Solving the torsion problem for isotropic material with a rectangular cross section using the FEM and FVM methods with triangular elements

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December 31, 2017

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For MAE 207, Computational methods. UCI. Fall 2006

1 Introduction

We consider bar made of isotropic material with rectangular cross section subjected to twisting torque $T$. The following diagram illustrate the basic geometry.
Experiments show that rectangular cross sections do wrap and that cross sections do not remain plane as shown in this diagram (in the case of a circular cross section, cross section do NOT wrap).
This is another diagram showing a bar under torsion

- Cross sections do not remain plane during torsion of rectangular cross section
1.1 Problem setup

1.1.1 What are the assumptions?

1. The twist rate (called $k$ in this problem) and defined as $\frac{d\alpha}{dz}$ where $\alpha$ is the twist angle is assumed to be constant.

2. Cross section can wrap also in the $z$ direction (i.e. the cross section does not have to remain in the $xy$ plane) but if this happens, all cross sections will wrap in the $z$ section by the same amount.

3. Material is isotropic

1.1.2 What is the input and what is the output?

The input to the problem are the following (these are the known or given):

1. The width $b$ and height $a$ of the cross section.
2. Material Modulus of rigidity or shear modulus $G$ which is the ratio of
the shearing stress $\tau$ to the shearing strain $\gamma$.

3. The applied torque $T$.

4. $J$ the torsion constant for the a rectangular cross section. For a rectan-
gular section of dimensions $a, b$ it is given by

$$J = \frac{16}{3} a^3 b \left( 1 - \frac{192}{a^5} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^5} \tanh \left( \frac{n \pi b}{2a} \right) \right) \quad (1)$$

Hence the torsional rigidity $GJ$ is known since $G$ is given (material) and
$J$ is from above (geometry).

1.1.3 The output from the problem (the things we need to calculate)

1. The stress distribution in the cross section (stress tensor field). Once
this is found then using the material constitutive relation we can the
strain tensor field.

2. The angle of twist $\alpha$ as a function of $z$ (the length of the beam).

2 Analytical solution using Prandtl stress function

First we solve for the Prandtl stress function $\Phi (x, y)$ by solving the Poisson
equation

$$\nabla^2 \Phi (x, y) = -2GK$$

Where $G$ is the shear modulus and $k$ is the twist rate (which was assumed
to be constant).

The boundary conditions ($\Phi (x, y)$ at any point on the edge of the cross
section and at the ends of the beam) is an arbitrary constant. We take this
constant to be zero. Hence at the cross section boundary we have

$$\Phi = 0$$

The analytical solution to the above equation is from book Theory of
elasticity by S. P. Timoshenko and J. N. Goodier.
\[ \Phi(x, y) = \frac{32}{\pi^3} G k a^2 \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^3} (-1)^{(n-1)/2} \left( 1 - \frac{\cosh\left(\frac{n\pi y}{2a}\right)}{\cosh\left(\frac{n\pi b}{2a}\right)} \right) \cos\left(\frac{n\pi x}{2a}\right) \]

where the linear twist \( k \)

\[ k = \frac{T}{GJ} \]

Hence (2) becomes

\[ \Phi(x, y) = \frac{32 T a^2}{J^3} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^3} (-1)^{(n-1)/2} \left( 1 - \frac{\cosh\left(\frac{n\pi y}{2a}\right)}{\cosh\left(\frac{n\pi b}{2a}\right)} \right) \cos\left(\frac{n\pi x}{2a}\right) \]

Where \( J \) is given by (1)

### 2.1 Stress components

\[ \tau_{yz} = -\frac{\partial \Phi}{\partial x} = \frac{16Gk a}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} (-1)^{(n-1)/2} \left( 1 - \frac{\cosh\left(\frac{n\pi y}{2a}\right)}{\cosh\left(\frac{n\pi b}{2a}\right)} \right) \sin\left(\frac{n\pi x}{2a}\right) \]

Hence

\[ \tau_{yz} = -\frac{\partial \Phi}{\partial x} = \frac{16Gk a}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} (-1)^{(n-1)/2} \left( 1 - \frac{\cosh\left(\frac{n\pi y}{2a}\right)}{\cosh\left(\frac{n\pi b}{2a}\right)} \right) \sin\left(\frac{n\pi x}{2a}\right) \]

and

\[ \tau_{xz} = \frac{\partial \Phi}{\partial y} = \frac{16 T a}{J \pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} (-1)^{(n-1)/2} \left( -\cos\left(\frac{n\pi x}{2a}\right) \text{sech}\left(\frac{bn\pi}{2a}\right) \sinh\left(\frac{n\pi y}{2a}\right) \right) \]

Timoshenko gives the maximum sheer stress, which is \( \sqrt{\tau_{yz}^2 + \tau_{xz}^2} \) as

\[ \tau_{\text{max}} = \frac{16Gk a}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} \left( 1 - \frac{1}{\cosh\left(\frac{n\pi b}{2a}\right)} \right) \]

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2.2 Strain components

Given that \( E \) is Young’s modulus for the material, \( v \) is Poisson’s ratio for the material, and \( G = \frac{E}{2(1+v)} \) we can now obtain the strain components from the constitutive equations (stress-strain equations) since we have determined the stress components from the above solution.

\[
\varepsilon_x = \frac{1}{E} \left( \sigma_x - v (\sigma_y + \sigma_z) \right) = 0 \\
\varepsilon_y = \frac{1}{E} \left( \sigma_y - v (\sigma_x + \sigma_z) \right) = 0 \\
\varepsilon_z = \frac{1}{E} \left( \sigma_z - v (\sigma_x + \sigma_y) \right) = 0 \\
\gamma_{xy} = \frac{2(1+v)}{E} \tau_{xy} = \frac{1}{G} \tau_{xy} = 0 \\
\gamma_{yz} = \frac{2(1+v)}{E} \tau_{yz} = \frac{1}{G} \tau_{yz} \\
\gamma_{xz} = \frac{2(1+v)}{E} \tau_{xz} = \frac{1}{G} \tau_{xz}
\]

Hence only \( \gamma_{yz} \) and \( \gamma_{xz} \) are non-zero.

2.3 Determining the twist angle \( \alpha \)

If we look at a cross section of the bar at some distance \( z \) from the end of the bar, the angle that this specific cross section has twisted due to the torque is \( \alpha \).
Before twist

After twist, particle at blue location moved to Red location

The twist angle
This angle is given by the solution to the equation

$$\frac{d\alpha(z)}{dz} = k$$

But $k$ is the linear twist and is given by $k = \frac{T}{GJ}$ hence the above equation becomes

$$\frac{d\alpha(z)}{dz} = \frac{T}{GJ}$$

Hence

$$\alpha(z) = \frac{T}{GJ} z + C_1$$

Where $C_1$ is the constant of integration. Assuming $\alpha = 0$ at $z = 0$ we obtain that

$$\alpha(z) = \frac{T}{GJ} z$$

and using the expression $J$ given in equation (1) above we can determine $\alpha$ for each $z$. 
2.4 Displacement calculations

\[ r = \sqrt{x^2 + y^2} \]
\[ u = r \alpha (-\sin \beta) \]
\[ v = r \alpha (\cos \beta) \]

we see that

Finding displacements for coordinates.
Valid for SMALL twist angle alpha

\[ r = \sqrt{x^2 + y^2} \]
\[ u = r \alpha (-\sin \beta) \]
\[ v = r \alpha (\cos \beta) \]
\[
\begin{align*}
\sin \beta &= \frac{y}{r} \\
\cos \beta &= \frac{x}{r}
\end{align*}
\]

Hence

\[
\begin{align*}
u &= -\alpha y \\
v &= \alpha x
\end{align*}
\]

Where \( \alpha = \frac{T}{GJz} \)

### 3 References

1. Mathematica Structural Mechanics help page
2. MIT course 16.20 lecture notes. MIT open course website.
3. Theory of elasticity by S. P. Timoshenko and J. N. Goodier. chapter 10