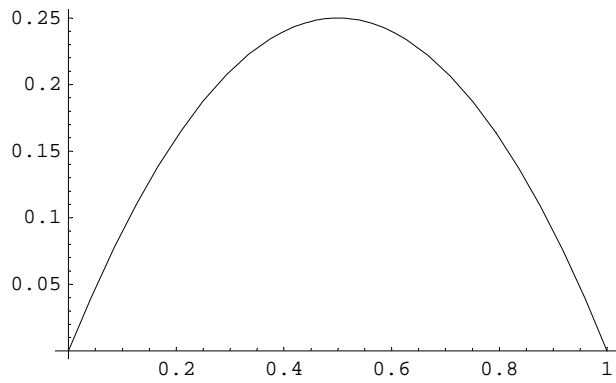
(* http://www.efunda.com/formulae/solid_mechanics/mat_mechanics/strain.cfm *)

$$\gamma_{yz} \equiv \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \quad \gamma_{xz} \equiv \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \quad \gamma_{xy} \equiv \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$$\mathbf{x} = \mathbf{y} = 0; \left(\frac{1}{2} \left(\left(\frac{\mathbf{x}}{\mathbf{a}/2} \right)^2 + \left(\frac{\mathbf{y}}{\mathbf{b}/2} \right)^2 \right) \right)^{1/2}$$

0

`Plot[g (1 - g), {g, 0, 1}];`



```

PlotColor[hue_] :=
  RGBColor @@ If[hue == 1, {1, 0, 0}, Module[{x = Mod[4 hue, 1]}, Switch[Mod[Floor[4 hue], 4],
    0, {0, x, 1}, 1, {0, 1, 1 - x}, 2, {x, 1, 0}, 3, {1, 1 - x, 0}]]];
a = 2.0; b = 2.0; L = 10.0; G = 1; dx = a / 8; dy = b / 8; dz = L / 40; J = 2.1849801331564294` ;
Clear[x, y, z, T]; k =  $\frac{T}{GJ}$ ;

$$\Phi = \frac{32 G k a}{\pi^3} \text{Sum}\left[\frac{1}{n^3} (-1)^{\frac{n-1}{2}} \left(1 - \frac{\text{Cosh}[n \pi y / (2 a)]}{\text{Cosh}[n \pi b / (2 a)]}\right) \text{Cos}\left[\frac{n \pi x}{2 a}\right], \{n, 1, 10, 2\}\right];$$


$$\tau_{yz} = -\partial_x \Phi; \tau_{xz} = \partial_y \Phi; \gamma_{yz} = \frac{\tau_{yz}}{G}; \gamma_{xz} = \frac{\tau_{xz}}{G};$$

Rotate[{x_, y_, z_},  $\theta$ ] := {x Cos[ $\theta$ ] - y Sin[ $\theta$ ], x Sin[ $\theta$ ] + y Cos[ $\theta$ ], z};
animation = {}; scale2 = 10.0; Tmax = scale2; dt = 1.0 / 360;
Prepare[T1_, z1_] := Module[{},  $\alpha = k z1 / \text{scale2} / T \rightarrow T1$ ;
  scale =  $\left(\frac{x^2 + y^2}{(x + \gamma_{xz} / 2)^2 + (y + \gamma_{yz} / 2)^2}\right)^{1/2} /. \{x \rightarrow a / 2, y \rightarrow b / 2\}$ ;
  Calculate[T1_, {x1_, y1_, z1_}] :=
    {Rotate[ $\{\text{scale}(x + \gamma_{xz} / 2), \text{scale}(y + \gamma_{yz} / 2), z1 + \text{scale}(x \gamma_{xz} + y \gamma_{yz}) / 2\}$ ,  $\alpha$  -
      ArcTan[ $\frac{1 + 0.09576080145730925` T}{1 - 0.09332536922126035` T}$ ] -  $\frac{\pi}{4}$ ],  $(3 (\tau_{xz}^2 + \tau_{yz}^2))^{1/2}$ ] /. {x  $\rightarrow$  x1, y  $\rightarrow$  y1, T  $\rightarrow$  T1};
  Do[slices = {}; Show[Graphics3D[ $\{\text{EdgeForm}[], \text{Table}[T = \text{Tmax}(1 - \text{Cos}[2 \pi t]) / 2$ ;
    Prepare[T, z]; slice1 = slice2; slice2 = Map[Calculate[T,  $\{\#[1], \#[2], z\}$ ] &,
      Flatten[ $\{\text{Table}\{x, -b / 2\}, \{x, -a / 2, a / 2 - dx, dx\}, \text{Table}\{a / 2, y\},$ 
         $\{y, -b / 2, b / 2 - dy, dy\}, \text{Table}\{x, b / 2\}, \{x, a / 2, dx - a / 2, -dx\}, \text{Table}$ 
         $\{-a / 2, y\}, \{y, b / 2, dy - b / 2, -dy\}\}$ , 1]]; slices = Append[slices, slice2];
    n = Length[slice2]; If[z == 0, Polygon[Map[ $\#[1]$  &, slice2]], Table[
      quad = {slice1[[i]], slice2[[i]], slice2[[Mod[i, n] + 1]], slice1[[Mod[i, n] + 1]]};
      {SurfaceColor[PlotColor[(Plus @@ Map[ $\#[2]$  &, quad] / 4) / 4.632030522701348]],
        Polygon[Map[ $\#[1]$  &, quad]]}, {i, 1, n}], {z, 0, L, dz}],
    ViewPoint  $\rightarrow$  {-1, 1, 1}, ViewVertical  $\rightarrow$  {0, 1, 0}, Axes  $\rightarrow$  True,
    AxesLabel  $\rightarrow$  {"x", "y", "z"},
    PlotRange  $\rightarrow$  {{-a, a}, {-b, b}, {0, L}}];];
  animation = Append[animation, slices],
  {t,
  0,
  0.5,
  dt}];

```

```

Zero[x_, n_] := Module[{str = ToString[x]},
  StringJoin@@Append[Table["0", {n - StringLength[str]}], str]];
POVFormat[x_Real] := ToString[x, CForm]; POVFormat[x_Integer] := ToString[x];
POVFormat[{{x_, y_, z_},  $\sigma$ _}] := StringJoin["<", POVFormat[x],
  ",", POVFormat[y], ",", POVFormat[z], ",", POVFormat[ $\sigma$ ], ">"];
POVFormat[xlist_List] := StringJoin["{", StringJoin@@
  Map[POVFormat[#] <> ", " &, Drop[xlist, -1]], POVFormat[Last[xlist]], "}"];
TakeLoop[list_, n_] := Take[Join[list, list], n]; nslice = Length[animation[[1]];
WriteVar[name_, x_] := WriteString[wf, StringJoin["#declare ", name, "=array[",
  ToString[Length[x]], "][", ToString[Length[x[[1]]], " ]", POVFormat[x], ";\n"]];
Do[T = Tmax (1 - Cos[2  $\pi$  t]) / 2; frame = Round[t / dt + 1]; slices = animation[[frame]]; wf =
  OpenWrite[StringJoin["C:/Files/PovRay/Physics/Torsion/", Zero[frame, 3], ".inc"]];
Do[WriteVar[StringJoin["wall", ToString[iwall]],
  Table[TakeLoop[slices[[islice]], 8 {iwall - 1, iwall} + 1], {islice, 1, nslice}]],
  {iwall, 1, 4}]; Prepare[T, 0]; WriteVar["cap1",
  Table[Calculate[T, {x, y, 0}], {y, -b/2, b/2, dy}, {x, -a/2, a/2, dx}]];
Prepare[T, L]; WriteVar["cap2", Table[Calculate[T, {x, y, L}],
  {y, -b/2, b/2, dy}, {x, -a/2, a/2, dx}]];
Close[wf], {t, 0, 0.5, dt}];

CopyScaleExport["C:/Files/Animated Gifs/Torsion", 140, "*.bmp"];
CopyScaleExport["C:/Files/Animated Gifs/Torsion", 70, "*.bmp"];

Export["C:/Files/Animated Gifs/Torsion/Torsion.gif", Map[Import[#[[1]]] &, Partition[
  FileNames["*.bmp", "C:/Files/Animated Gifs/Torsion"], 1, 1]], ConversionOptions ->
  {"AnimationDisplayTime" -> 0, "Loop" -> True, "GlobalColorReduction" -> True}];

```
