MAE 171 Digital Control Systems

Homework # 7 Solution

Problem 2.

$$G(z) = \mathcal{Z}\left[\frac{1-e^{-Ts}}{s}\frac{1}{s(s+1)}\right] = (1-z^{-1})\,\mathcal{Z}\left[\frac{1}{s^2(s+1)}\right]$$

$$= \frac{(T-1+e^{-T})z^{-1} + (1-e^{-T}-Te^{-T})z^{-2}}{(1-z^{-1})(1-e^{-T}z^{-1})}$$

Since $T = 0.1$ sec, $G(z)$ becomes

$$G(z) = \frac{0.004837z^{-1}(1+0.9674z^{-1})}{(1-z^{-1})(1-0.9048z^{-1})} = \frac{0.004837(z+0.9674)}{(z-1)(z-0.9048)}$$

Since the number of samples per cycle of damped sinusoidal oscillation is specified as 8, one of the dominant closed-loop poles must be on the line having an angle of 45° and passing through the origin. Thus, the desired dominant closed-loop pole location in the upper half z plane can be determined as the intersection of the line having an angle of 45° and the $\zeta = 0.5$ locus.

Thus, $z_d = re^{j\theta}$ with $\theta = 45°$ or $\frac{\pi}{4}$, and r calculated from

$$r = \exp\left(-\frac{\zeta\theta}{\sqrt{1-\zeta^2}}\right) = e^{-\frac{\pi}{4\sqrt{3}}} \quad \text{with } \zeta = 0.5$$

$$= 0.6354$$

The desired dominant closed-loop pole in the upper half z plane is located at

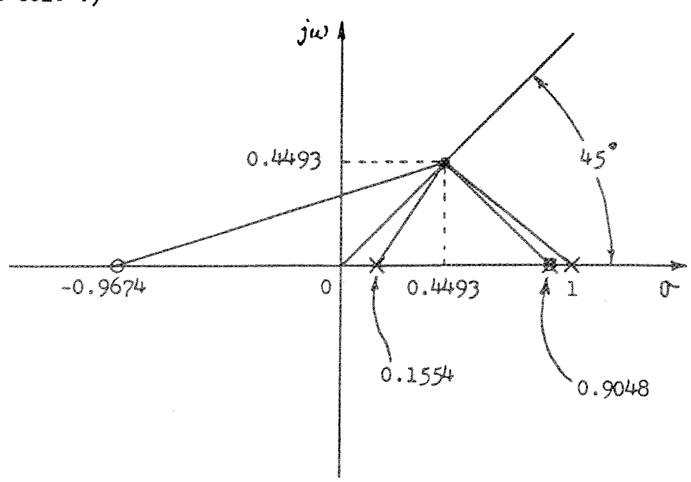
$$z = 0.6354 \angle 45° = 0.4493 + j0.4493$$

In order to have a closed-loop pole at this location, we need to add a phase lead angle of 78.59°. The digital controller must give this necessary phase lead angle.

We shall choose the digital controller $G_D(z)$ to be

$$G_D(z) = K\frac{z+\alpha}{z+\beta} = K\frac{z-0.9048}{z+\beta}$$

(Here we chose $\alpha = -0.9048$.) Then, from the angle condition we find that the controller pole must be located at $z = 0.1554$, or $\beta = -0.1554$. (See the diagram below.)



The controller $G_D(z)$ is now given by

$$G_D(z) = K \frac{z - 0.9048}{z - 0.1554}$$

The open-loop pulse transfer function becomes

$$G_D(z)G(z) = K \frac{0.004837(z + 0.9674)}{(z - 0.1554)(z - 1)}$$

Using the magnitude condition, the gain K can be determined as follows:

$$\left| K \frac{0.004837(z + 0.9674)}{(z - 0.1554)(z - 1)} \right|_{z = 0.4493 + j0.4493} = 1$$

or

$$K = 53.08$$

Thus, the digital controller has the following pulse transfer function:

$$G_D(z) = 53.08 \frac{z - 0.9048}{z - 0.1554}$$

The static velocity error constant K_v is determined as follows:

$$K_v = \lim_{z \to 1} \frac{1 - z^{-1}}{0.1} (53.08) \frac{z - 0.9048}{z - 0.1554} \frac{(0.004837)(z + 0.9674)}{(z - 1)(z - 0.9048)}$$

$$= 5.98$$

To boost K_v by a factor of 2, we add a lag section $\frac{z-\gamma}{z-\delta}$.
The complete compensator will then be $C_D(z) = K \frac{z+\alpha}{z+\beta} \frac{z-\gamma}{z-\delta}$.

We calculate

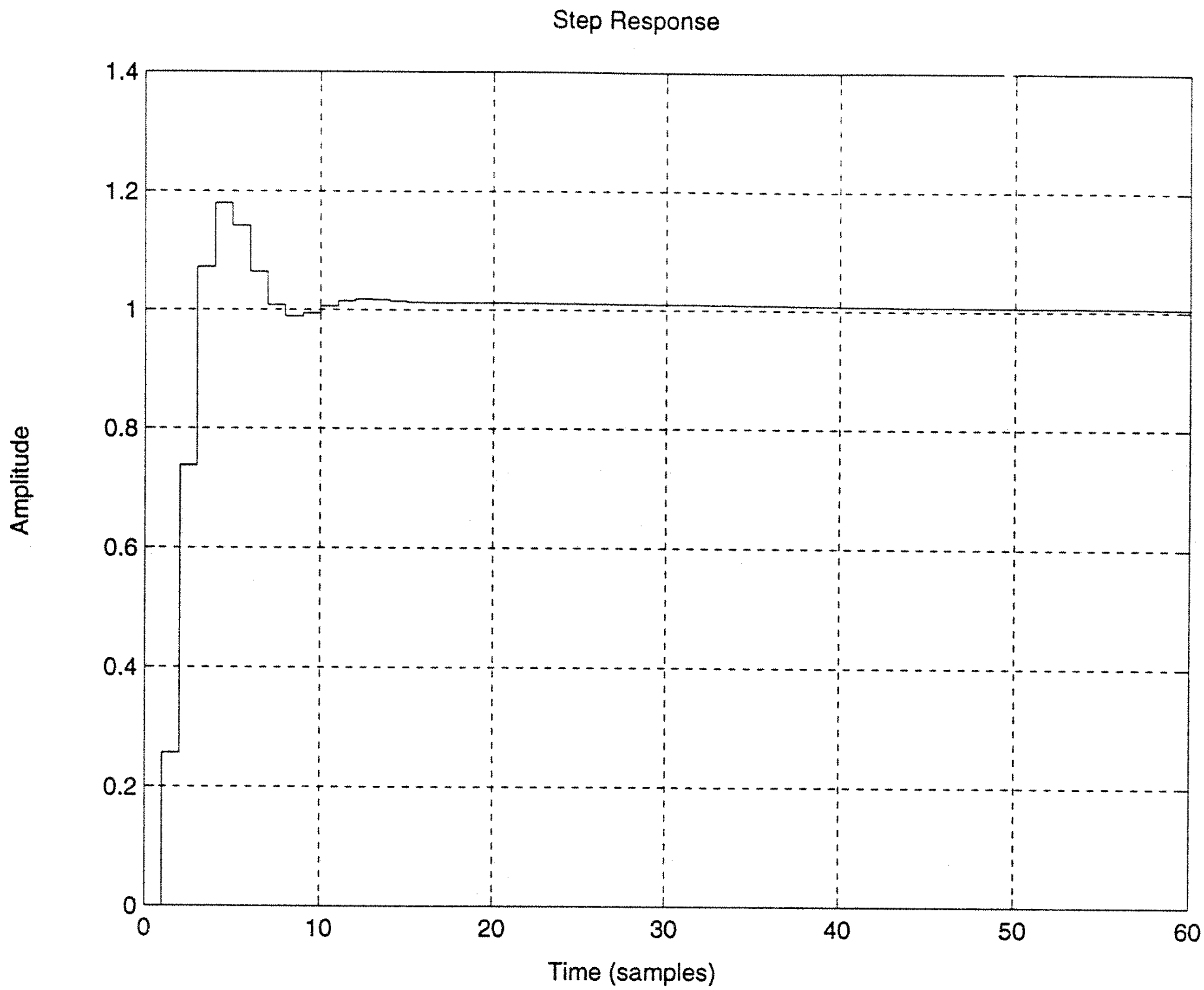
$$\gamma = 1 - \frac{K_v^{\text{after}}}{K_v^{\text{before}}} (1-\delta)$$

By taking $\delta = 0.99$ and $\frac{K_v^{\text{after}}}{K_v^{\text{before}}} = 2$, we obtain

$$\gamma = 1 - 2(1 - 0.99) = 0.98$$

Therefore,

$$G_D(z) = 53.08 \frac{z - 0.9048}{z - 0.1554} \frac{z - 0.98}{z - 0.99}$$



$$G_D(z) = \frac{N_c(z)}{D_c(z)} = \frac{53.08 z^2 - 100.452 z + 47.0662}{z^2 - 1.1454 z + 0.1538}$$

$$G(z) = \frac{N_p(z)}{D_p(z)} = \frac{0.004837 z + 0.004679}{z^2 - 1.9048 z + 0.9048}$$

$$T(z) = \frac{G(z) G_D(z)}{1 + G(z) G_D(z)} = \frac{N_c N_p}{N_c N_p + D_c D_p}$$

$$= \frac{0.2567 z^2 - 0.0036 z - 0.2438}{z^3 - 1.8905 z^2 + 1.2978 z - 0.398}$$

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clear;
nc = conv(53.08*[1 -0.9048],[1 -0.98]);
dc = conv([1 -0.1554],[1 -0.99]);
np = 0.004837*[1 0.9674];
dp = conv([1 -1],[1 -0.9048]);
num = conv(nc,np);
den = [0 conv(nc,np)]+conv(dc,dp);
dstep(num,den); hold;
grid;
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Problem 1

$$G(z) = \frac{K(z+1)}{(z-1)(z-0.6065)}$$

The characteristic equation for the system is

$$z^2 + (K - 1.6065)z + 0.6065 + K = 0$$

The critical value of gain K for stability can be determined easily by use of the Jury stability criterion. Define

$$P(z) = z^2 + (K - 1.6065)z + 0.6065 + K$$
$$= a_0 z^2 + a_1 z + a_2 = 0$$

Then

$$a_0 = 1, \qquad a_1 = K - 1.6065, \qquad a_2 = 0.6065 + K$$

The conditions for stability are

1. $|a_2| < a_0$
2. $P(1) > 0$
3. $P(-1) > 0$

Thus we require

$$|0.6065 + K| < 1$$
$$P(1) = 1 + K - 1.6065 + 0.6065 + K = 2K > 0$$
$$P(-1) = 1 - K + 1.6065 + 0.6065 + K = 3.213 > 0$$

Hence

$$0 < K < 0.3935$$

The critical value of gain K for stability is 0.3935.

Since

$$G(z) = \frac{K(z+1)}{(z-1)(z-0.6065)}$$

we have

$$\angle G(z) = \angle z+1 - \angle z-1 - \angle z - 0.6065$$

Define

$$z = \sigma + j\omega$$

The angle condition is

$$\angle \sigma + j\omega + 1 - \angle \sigma + j\omega - 1 - \angle \sigma + j\omega - 0.6065 = 180^\circ$$

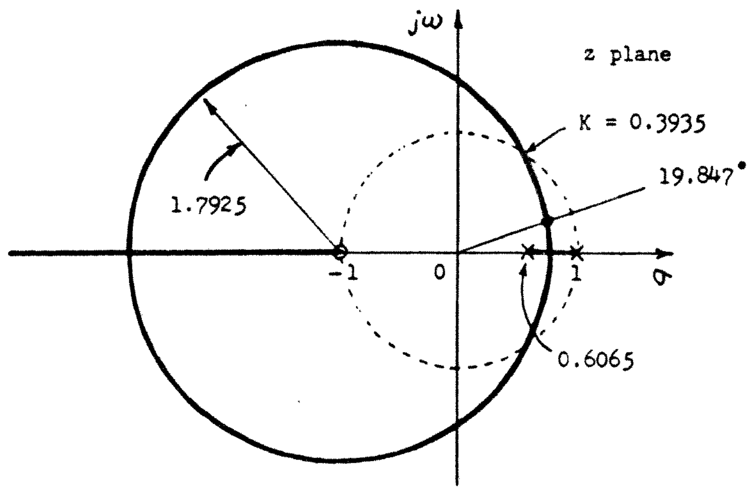
Hence

$$\tan^{-1}\frac{\omega}{\sigma+1} - \tan^{-1}\frac{\omega}{\sigma-1} = 180^\circ + \tan^{-1}\frac{\omega}{\sigma - 0.6065}$$

Taking the tangent of both sides of this equation and simplifying, we get

$$\omega = 0 \qquad \text{and} \qquad (\sigma+1)^2 + \omega^2 = (1.7925)^2$$

Thus, the root loci consist of a part of the real axis (between -1 and $-\infty$) and a circle with center at $\sigma = -1$, $\omega = 0$ and the radius equal to 1.7925, as shown below.



The value of gain K that will yield the damping ratio ζ of the closed-loop poles equal to 0.5 can be determined from Eqs. (4-53) and (4-54). Since T is given as 0.1 sec, we have

$$|z| = e^{-0.1 \times 0.5\omega_n} = e^{-0.05\omega_n}$$

$$\angle z = 0.1 \times \omega_n \sqrt{1 - 0.5^2} = 0.0866\ \omega_n$$

By trial and error we find that the point that corresponds to $\zeta = 0.5$ and $\omega_n = 4$ rad/sec, that is, the point for which

$$|z| = e^{-0.05 \times 4} = 0.8187$$

$$\angle z = 0.0866 \times 4 = 0.3464 \text{ rad} = 19.847°$$

is on the root locus. This point is

$$z = e^{-0.05 \times 4} \angle 19.847° = 0.8187 \angle 19.847°$$

$$= 0.7701 + j0.2780$$

The value of gain K that corresponds to this closed-loop pole is found from the magnitude condition

$$\left| \frac{K(z+1)}{(z-1)(z-0.6065)} \right|_{z = 0.7701 + j0.2780} = 1$$

as follows:

$$K = \frac{0.3606 \times 0.3226}{1.7918} = 0.0649$$

When gain K is set to 0.0649, or K = 0.0649, the damping ratio ζ of the dominant closed-loop poles is 0.5. With this gain value, the damped natural frequency ω_d is found as

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 4\sqrt{1 - 0.5^2} = 3.464$$

The number of samples per cycle of the damped sinusoidal oscillation is

$$\frac{360°}{19.847°} = 18.14$$

or 18.14 samples per cycle.

Problem 3 $\qquad G(s) = \dfrac{s+2}{(s+3)(s+1)}$, $T = 1$ s.

(a) standard z-transform

$$G(s) = \frac{1/2}{(s+1)} + \frac{1/2}{(s+3)}$$

$$D(z) = G(z) = \frac{1/2}{1-\epsilon^{-1}z^{-1}} + \frac{1/2}{1-\epsilon^{-3}z^{-1}} = \frac{1 - \frac{1}{2}(\epsilon^{-1}+\epsilon^{-3})z^{-1}}{1-(\epsilon^{-1}+\epsilon^{-3})z^{-1}+\epsilon^{-4}z^{-2}}$$

(b) Step-invariance

$$D(z) = (1-z^{-1})\,\mathcal{Z}\left[\frac{G(s)}{s}\right] = (1-z^{-1})\,\mathcal{Z}\left[\frac{\frac{2}{3}}{s} - \frac{\frac{1}{2}}{s+1} - \frac{\frac{1}{6}}{s+3}\right]$$

$$= (1-z^{-1})\left[\frac{\frac{2}{3}}{1-z^{-1}} - \frac{\frac{1}{2}}{1-\epsilon^{-1}z^{-1}} - \frac{\frac{1}{6}}{1-\epsilon^{-3}z^{-1}}\right]$$

$$= \frac{2}{3} - \frac{1}{2}\,\frac{1-z^{-1}}{1-\epsilon^{-1}z^{-1}} - \frac{1}{6}\,\frac{1-z^{-1}}{1-\epsilon^{-3}z^{-1}}$$

(c) Backward difference

$$s = \frac{1-z^{-1}}{T} = 1 - z^{-1}$$

$$D(z) = \frac{3-z^{-1}}{(4-z^{-1})(2-z^{-1})} = \frac{1}{2}\left[\frac{1}{4-z^{-1}} + \frac{1}{2-z^{-1}}\right]$$

(d) Forward difference

$$s = \frac{z-1}{T} = z-1$$

$$D(z) = \frac{z+1}{(z+2)\,z} = \frac{1}{2}\left[\frac{1}{z} + \frac{1}{z+2}\right]$$

(e) Bilinear z-transform , $s = \frac{2}{T}\,\frac{1-z^{-1}}{1+z^{-1}}$, $T = 1$ s.

$$D(z) = G(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}\cdot\frac{2}{T}} = \frac{\left(2\,\frac{1-z^{-1}}{1+z^{-1}} + 2\right)}{\left(2\,\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 4\left(2\,\frac{1-z^{-1}}{1+z^{-1}}\right) + 3}$$

$$= \frac{4+4z^{-1}}{15-2z^{-1}-z^{-2}}$$

(f) Matched z-transform

$$D(z) = G(s)\Big|_{s+\alpha = 1-\epsilon^{-\alpha}z^{-1}} \quad , \ T=1$$

$$= \frac{1 - \cancel{\epsilon^{-2}}\,z^{-1}}{1-(\epsilon^{-1}+\epsilon^{-3})z^{-1}+\epsilon^{-4}z^{-2}} \qquad \cancel{0.5(\epsilon^{-1}+\epsilon^{-3})}$$