# HW4, MAE 270A. Linear systems I. Fall 2005. UCI 

Nasser M. Abbasi

June 21, 2014

## Contents

1 Problem 3.12 ..... 1
2 Problem 3.13 ..... 6
2.1 A1. ..... 7
$2.2 \quad$ A2 ..... 7
$2.3 \quad$ A3 ..... 8
$2.4 \quad \mathrm{~A} 4$ ..... 11

## 1 Problem 3.12

Given

$$
A=\left(\begin{array}{llll}
2 & 1 & 0 & 0 \\
0 & 2 & 1 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 1
\end{array}\right) b=\left(\begin{array}{l}
0 \\
0 \\
1 \\
1
\end{array}\right) \bar{b}=\left(\begin{array}{l}
1 \\
2 \\
3 \\
1
\end{array}\right)
$$

What are the representation of $A$ w.r.t. the basis $\left\{b, A b, A^{2} b, A^{3} b\right\}$, and the basis $\left\{\bar{b}, A \bar{b}, A^{2} \bar{b}, A^{3} \bar{b}\right\}$ respectively?

Answer
We are given the matrix $A$ and we want to find its repesentation under new basis, say $Q=$ $\left\{q_{1}, q_{2}, \cdots, q_{n}\right\}$

Let the new represntation Matrix be called $\bar{A}$.
The ith column of $\bar{A}$ is the reprsentation of the base $q_{i}$ under $Q$.
Now, the vector $\mathbf{q}_{i}$ itself is independent of its representation under different coordinates systems. Hence we can write

$$
\mathbf{q}_{i}=A\left(q_{i} \text { representation under } A\right)=Q\left(q_{i} \text { representation under } \mathrm{Q}\right)
$$

Hence let $\mathbf{q}_{i}$ representation under Q be the column $x$, and let $\mathbf{q}_{i}$ representation under $A$ be column $y$, then the above equation can be written as

$$
\begin{equation*}
\mathbf{q}_{i}=A(y)=Q(x) \tag{1}
\end{equation*}
$$

Now we solve for $x$ in the above equation since we know $A$ and $y$ and $Q$.
This will give us one column of $\bar{A}$. We do the above again for each vector $\mathbf{q}_{i}$, and at the end we have $\bar{A}$

Hence the algorithm is
input: Old basis $A$, new basis matrix $Q$, vector $y$ which is the representation of $\mathbf{q}_{i}$ under A output: column $x$ which is the representation of $\mathbf{q}_{i}$ under $Q$ by solving equation (1).
Collect all these $x^{\prime} s$ to build up the matrix $\bar{A}$
Hence we apply the algorithm for each of these vectors, one at a time.
First we build the matrix $Q$ (the new basis), this is done by just calculating $b, A b, A^{2} b, A^{3} b$, this reults in

$$
Q=\left(\begin{array}{llll}
0 & 0 & 1 & 6 \\
0 & 1 & 4 & 12 \\
1 & 2 & 4 & 8 \\
1 & 1 & 1 & 1
\end{array}\right)
$$

Hence to find the representation of $y=b$ under $Q$, we solve for $x$ in equation (1). We do this for each $y=b, y=A b, y=A^{2} b$ and $y=A^{3} b$ as follows

$$
\begin{aligned}
& A b=Q x \\
& A b=\left(\begin{array}{llll}
b & A b & A^{2} b & A^{3} b
\end{array}\right)\left(\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3} \\
x_{4}
\end{array}\right) \Rightarrow x=\left(\begin{array}{l}
0 \\
1 \\
0 \\
0
\end{array}\right) \\
& A(A b)=Q x \\
& A^{2} b=\left(\begin{array}{llll}
b & A b & A^{2} b & A^{3} b
\end{array}\right)\left(\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3} \\
x_{4}
\end{array}\right) \Rightarrow x=\left(\begin{array}{l}
0 \\
0 \\
1 \\
0
\end{array}\right) \\
& A\left(A^{2} b\right)=Q x \\
& A^{3} b=\left(\begin{array}{llll}
b & A b & A^{2} b & A^{3} b
\end{array}\right)\left(\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3} \\
x_{4}
\end{array}\right) \Rightarrow x=\left(\begin{array}{l}
0 \\
0 \\
0 \\
1
\end{array}\right) \\
& A\left(A^{3} b\right)=Q x \\
& A^{4} b=\left(\begin{array}{llll}
b & A b & A^{2} b & A^{3} b
\end{array}\right)\left(\begin{array}{c}
x_{1} \\
x_{2} \\
x_{3} \\
x_{4}
\end{array}\right) \Rightarrow x=\left(\begin{array}{c}
-8 \\
20 \\
-18 \\
7
\end{array}\right)
\end{aligned}
$$

Solving for the last equation above is shown here.

$$
\left(\begin{array}{l}
24 \\
32 \\
16 \\
1
\end{array}\right)=x_{1}\left(\begin{array}{l}
0 \\
0 \\
1 \\
1
\end{array}\right)+x_{2}\left(\begin{array}{l}
0 \\
1 \\
2 \\
1
\end{array}\right)+x_{3}\left(\begin{array}{l}
1 \\
4 \\
4 \\
1
\end{array}\right)+x_{4}\left(\begin{array}{l}
6 \\
12 \\
8 \\
1
\end{array}\right)
$$

Hence we get these 4 equations

$$
\begin{align*}
24 & =x_{3}+6 x_{4}  \tag{A}\\
32 & =x_{2}+4 x_{3}+12 x_{4}  \tag{B}\\
16 & =x_{1}+2 x_{2}+4 x_{3}+8 x_{4}  \tag{C}\\
1 & =x_{1}+x_{2}+x_{3}+x_{4} \tag{D}
\end{align*}
$$

$\mathrm{C}-\mathrm{D} \Longrightarrow$

$$
\begin{equation*}
15=x_{2}+3 x_{3}+7 x_{4} \tag{E}
\end{equation*}
$$

$\mathrm{B}-\mathrm{E} \Longrightarrow$

$$
\begin{equation*}
17=x_{3}+5 x_{4} \tag{F}
\end{equation*}
$$

$\mathrm{A}-\mathrm{F} \Longrightarrow$

$$
7=x_{4}
$$

Sub. into F we get $x_{3}=17-5 \times 7=-18$, sub into E we get $x_{2}=15-3(-18)-7(7)=20$, sub into $D$ we get $x_{1}=1-20-(-18)-7=-8$

Hence $x=\left(\begin{array}{c}-8 \\ 20 \\ -18 \\ 7\end{array}\right)$
Hence $\bar{A}$, the form that the matrix $A$ takes w.r.t the basis $Q=\left\{b, A b, A^{2} b, A^{3} b\right\}$ is

$$
\bar{A}=\left(\begin{array}{cccc}
0 & 0 & 0 & -8  \tag{2}\\
1 & 0 & 0 & 20 \\
0 & 1 & 0 & -18 \\
0 & 0 & 1 & 7
\end{array}\right)
$$

Now that we have found the representation of $A$ under $Q$, we are given a new basis $Q^{\prime}=$ $\left\{\bar{b}, A \bar{b}, A^{2} \bar{b}, A^{3} \bar{b}\right\}$ and asked again find representation of $A$ under $Q^{\prime}$

Clearly the problem is not asking us to do this computation all over again for the new basis, becuase the algorithm would be the same and we can just repeat all the above steps again but using the new basis. So we need to try to find a short cut solution.

Start by finding the charactertic polynomial for $A$

$$
\begin{align*}
|A-\lambda I| & =0 \\
\left|\begin{array}{ccccc}
2-\lambda & 1 & 0 & 0 \\
0 & 2-\lambda & 1 & 0 \\
0 & 0 & 2-\lambda & 0 \\
0 & 0 & 0 & 1-\lambda
\end{array}\right| & =0 \\
(2-\lambda)((2-\lambda)((2-\lambda)(1-\lambda))) & =0 \\
\lambda^{4}-7 \lambda^{3}+18 \lambda^{2}-20 \lambda+8 & =0 \tag{3}
\end{align*}
$$

We know that a matrix companion form can be written directly if we know the matrix charaterstic equation.

If $\Delta(\lambda)=\lambda^{4}+\alpha_{4} \lambda^{3}+\alpha_{3} \lambda^{2}+\alpha_{2} \lambda+\alpha_{1}$ then the matrix $A$ companion form can be written as (see page 55 in text)

$$
\left(\begin{array}{cccc}
0 & 0 & 0 & -\alpha_{1}  \tag{4}\\
1 & 0 & 0 & -\alpha_{2} \\
0 & 1 & 0 & -\alpha_{3} \\
0 & 0 & 1 & -\alpha_{4}
\end{array}\right)
$$

Hence by comparing (4) and (3) we see that $\alpha_{1}=8, \alpha_{2}=-20, \alpha_{3}=18, \alpha_{4}=-7$, hence the compaion form for $A$ is

$$
\left(\begin{array}{cccc}
0 & 0 & 0 & -8  \tag{5}\\
1 & 0 & 0 & 20 \\
0 & 1 & 0 & -18 \\
0 & 0 & 1 & -7
\end{array}\right)
$$

Compare matrix (5) and (2) we see they are the same. (I do not think I needed to show that (5) and (2) are the same here, this is just a confirmation).

Using Cayley-Hamilton, then

$$
\begin{aligned}
\lambda^{4}-7 \lambda^{3}+18 \lambda^{2}-20 \lambda+8 & =0 \\
& \Rightarrow \\
A^{4}-7 A^{3}+18 A^{2}-20 A+8 & =0
\end{aligned}
$$

Hence

$$
\begin{equation*}
A^{4}=7 A^{3}-18 A^{2}+20 A-8 \tag{6}
\end{equation*}
$$

If we multiply the above by $b$ we get

$$
\begin{aligned}
A^{4} b & =7 A^{3} b-18 A^{2} b+20 A b-8 b \\
& =7\left(A^{3} b\right)-18\left(A^{2} b\right)+20(A b)-8(b)
\end{aligned}
$$

If we multiply (6) by any other vector, say $\bar{b}$ we also get

$$
A^{4} \bar{b}=7\left(A^{3} \bar{b}\right)-18\left(A^{2} \bar{b}\right)+20(A \bar{b})-8(\bar{b})
$$

But we found that the representation of $A$ under $Q=\left\{b, A b, A^{2} b, A^{3} b\right\}$ is

$$
\left\{\left(\begin{array}{l}
0 \\
1 \\
0 \\
0 \\
0 \\
1 \\
0
\end{array}\right),\left(\begin{array}{l}
0 \\
0 \\
0 \\
1
\end{array}\right),\left(\begin{array}{l}
0 \\
\text { (coeff. of polynomial expressing } \left.A^{4} b \text { in L.I. combination of } A^{3} b, A^{2} b, A b, b\right)
\end{array}\right\}\right.
$$

Hence the representation of $A$ under $Q^{\prime}=\left[\bar{b}, A \bar{b}, A^{2} \bar{b}, A^{3} \bar{b}\right]$ is also
$\left\{\left(\begin{array}{l}0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1\end{array}\right),\left(\begin{array}{l}0 \\ 0 \\ 0 \\ 0\end{array}\right),\left(\begin{array}{l}\left.\left.\text { coeff. of polynomial expressing } A^{4} \bar{b} \text { in L.I. combination of } A^{3} \bar{b}, A^{2} \bar{b}, A \bar{b}, \bar{b}\right)\right\}\end{array}\right\}\right.$
But with the help of the cayley-hamilton theorm, we see that the coeff. of these polynomials (i.e. the values in the last column of $\bar{A}$ ) are the same if the basis were $\left\{A^{3} b, A^{2} b, A b, b\right\}$ or $\left\{A^{3} \bar{b}, A^{2} \bar{b}, A \bar{b}, \bar{b}\right\}$.

Hence the representation of $A$ under $Q^{\prime}=\left[\bar{b}, A \bar{b}, A^{2} \bar{b}, A^{3} \bar{b}\right]$ is the same as the representation of $A$ under $Q=\left[b, A b, A^{2} b, A^{3} b\right]$

## 2 Problem 3.13

Find jordan form of the following matrices

$$
A_{1}=\left(\begin{array}{ccc}
1 & 4 & 10 \\
0 & 2 & 0 \\
0 & 0 & 3
\end{array}\right), A_{2}=\left(\begin{array}{ccc}
0 & 1 & 0 \\
0 & 0 & 1 \\
-2 & -4 & -3
\end{array}\right), A_{3}=\left(\begin{array}{ccc}
1 & 0 & -1 \\
0 & 1 & 0 \\
0 & 0 & 2
\end{array}\right), A_{4}=\left(\begin{array}{ccc}
0 & 4 & 3 \\
0 & 20 & 16 \\
0 & -25 & -20
\end{array}\right)
$$

Answer

### 2.1 A1

For $A_{1}$,

$$
\begin{gathered}
|A-\lambda I|=0 \\
\left|\left(\begin{array}{ccc}
1-\lambda & 4 & 10 \\
0 & 2-\lambda & 0 \\
0 & 0 & 3-\lambda
\end{array}\right)\right|=(1-\lambda)((2-\lambda)(3-\lambda))=0
\end{gathered}
$$

Hence $\lambda_{1}=1, \lambda_{2}=2, \lambda_{3}=3 \Rightarrow 3$ distinct roots. Hence Jordan form is the standard diagoonal form

$$
\left(\begin{array}{ccc}
\lambda_{1} & 0 & 0 \\
0 & \lambda_{2} & 0 \\
0 & 0 & \lambda_{3}
\end{array}\right)=\left(\begin{array}{lll}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 3
\end{array}\right)
$$

## $2.2 \quad \mathrm{~A} 2$

For $A_{2}$,

$$
\begin{aligned}
|A-\lambda I| & =0 \\
\left|\left(\begin{array}{ccc}
-\lambda & 1 & 0 \\
0 & -\lambda & 1 \\
-2 & -4 & -3-\lambda
\end{array}\right)\right| & =(-\lambda)((-\lambda)(-3-\lambda)-(1 \times-4))-1(-(1 \times-2)) \\
0 & =(-\lambda)\left(\left(3 \lambda+\lambda^{2}\right)+4\right)-2 \\
0 & =-\lambda^{3}-3 \lambda^{2}-4 \lambda-2 \\
0 & =\lambda^{3}+3 \lambda^{2}+4 \lambda+2
\end{aligned}
$$

Try factor $(\lambda+1)$ out, then we have $\lambda^{3}+3 \lambda^{2}+4 \lambda+2=(\lambda+1)(f(\lambda))$, hence $f(\lambda)=$ $\frac{\lambda^{3}+3 \lambda^{2}+4 \lambda+2}{(\lambda+1)}=\lambda^{2}+2 \lambda+2$ hence $|A-\lambda I|=0=(\lambda+1)\left(\lambda^{2}+2 \lambda+2\right)$. But roots of the quadratic are $\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a}=\frac{-2 \pm \sqrt{4-4 \times 2}}{2}=\frac{-2 \pm \sqrt{-4}}{2}=\frac{-2 \pm 2 j}{2}=-1 \pm j$

Hence $\lambda_{1}=-1, \lambda_{2}=-1+j, \lambda_{3}=-1-j \Rightarrow 3$ distinct roots, hence Jordan form is

$$
\left(\begin{array}{ccc}
\lambda_{1} & 0 & 0 \\
0 & \lambda_{2} & 0 \\
0 & 0 & \lambda_{3}
\end{array}\right)=\left(\begin{array}{ccc}
-1 & 0 & 0 \\
0 & -1+j & 0 \\
0 & 0 & -1-j
\end{array}\right)
$$

## $2.3 \quad$ A3

For $A_{3}$

$$
\begin{gathered}
|A-\lambda I|=0 \\
\left|\left(\begin{array}{ccc}
1-\lambda & 0 & -1 \\
0 & 1-\lambda & 0 \\
0 & 0 & 2-\lambda
\end{array}\right)\right|=(1-\lambda)((1-\lambda)(2-\lambda))=0
\end{gathered}
$$

Hence $\lambda_{1}=1, \lambda_{2}=1, \lambda_{3}=2$. The eigenvalues are not all distinct.
For $\lambda=\lambda_{1}$, we have the matrix $(A-I)=\left(\begin{array}{ccc}0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1\end{array}\right)$ which has rank of 1 , hence nullity is $3-1=2$, hence we can find 2 independent vectors in the null space of the matrix $\left(A-\lambda_{1} I\right)$, so we
get 2 jordan blocks for $\lambda=1$ of order 1. Hence the Jordan form is

$$
\left(\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 2
\end{array}\right) \text { or }\left(\begin{array}{lll}
2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right)
$$

One can also solve the above as follows:
The null space of $A-\lambda_{1} I$ is 2 , hence we can find 2 L.I. vectors in this null space as follows. Let $u_{1}$ be the first eigenvector associated with eigenvalue $\lambda_{1}$.

$$
\begin{aligned}
A u_{1} & =\lambda_{1} u_{1} \\
\left(\begin{array}{ccc}
1 & 0 & -1 \\
0 & 1 & 0 \\
0 & 0 & 2
\end{array}\right)\left(\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right) & =\lambda_{1}\left(\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right) \\
\left(\begin{array}{c}
x_{1}-x_{3} \\
x_{2} \\
2 x_{3}
\end{array}\right) & =\left(\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right)
\end{aligned}
$$

Hence $x_{3}=0$, and so $x_{1}$ and $x_{2}$ can take any values. Hence $u_{1}=\left(\begin{array}{l}1 \\ 0 \\ 0\end{array}\right)$
Now $u_{2}$ is the second eigenvector associated with $\lambda=1$. It will have similar solution to the above, but since it is L.I. to $u_{1}$, we now pick $x_{1}=0$ and $x_{2}=1$, hence $u_{2}=\left(\begin{array}{l}0 \\ 1 \\ 0\end{array}\right)$ or we could have found $u_{2}$ as follows: write $A u_{2}=\lambda_{1} u_{2}+u_{1} \Rightarrow\left(\begin{array}{ccc}1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2\end{array}\right)\left(\begin{array}{l}x_{1} \\ x_{2} \\ x_{3}\end{array}\right)=\lambda_{1}\left(\begin{array}{l}x_{1} \\ x_{2} \\ x_{3}\end{array}\right)+\left(\begin{array}{l}1 \\ 0 \\ 0\end{array}\right)$ hence $\left(\begin{array}{c}x_{1}-x_{3} \\ x_{2} \\ 2 x_{3}\end{array}\right)=\left(\begin{array}{c}x_{1}+1 \\ x_{2} \\ x_{3}\end{array}\right)$ hence $x_{3}=0$, then $x_{1}=0$ and $x_{2}$ any value, say 1. Hence $u_{2}=\left(\begin{array}{l}0 \\ 1 \\ 0\end{array}\right)$ as above.

Now to find $u_{3}$. This is the eigevector associated with $\lambda=2$. Hence

$$
\begin{aligned}
A u_{3} & =\lambda_{3} u_{3} \\
\left(\begin{array}{ccc}
1 & 0 & -1 \\
0 & 1 & 0 \\
0 & 0 & 2
\end{array}\right)\left(\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right) & =2\left(\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right) \\
\left(\begin{array}{c}
x_{1}-x_{3} \\
x_{2} \\
2 x_{3}
\end{array}\right) & =\left(\begin{array}{l}
2 x_{1} \\
2 x_{2} \\
2 x_{3}
\end{array}\right)
\end{aligned}
$$

Hence $x_{2}=0$ and $x_{3}$ is any value, say 1 hence $x_{1}=-1$ hence $u_{3}=\left(\begin{array}{c}-1 \\ 0 \\ 1\end{array}\right)$
Note that this is L.I. to both $u_{1}$ and $u_{2}$
Hence $Q=\left(u_{1}, u_{2}, u_{3}\right)=\left(\begin{array}{ccc}1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{array}\right)$ hence

$$
\begin{aligned}
Q^{-1} A Q & =J=\left(\begin{array}{lll}
1 & 0 & -1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right)^{-1}\left(\begin{array}{lll}
1 & 0 & -1 \\
0 & 1 & 0 \\
0 & 0 & 2
\end{array}\right)\left(\begin{array}{lll}
1 & 0 & -1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right) \\
& =\left(\begin{array}{lll}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right)\left(\begin{array}{lll}
1 & 0 & -2 \\
0 & 1 & 0 \\
0 & 0 & 2
\end{array}\right) \\
& =\left(\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 2
\end{array}\right)
\end{aligned}
$$

Which matches earlier solution.
$2.4 \quad$ A4
$A_{4}=\left(\begin{array}{ccc}0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20\end{array}\right)$

$$
\begin{aligned}
|A-\lambda I| & =0 \\
\left|\left(\begin{array}{ccc}
-\lambda & 4 & 3 \\
0 & 20-\lambda & 16 \\
0 & -25 & -20-\lambda
\end{array}\right)\right| & =(-\lambda)((20-\lambda)(-20-\lambda)-(16 \times-25))=0 \\
0 & =(-\lambda)\left(-400+\lambda^{2}+400\right) \\
0 & =(\lambda)\left(\lambda^{2}\right)
\end{aligned}
$$

Hence $\lambda=0,0,0$ hence we have multipicity of 3 .
Matrix $(A-\lambda I)=A$ since $\lambda=0$. Rank of A is 2 , hence null space is 1 , hence we find one eigenvector

$$
\begin{aligned}
A v & =\lambda v=0 \\
\left(\begin{array}{ccc}
0 & 4 & 3 \\
0 & 20 & 16 \\
0 & -25 & -20
\end{array}\right)\left(\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right) & =\left(\begin{array}{l}
0 \\
0 \\
0
\end{array}\right) \\
\left(\begin{array}{c}
4 x_{2}+3 x_{3} \\
20 x_{2}+16 x_{3} \\
-25 x_{2}-20 x_{3}
\end{array}\right) & =\left(\begin{array}{l}
0 \\
0 \\
0
\end{array}\right)
\end{aligned}
$$

But this gives $\left(\begin{array}{l}x_{1} \\ x_{2} \\ x_{3}\end{array}\right)=\left(\begin{array}{l}0 \\ 0 \\ 0\end{array}\right)$ hence can not use.

Matrix $(A-\lambda I)^{2}=A^{2}=\left(\begin{array}{ccc}0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20\end{array}\right)\left(\begin{array}{ccc}0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20\end{array}\right)=\left(\begin{array}{lll}0 & 5 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}\right)$
This has rank $=1$, hence nullity $=2$, so we still need one more eigevector. keep going.
Matrix $(A-\lambda I)^{3}=\left(\begin{array}{ccc}0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20\end{array}\right)\left(\begin{array}{ccc}0 & 5 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}\right)=\left(\begin{array}{lll}0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}\right)$ this has rank=0 and nullspace $=3$, hence we can find the last eigenvector from this null space.

$$
\begin{aligned}
(A-\lambda I)^{3} u_{3} & =0 \\
\left(\begin{array}{lll}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right)\left(\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right) & =\left(\begin{array}{l}
0 \\
0 \\
0
\end{array}\right)
\end{aligned}
$$

So pick vector $u_{3}$ such that the above is true and also that $(A-\lambda I)^{2} u_{3} \neq 0$
Try $u_{3}=\left(\begin{array}{l}0 \\ 0 \\ 1\end{array}\right) \Rightarrow\left(\begin{array}{lll}0 & 5 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}\right)\left(\begin{array}{l}0 \\ 0 \\ 1\end{array}\right)=\left(\begin{array}{l}4 \\ 0 \\ 0\end{array}\right) \neq 0$ hence we pick $u_{3}=\left(\begin{array}{l}0 \\ 0 \\ 1\end{array}\right)$
So now we write $(A-\lambda I)^{2} u_{3}=u_{2} \Rightarrow\left(\begin{array}{lll}0 & 5 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}\right)\left(\begin{array}{l}0 \\ 0 \\ 1\end{array}\right)=\left(\begin{array}{l}4 \\ 0 \\ 0\end{array}\right)=u_{2}$
And we finally write $(A-\lambda I)^{1} u_{3}=u_{1} \Rightarrow\left(\begin{array}{ccc}0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20\end{array}\right)\left(\begin{array}{l}0 \\ 0 \\ 1\end{array}\right)=\left(\begin{array}{c}3 \\ 16 \\ -20\end{array}\right)$

Hence $Q=\left(u_{1}, u_{2}, u_{3}\right)=\left(\begin{array}{ccc}3 & 4 & 0 \\ 16 & 0 & 0 \\ -20 & 0 & 1\end{array}\right)$ hence

$$
\begin{aligned}
& Q^{-1} A Q=J=\left(\begin{array}{ccc}
3 & 4 & 0 \\
16 & 0 & 0 \\
-20 & 0 & 1
\end{array}\right)^{-1}\left(\begin{array}{ccc}
0 & 4 & 3 \\
0 & 20 & 16 \\
0 & -25 & -20
\end{array}\right)\left(\begin{array}{ccc}
3 & 4 & 0 \\
16 & 0 & 0 \\
-20 & 0 & 1
\end{array}\right) \\
& \\
& =\left(\begin{array}{lll}
16 & 0 & 0 \\
-20 & 0 & 1
\end{array}\right)\left(\begin{array}{lll}
4 & 0 & 3 \\
0 & 0 & 16 \\
0 & 0 & -20
\end{array}\right) \\
& \text { Hence jordan form is }\left(\begin{array}{lll}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{array}\right) \\
&
\end{aligned}
$$

