

Point Collocation solution for $u'''' + u = 1$

A Mathematica based solution to $u''''+u=1$ using the point collocation method.

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Boundary conditions are $u(0)=u(1)=u''(0)=u''(1)=0$

First Obtain the approximate numerical solution using Collocation Method

```
Remove["Global`*"]
<< Graphics`
nBasis = 5; nPoints = 5;
trial[x_] := Sum[a_n (x (x - 1))^n, {n, 1, nBasis}
residual[x_] := trial''''[x] + trial[x] - 1
A = Table[residual[x] /. x -> {n/(2 nPoints - 1)}, {n, 1, nPoints - 1}];
A = Flatten[Append[A, {2 a_1 + 2 a_2}]];
sol = Flatten[Solve[A == 0, Table[a_n, {n, 5}]]];
```

```
Print["Coefficients of trial solution are "]
```

```
sol // N
```

```
Coefficients of trial solution are
```

```
{a1. → -0.0412493, a2. → 0.0412493, a3. → 0.000147289, a4. → -0.0000245233, a5. → -3.28259 × 10-8}
```

```
numericalSolution[x_] := Sum[sol[[n, 2]] (x (x - 1)) ^ n, {n, 1, nBasis}]
```

```
numericalSolutionDiff4[x_] := D[numericalSolution[x], {x, 4}]
```

Now get the analytical solution using DSolve and compare to the numerical solution

```
eq = u''''[x] + u[x] == 1;
```

```
bc = {u[0] == 0, u[1] == 0, u''[0] == 0, u''[1] == 0};
```

```
DSolve[{eq, u[0] == 0, u[1] == 0, u''[0] == 0, u''[1] == 0}, u[x], x];
```

```
mmaAnalyticSol = (u[x] /. %)[[1]];
```

```
mmaAnalyticMoment := D[mmaAnalyticSol, {x, 2}];
```

$$a = \frac{1}{\sqrt{2}};$$

$$d = 2 e^{-a} \cos[a] + e^{-2a} + 1;$$

$$\text{textAnalyticSol}[x_] := 1 + \left(\frac{-e^{-a} (\cos[a] + e^{-a})}{d} \right) e^{ax} \cos[ax] + \left(\frac{-e^{-a} \sin[a]}{d} \right) e^{ax} \sin[ax] + \left(\frac{-(e^{-a} \cos[a] + 1)}{d} \right) e^{-ax} \cos[ax] + \left(\frac{-e^{-a} \sin[a]}{d} \right) e^{-ax} \sin[ax]$$

$$\text{textDispErr}[x_] := \text{Abs} \left[\frac{\text{numericalSolution}[x] - \text{textAnalyticSol}[x]}{\text{textAnalyticSol}[x]} \right]$$

$$\text{textAnalyticMoment}[x_] := \text{textAnalyticSol}''[x]$$

$$\text{textMomentErr}[x_] := \text{Abs} \left[\frac{(\text{numericalSolution}''[x] - \text{textAnalyticSol}''[x])}{\text{textAnalyticMoment}[.5]} \right]$$

```
pp1 = Plot[textDispErr[x], {x, 0, 1}, PlotStyle → {Red},
```

```
PlotLabel -> "textbook Displacement Error %", DisplayFunction -> Identity]
```

```
pp2 = Plot[textMomentErr[x], {x, 0, 1}, PlotLabel -> "textbook Moment Error %",
```

```
PlotRange → All, PlotStyle → {Red}, DisplayFunction -> Identity]
```

$$\text{mmaDispErr}[pt_] := \text{Abs} \left[\frac{\text{numericalSolution}[pt] - \text{mmaAnalyticSol} /. x \rightarrow pt}{\text{mmaAnalyticSol} /. x \rightarrow .5} \right]$$

$$\text{mmaMomentErr}[pt_] := \text{Abs} \left[\frac{(D[\text{numericalSolution}[x], \{x, 2\}] - \text{mmaAnalyticMoment}) /. x \rightarrow pt}{\text{mmaAnalyticMoment} /. x \rightarrow .5} \right]$$

```
pp3 = Plot[mmaDispErr[x], {x, 0, 1},
```

```
PlotLabel -> "mma Displacement Error %", DisplayFunction -> Identity]
```

```
pp4 = Plot[mmaMomentErr[x], {x, 0, 1}, PlotLabel -> "mma Moment Error %",
```

```
PlotRange → All, DisplayFunction -> Identity]
```

Print the Exact solution and the numerical solution

`FullSimplify[mmaAnalyticSol]`

$$\frac{e^{-\frac{x}{\sqrt{2}}} \left(e^{\frac{x}{\sqrt{2}}} \left(1 + e^{\sqrt{2}} + 2 e^{\frac{1}{\sqrt{2}}} \cos \left[\frac{1}{\sqrt{2}} \right] \right) - e^{\frac{1-x}{\sqrt{2}}} \left(1 + e^{\sqrt{2} x} \right) \cos \left[\frac{-1+x}{\sqrt{2}} \right] - \left(e^{\sqrt{2}} + e^{\sqrt{2} x} \right) \cos \left[\frac{x}{\sqrt{2}} \right] \right)}{1 + e^{\sqrt{2}} + 2 e^{\frac{1}{\sqrt{2}}} \cos \left[\frac{1}{\sqrt{2}} \right]}$$

Print the numerical solution

`FullSimplify[N[numericalSolution[x]]]`

$$-3.28259 \times 10^{-8} (-7.02822 + x) (-1.61671 + x) (-1. + x) x$$

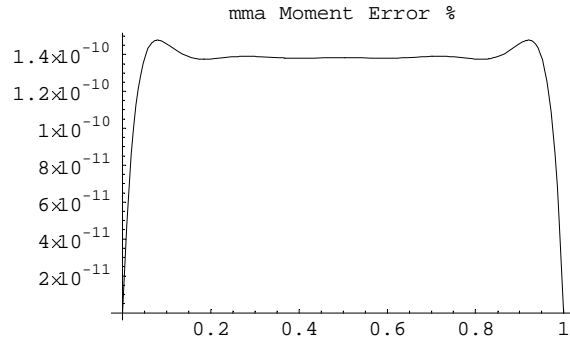
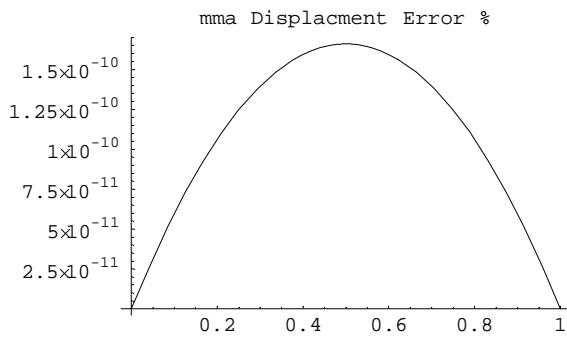
$$(0.61671 + x) (6.02822 + x) (750.815 + (-1. + x) x) (39.6206 + (-1. + x) x)$$

(*Verify DSolve solution matches textbook solution*)

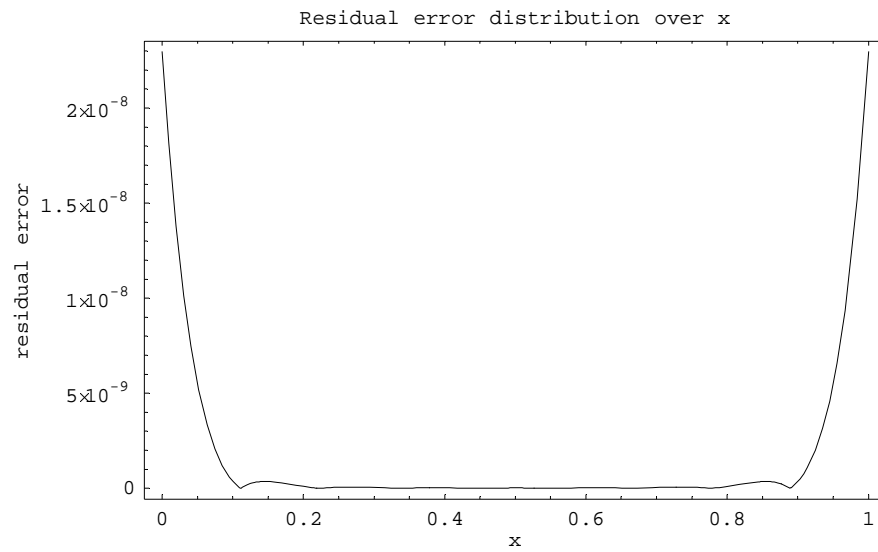
`mmaAnalyticSol - textAnalyticSol[x] // FullSimplify`

0

`Show[GraphicsArray[{pp3, pp4}], DisplayFunction -> $DisplayFunction]`



```
Plot[Evaluate[Abs[numericalSolutionDiff4[x] + numericalSolution[x] - 1]],  
{x, 0, 1}, PlotRange -> All, PlotLabel -> "Residual error distribution over x",  
FrameLabel -> {"x", "residual error"}, Frame -> True]
```



Reference

1. *Methods of computer modeling in engineering & the sciences. Volume 1*
by Professor Atluri, S.N.
2. *Lecture notes. MAE207. Instructor: Professor Atluri. UCI. Spring 2006*