

Problem 3.19

Solution

λ is an eigenvalue of A with eigenvector x means

$$Ax = \lambda x \quad \text{--- (1)}$$

so we need to show that

$$f(A)x = f(\lambda)x \quad \text{--- (2)}$$

start by considering $f(A)$ which is a polynomial in A .

i.e. let $f(A) = a_0 I + a_1 A + a_2 A^2 + \dots + a_{n-1} A^{n-1}$

multiply LHS and RHS by $x \Rightarrow$

$$f(A)x = a_0 Ix + a_1 Ax + a_2 A^2x + \dots + a_{n-1} A^{n-1}x \quad \text{--- (3)}$$

now sub (1) into above; but first need to show that

$$A^m x = \lambda^m x \quad \text{to be able to do this.}$$

from (1), multiply by A we get

$$\begin{aligned} A^2 x &= A(\lambda x) \\ &= \lambda(Ax) = \lambda(\lambda x) = \lambda^2 x \end{aligned}$$

Keep doing this, we see that

$$\boxed{A^m x = \lambda^m x}$$

sub above into (3) we get

$$f(A)x = a_0 x + a_1 \lambda x + a_2 \lambda^2 x + a_3 \lambda^3 x + \dots + a_{n-1} \lambda^{n-1} x$$

$$f(A)x = (a_0 + a_1 \lambda + a_2 \lambda^2 + \dots + a_{n-1} \lambda^{n-1}) x$$

$$\boxed{f(A)x = f(\lambda)x} \quad \text{where } \overbrace{f(\lambda)}^{\text{---}}$$

Problem 3.21

given $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ Find A^{10} , A^{103} , e^{A^t}

Solution

Since this is an upper triangle matrix \rightarrow its eigenvalues can be read from diagonal.

$$\therefore \boxed{\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 0}$$

Now let $f(A) = A^{10} \Rightarrow$ Cayley-Hamilton $\Rightarrow f(\lambda) = \lambda^{10}$

let polynomial that agrees on the spectrum of A be $h(\lambda)$.

$$\text{let } h(\lambda) = a_0 + a_1\lambda + a_2\lambda^2$$

$$\lambda_1 = 1 \quad f(\lambda_1) = \lambda_1^{10}$$

$$f(1) = 1^{10} = 1$$

$$\therefore h(\lambda) = 1 = a_0 + a_1(1) + a_2(1)^2 = \boxed{a_0 + a_1 + a_2 = 1} \quad \textcircled{1}$$

$$\lambda_2 = 1$$

$$f'(\lambda) = 10\lambda^9$$

$$h'(\lambda) = a_1 + 2a_2\lambda$$

$$\Rightarrow f'(1) = 10$$

$$h'(1) = \boxed{a_1 + 2a_2 = 10} \quad \textcircled{2}$$

$$\lambda_3 = 0$$

$$\left. \begin{array}{l} f(\lambda_3) = 0^{10} = 0 \\ h(\lambda_3) = a_0 \end{array} \right\} \Rightarrow \boxed{a_0 = 0} \quad \textcircled{3}$$

$$\text{From } \textcircled{3} + \textcircled{1} \Rightarrow \boxed{a_1 + a_2 = 1} \quad \textcircled{4}$$

$$\textcircled{2} - \textcircled{4} \Rightarrow \boxed{a_2 = 9} \text{ puts into } \textcircled{1} \Rightarrow a_1 = 1 - 9 = -8$$

$$\text{hence } \boxed{h(\lambda) = -8\lambda + 9\lambda^2}$$

$$\therefore f(A) = \boxed{A^{10} = -8A + 9A^2}$$

$$A^2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{so } A^{10} = -8 \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} + 9 \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -8 & -8 & 0 \\ 0 & 0 & -8 \\ 0 & 0 & -8 \end{pmatrix} + \begin{pmatrix} 9 & 9 & 9 \\ 0 & 0 & 9 \\ 0 & 0 & 9 \end{pmatrix}$$

$$\boxed{A^{10} = \begin{pmatrix} 1 & 1 & 9 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}}$$

For A^{103}

$$f(A) = A^{103}, \quad f(\lambda) = \lambda^{103}$$

$$\lambda_1 = 1 \rightarrow f(1) = 1^{103} = 1$$

$$h(1) = \boxed{a_0 + a_1 + a_2 = 1} \quad \text{--- (1)}$$

$$\lambda_2 = 1$$

$$f'(\lambda) = 103\lambda^{102} \rightarrow f'(1) = 103$$

$$h'(\lambda) = a_1 + 2a_2 \lambda \rightarrow h'(1) = a_1 + 2a_2 \quad \left. \right\} \Rightarrow \boxed{a_1 + 2a_2 = 103} \quad \text{--- (2)}$$

$$\lambda_3 = 0$$

$$\left. \begin{array}{l} f(0) = 0^{103} = 0 \\ h(0) = a_0 \end{array} \right\} \boxed{a_0 = 0} \quad \text{--- (3)}$$

$$\text{From (3)+(1)} \Rightarrow \boxed{a_1 + a_2 = 1} \quad \text{--- (4)}$$

$$(2) - (4) \Rightarrow \boxed{a_2 = 103 - 1 = 102} \quad \text{plus into (1)} \Rightarrow \boxed{a_1 = 1 - 102 = -101}$$

$$\therefore h(\lambda) = -101\lambda + 102\lambda^2$$

$$\therefore f(A) = -101A + 102A^2.$$

$$\therefore A^{103} = -101A + 102A^2$$

$$= -101 \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} + 102 \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -101 & -101 & 0 \\ 0 & 0 & -101 \\ 0 & 0 & -101 \end{pmatrix} + \begin{pmatrix} 102 & 102 & 102 \\ 0 & 0 & 102 \\ 0 & 0 & 102 \end{pmatrix}$$

$$\boxed{A^{103} = \begin{pmatrix} 1 & 1 & 102 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}}$$



e^{At}

$$\textcircled{1} \quad f(A) = e^{At}, \Rightarrow f(N) = e^{\lambda t}, \quad h(\lambda) = a_0 + a_1 \lambda + a_2 \lambda^2$$

$\lambda_1 = 1$

$$f(\lambda) = e^\lambda, \quad h(1) = a_0 + a_1 + a_2 \Rightarrow \boxed{a_0 + a_1 + a_2 = e^t} \quad \textcircled{1}$$

$\lambda_2 = 1$

$$f'(t) = te^t.$$

$$f'(1) = te^t$$

$$h'(1) = a_1 + 2a_2$$

$$h'(1) = a_1 + 2a_2$$

$$\boxed{te^t = a_1 + 2a_2}$$

$$\boxed{a_0 + a_1 + a_2 = e^t}$$

— \textcircled{2}

$\lambda_3 = 0$

$$\begin{cases} f(0) = 1 \\ h(0) = a_0 \end{cases}$$

$$\boxed{a_0 = 1} \quad \textcircled{3}$$

$$\text{Plug } \textcircled{3} \text{ in } \textcircled{1} \Rightarrow \boxed{a_1 + a_2 = e^t - 1} \quad \textcircled{4}$$

$$\textcircled{2} - \textcircled{4} \Rightarrow (1 - a_1 - 2a_2) - (a_1 + a_2) = te^t - e^t + 1 \Rightarrow \boxed{a_2 = te^t - e^t + 1}$$

$$\text{so from } \textcircled{1} \Rightarrow 1 + a_1 + te^t - e^t = e^t \Rightarrow \boxed{a_1 = 2e^t - te^t - 2}$$

$$\text{so } h(t) = 1 + (2e^t - te^t - 2)t + (te^t - e^t + 1)t^2$$

$$\text{so } f(A) = e^{At} = I + (2e^t - te^t - 2)A + (te^t - e^t + 1)A^2 \rightarrow$$

$$\cancel{I + (e^t - 2) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \cancel{\begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{pmatrix}} + \cancel{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}}$$

$$\therefore = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t - 1 & 0 \\ 0 & 0 & e^t - 1 \end{pmatrix}$$

$$f(A) = e^{At} =$$

$$\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (2e^t - te^t - 2) \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} + (te^t - e^t + 1) \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 1 + 2e^t - te^t - 2 + te^t - e^t + 1 & 2e^t - te^t - 2 + te^t - e^t + 1 & te^t - e^t + 1 \\ 0 & 1 & 2e^t - te^t - 2 + te^t - e^t + 1 \\ 0 & 0 & 1 + 2e^t - te^t - 2 + te^t - e^t + 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^t & e^t - 1 & te^t - e^t + 1 \\ 0 & 1 & e^t - 1 \\ 0 & 0 & e^t \end{pmatrix}$$

Ps. This could also be solved by using

$$\int e^{At} = (SI - A)^{-1} \quad \text{and then}$$

taking Inverse Laplace transform of resulting matrix (term-by-term).