HW3, MAE 270A. Linear systems I. Fall 2005. UCI

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Contents

1 Problem 3.5

Find the rank and nullities of the following matrices

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad A_2 = \begin{pmatrix} 4 & 1 & -1 \\ 3 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix} \quad A_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Solution

Solution method: First find the rank of the matrix, then to find the nullity, use the relation

$$\eta(A) = \textit{number of columns of } A - Rank(A)$$

The rank of a matrix can be found using one of the following methods

1. By inspection (works for small matrices), look at the rows (or columns) of the matrix, whichever is smaller, and see if there is any linear dependence between any pair of rows (or columns).
2. Find the largest non-zero square minor. The dimension of this minor is the rank of the matrix.
3. Find the eigenvalues (λ) of the matrix. The number of unique λ is the rank of the matrix. This works only on square matrices.

Hence for A_1 using the second method above, we see that the minor found by omitting the first columns and the second row is $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \neq 0$ but the full determinant is clearly zero (since A_1

has one columns which is all zeros, since any square matrix which has all zero's in one of its rows or columns must have zero determinant). Hence the largest size of a minor which is not zero is 2,

$\boxed{\text{hence the rank is 2}}$

Since the rank is 2, then $\boxed{\eta(A) = 3 - 2 = 1}$

For A_2, $\begin{vmatrix} 4 & 1 & -1 \\ 3 & 2 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 4(0) - 1(0) - 1(3-2) = -1 \neq 0$ $\boxed{\text{hence the rank is 3}}$

Therefor $\boxed{\eta(A) = 3 - 3 = 0}$

For A_3, the rank can not be more than 3, and since the last row contains zeros in the first 3 elements, I will select the minor to test on as the last 3 columns, hence $\begin{vmatrix} 2 & 3 & 4 \\ -1 & 2 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 2(2) - 3(-1) + 4(0) = 4 + 3 = 7 \neq 0$, hence $\boxed{\text{the rank is 3}}$

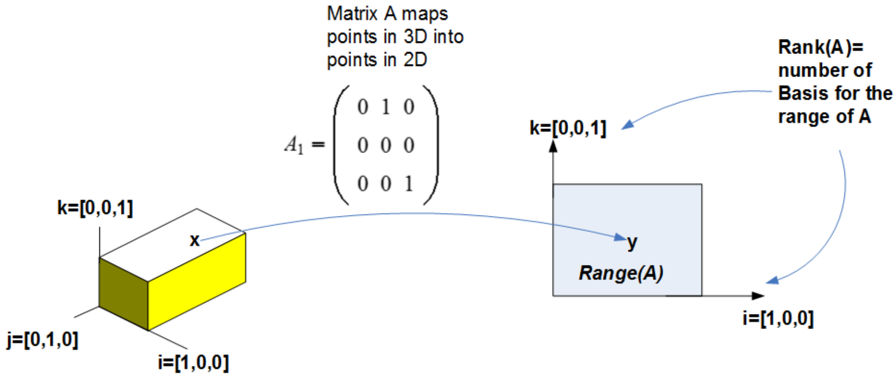
Hence $\boxed{\eta(A) = 4 - 3 = 1}$

2 Problem 3.6

question: Find bases of the range spaces and the null spaces of the matrices in problem 3.5

solution method: To find the bases for the range space, all what we need to do is to find n linearly independent vectors where n is the rank of A found above. To find bases for the null space of A, since we know the rank of the null space, and the dimension of the rank space, we need to find m linearly independent vectors where m is the rank of the null space of A.

A1: Since A_1 has rank of 2, this means that range(A) has dimension 2. In other words A maps a 'point' in a 3D volume to a 'point' of in a 2D flat plane. Hence we can use either i, j or i, k or j, k as the basis of the range of A. But since A has zeros in its second row, this means that there are no points in the range of A which has a component along the j dimension. Hence the only plane left is the one spanned by i, k or $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. This is illustrated by the following diagram



since the rank of the null space is 1, then we need to find one vector x s.t. $Ax = 0$

Hence $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies x_2 = 0, x_3 = 0$, hence a basic is $\begin{pmatrix} any \\ 0 \\ 0 \end{pmatrix}$ where 'any' could be any number. select 1 to make it normalized, hence a basis is for the null space is $\boxed{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$

A2: Since A_2 has rank of 3, and this is equal to the number of columns in A, then A maps a point in 3D vector space ($R3$) to a point in 3D vector space ($R3$). Then we can use i, j, k as its bases. i.e. $\boxed{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$

Since the rank of the null space of A is zero, then the null space of A is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, hence no basis for the null space as it is empty.

A3: Since A_3 has rank of 3, then we need to find 3 linearly independent vectors to span the range of A. Select i, j, k as its bases. i.e. $\boxed{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}}$

The null space of A_3 is 1, then we need to find one vector such $Ax = 0$ and normalize it as needed.

$$\begin{pmatrix}1 & 2 & 3 & 4\\0 & -1 & -2 & 2\\0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Longrightarrow \begin{pmatrix}x_1 + 2x_2 + 3x_3 + 4x_4 = 0\\-x_2 - 2x_3 + 2x_4 = 0\\x_4 = 0\end{pmatrix} \Longrightarrow \begin{pmatrix}x_1 + 2x_2 + 3x_3 = 0\\-x_2 - 2x_3 = 0\end{pmatrix}$$

Hence $x_3 = -\frac{x_2}{2}$ and so $x_1 + 2x_2 + 3\left(-\frac{x_2}{2}\right) = 0 \Longrightarrow x_1 + \frac{1}{2}x_2 = 0 \Longrightarrow x_1 = -\frac{1}{2}x_2$

Hence if we take $x_1 = 1$, then $x_2 = -2x_1 = -2$ and then $x_3 = 1$

Hence a basis is $\boxed{\begin{pmatrix}1\\-2\\1\\0\end{pmatrix}}$

3 Problem 3.7

Consider the linear algebraic equation $\begin{pmatrix}2 & -1\\-3 & 3\\-1 & 2\end{pmatrix} x = \begin{pmatrix}1\\0\\1\end{pmatrix} = y$, it has 3 equations and 3 unknowns.

Does a solution exist in the equation? Is this solution unique, Does a solution exist if $y = \begin{pmatrix}1\\1\\1\end{pmatrix}$

Solution

A solution exist if A spans a space which contains y. Since A has 2 columns we see that it takes points from 2D space and send these points to its range space. Since the rank of A is 2 (since

$\begin{vmatrix} 2 & -1 \\ -3 & 3 \end{vmatrix} = 3 \neq 0$) then the dimension of its range space is 2, i.e. it maps points from 2D space to 2D space. Solve this is by solving for $Ax = y$ and to see if we can find a vector x to satisfy this equation as follows

$$\begin{pmatrix} 2 & -1 \\ -3 & 3 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \Longrightarrow \begin{pmatrix} 2x_1 - x_2 = 1 \\ -3x_1 + 3x_2 = 0 \\ -x_1 + 2x_2 = 1 \end{pmatrix}$$

From second equation we see that $x_1 = x_2$, substitute this in either equation 1, we get that $x_1 = 1$, hence $x_2 = 1$.Sub this solution in equation 3 we see that is also satisfy it.

Hence we found a point $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, which is mapped by A to point $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ hence a solution exist.

$\eta(A) = 2 - Rank(A) = 2 - 2 = 0$

Since the null space is empty, then this $\boxed{\text{solution is unique}}$.

When $y = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, we need to try to see if there is an x such that $Ax = y$

$$\begin{pmatrix} 2 & -1 \\ -3 & 3 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Longrightarrow \begin{pmatrix} 2x_1 - x_2 = 1 \\ -3x_1 + 3x_2 = 1 \\ -x_1 + 2x_2 = 1 \end{pmatrix}$$

From first equation, $x_2 = 2x_1 - 1$ sub into second equation we get $-3x_1 + 3(2x_1 - 1) = 1 \Longrightarrow -3x_1 + 6x_1 - 3 = 1 \Longrightarrow 3x_1 = 4 \Longrightarrow x_1 = \frac{4}{3}$

Hence $x_2 = 2\left(\frac{4}{3}\right) - 1 \Longrightarrow x_2 = \frac{5}{3}$

Now sub this solution into the third equation we get $-\left(\frac{4}{3}\right) + 2\left(\frac{5}{3}\right) = 1 \Longrightarrow -\frac{4}{3} + \frac{10}{3} = 1 \Longrightarrow 2 = 1$ which is not valid. $\boxed{\text{Hence no solution exist.}}$

4 Problem 3.38

Problem: Consider $Ax = y$, where A is an $m \times n$ and has rank m. is $\left(A^T A\right)^{-1} A^T y$ a solution? if not, under what condition will it be a solution? is $A^T \left(AA^T\right)^{-1} y$ a solution?

Solution

First we need to determine if $\left(A^T A\right)^{-1}$ exist.

Since A is an $m \times n$ then $A^T A$ is $(n \times m) \times (m \times n) \rightarrow n \times n$ matrix.

Hence $A^T A$ is a square matrix of dimension n. Since we are told the rank is m, and the rank of a matrix is the smaller of its dimensions (the smaller of its rows or columns if they are not the same), hence there exist only m linearly independent rows, and not n linearly independent rows.

hence $\left(A^T A\right)$ is NOT invertible $\boxed{\left(A^T A\right)^{-1} A^T y \text{ is not a solution.}}$

$\boxed{\text{It will be a solution under the condition that } m = n}$. Since in this case $A^T A$ can be inverted.

Second part: is $A^T \left(AA^T\right)^{-1} y$ a solution?

Since A is an $m \times n$ then AA^T is $(m \times n) \times (n \times m) \rightarrow m \times m$ matrix.

Hence, since we are told the rank is m then there exist m linearly independent rows, hence $\left(AA^T\right)$ is invertible. Hence $\left(AA^T\right)^{-1}$ exist, and so $\left(AA^T\right)^{-1} A^T y$ can be computed. And in addition, if we multiply this by A we get y[1] hence $\boxed{\text{it is a solution.}}$

[1] $AA^T \left(AA^T\right)^{-1} y = \mathbf{I} y = y$