HW2, MAE 270A. Linear systems I. Fall 2005. UCI

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Contents

1 Problem 2.10

Consider the system described by

$$y'' + 2y' - 3y = u' - u$$

What are the transfer function and the impulse response of the system?

Answer

The transfer function is defined as the ratio of the Laplace transform of the output to the laplace transform of the input assuming zero initial conditions. i.e. assume that $y'(0^-) = 0$, $y(0^-) = 0$, $u(0^-) = 0$

Taking the laplace transform of the above differential equation we obtain

$$\begin{aligned} s^2Y(s) + 2sY(s) - 3Y(s) &= sU(s) - U(s) \\ Y(s)\left(s^2 + 2s - 3\right) &= U(s)(s-1) \\ \frac{Y(s)}{U(s)} &= \frac{s-1}{s^2+2s-3} \\ &= \frac{s-1}{(s-1)(s+3)} \\ &= \frac{1}{s+3} \end{aligned}$$

Hence

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s+3}$$

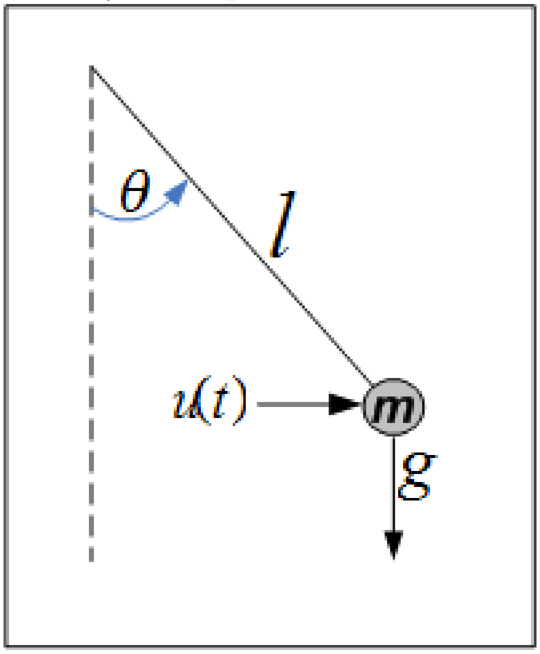
Hence the impulse response is the inverse laplace transform of $G(s)$, which is for this simple case can written directly

$$g(t) = e^{-3t}$$

Note: the above solution is valid for $t \geq 0$. For $t < 0$ the impulse response is zero.

2 Problem 2.15 part(a)

Find state space equations to describe the pendulum system in following figure. Write down the linearized dynamic equations and the transfer function from $u(t)$ to $\theta(t)$



Pendulum

Answer

The general state space representation for this system is

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{h}(x,u,t) \\ \mathbf{y}(t) &= \mathbf{f}(x,u,t) \end{aligned} \tag{1}$$

To simplify notations, I will not list time as an independent variables since it is implicit in x and u in this problem.

Now, assume we have a nominal solution x_0 and a nominal input u_0 and let the perturbation from these be $\bar{x}$ and $\bar{u}$ respectively. Hence (1) can be written as

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{h}(x_0+\bar{x}, u_0+\bar{u}) = \mathbf{h}(x_0,u_0) + \bar{x}\left.\frac{\partial \mathbf{h}(x,u)}{\partial x}\right|_{x_0,u_0} + \bar{u}\left.\frac{\partial \mathbf{h}(x,u)}{\partial u}\right|_{x_0,u_0} \\ \mathbf{y}(t) &= \mathbf{f}(x_0+\bar{x}, u_0+\bar{u}) = \mathbf{f}(x_0,u_0) + \bar{x}\left.\frac{\partial \mathbf{f}(x,u)}{\partial x}\right|_{x_0,u_0} + \bar{u}\left.\frac{\partial \mathbf{f}(x,u)}{\partial u}\right|_{x_0,u_0} \end{aligned} \tag{2}$$

Now pick a nominal solution when the system is in its stable equilibrium position (when the pendulum is hanging down at rest).

Hence $x_0 = 0$ and $u_0 = 0$. For this state and input we obtain $\mathbf{h}(x_0,u_0) = 0$ since $\dot{\mathbf{x}}_0(t) = 0$ as there is no state change with time, also we obtain that $\mathbf{f}(x_0,u_0) = 0$ since $\mathbf{y}_0(t)$ since the mass is not moving. Hence (2) becomes

$$
\begin{aligned}
\dot{\mathbf{x}}(t) &= \bar{x}\left.\frac{\partial \mathbf{h}(x,u)}{\partial x}\right|_{x_0,u_0} + \bar{u}\left.\frac{\partial \mathbf{h}(x,u)}{\partial u}\right|_{x_0,u_0} \\
\mathbf{y}(t) &= \bar{x}\left.\frac{\partial \mathbf{f}(x,u)}{\partial x}\right|_{x_0,u_0} + \bar{u}\left.\frac{\partial \mathbf{f}(x,u)}{\partial u}\right|_{x_0,u_0}
\end{aligned} \tag{3}
$$

Now since $\bar{x} = x - x_0$ and $x_0 = 0$ then $\bar{x} = x$
Similarly, $\bar{u} = u - u_0$ and $u_0 = 0$ then $\bar{u} = u$ hence (3) can be written as

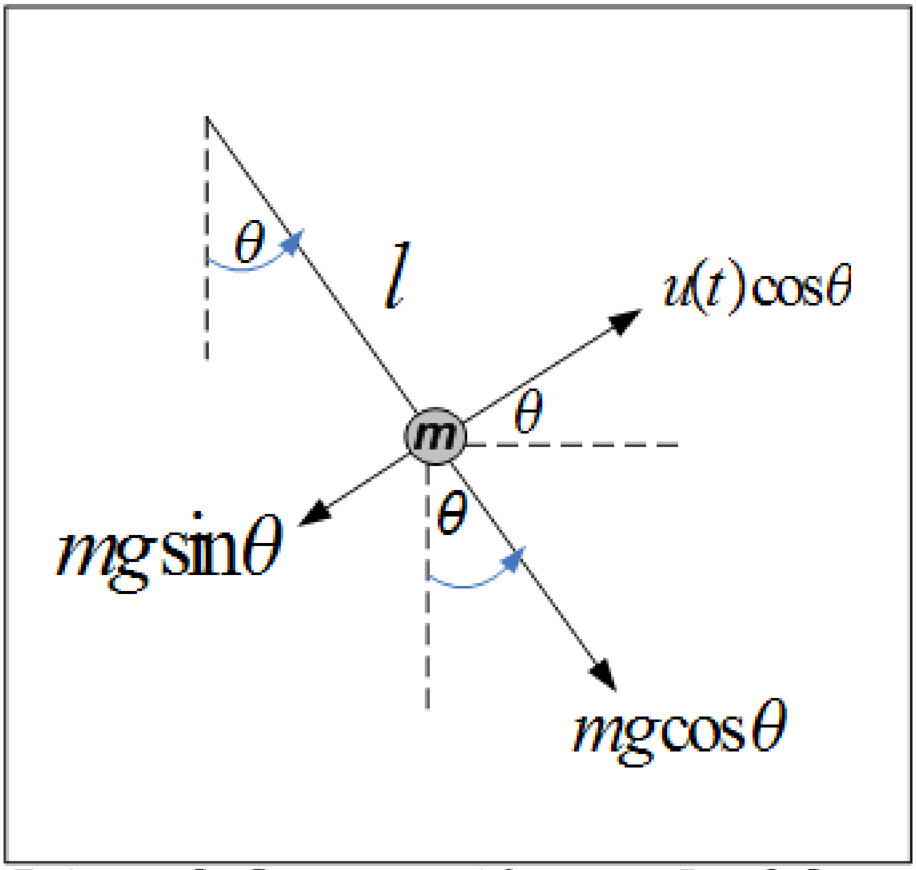
$$
\begin{aligned}
\dot{\mathbf{x}}(t) &= x(t)\left.\frac{\partial \mathbf{h}(x,u)}{\partial x}\right|_{x_0,u_0} + u(t)\left.\frac{\partial \mathbf{h}(x,u)}{\partial u}\right|_{x_0,u_0} \\
\mathbf{y}(t) &= x(t)\left.\frac{\partial \mathbf{f}(x,u)}{\partial x}\right|_{x_0,u_0} + u(t)\left.\frac{\partial \mathbf{f}(x,u)}{\partial u}\right|_{x_0,u_0}
\end{aligned} \tag{4}
$$

Hence we just need to evaluate $\left.\frac{\partial \mathbf{h}(x,u)}{\partial x}\right|_{x_0,u_0}, \left.\frac{\partial \mathbf{h}(x,u)}{\partial u}\right|_{x_0,u_0}, \left.\frac{\partial \mathbf{f}(x,u)}{\partial x}\right|_{x_0,u_0}, \left.\frac{\partial \mathbf{f}(x,u)}{\partial u}\right|_{x_0,u_0}$ to obtain the linearized solution.

Since there are 2 states in this system and one input we obtain

$$
\begin{aligned}
\frac{\partial \mathbf{h}(x,u)}{\partial x} &= \begin{pmatrix} \frac{\partial h_1(x,u)}{\partial x_1} & \frac{\partial h_1(x,u)}{\partial x_2} \\ \frac{\partial h_2(x,u)}{\partial x_1} & \frac{\partial h_2(x,u)}{\partial x_2} \end{pmatrix} \\
\frac{\partial \mathbf{h}(x,u)}{\partial u} &= \begin{pmatrix} \frac{\partial h_1(x,u)}{\partial u} \\ \frac{\partial h_2(x,u)}{\partial u} \end{pmatrix} \\
\frac{\partial \mathbf{f}(x,u)}{\partial x} &= \begin{pmatrix} \frac{\partial f(x,u)}{\partial x_1} & \frac{\partial f(x,u)}{\partial x_2} \end{pmatrix} \\
\frac{\partial \mathbf{f}(x,u)}{\partial u} &= \left(\frac{\partial f(x,u)}{\partial u} \right)
\end{aligned} \tag{5}
$$

Now we need to find h_1, h_2, f and substitute these into (5) and then into (4) to find the solution.
First find the dynamic equation for this system. The forces on the mass are



External forces acting on Pendulum

Applying Newton second law $F = ma$, along the direction tangent to the motion we get

$$u(t)\cos\theta(t) - mg\sin\theta(t) = mL\ddot{\theta}(t)$$

Hence

$$\ddot{\theta}(t) = \frac{1}{mL}u(t)\cos\theta(t) - \frac{g}{L}\sin\theta(t)$$

Now convert to state space. Let $x_1 = \theta$ and $x_2 = \dot{\theta}$.
Hence

$$\dot{x}_1 = x_2$$

and

$$\begin{aligned}\dot{x}_2 = \ddot{\theta} &= \frac{1}{mL}u(t)\cos\theta(t) - \frac{g}{L}\sin\theta(t)\\ &= \frac{1}{mL}u(t)\cos x_1 - \frac{g}{L}\sin x_1\end{aligned}$$

Hence we can write

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}h_1(x_1,x_2,u)\\ h_2(x_1,x_2,u)\end{pmatrix} = \begin{pmatrix}x_2\\ \frac{1}{mL}u(t)\cos x_1 - \frac{g}{L}\sin x_1\end{pmatrix} \tag{6}$$

Now for the output equation:

$$\begin{aligned} y &= \theta \\ &= x_1 \end{aligned}$$

Hence

$$y = f(x,u) = (x_1)$$

Now that we know h_1, h_2, f, we can go back to (5) and evaluate that, we obtain

$$\begin{aligned} \frac{\partial \mathbf{h}(x,u)}{\partial x} &= \begin{pmatrix} 0 & 1 \\ -\frac{1}{mL}u(t)\sin x_1 - \frac{g}{L}\cos x_1 & 0 \end{pmatrix} \\ \frac{\partial \mathbf{h}(x,u)}{\partial u} &= \begin{pmatrix} 0 \\ \frac{1}{mL}\cos x_1 \end{pmatrix} \\ \frac{\partial \mathbf{f}(x,u)}{\partial x} &= \begin{pmatrix} 1 & 0 \end{pmatrix} \\ \frac{\partial \mathbf{f}(x,u)}{\partial u} &= (0) \end{aligned} \tag{7}$$

Now to obtain the solution (4) we need to evaluate (7) at the nominal solution x_0, u_0 and these are zero, i.e. $x_1 = x_{1,0} = 0, x_2 = x_{2,0} = 0, u = u_0 = 0$ then (7) becomes

$$\begin{aligned} \frac{\partial \mathbf{h}(x,u)}{\partial x} &= \begin{pmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{pmatrix} \\ \frac{\partial \mathbf{h}(x,u)}{\partial u} &= \begin{pmatrix} 0 \\ \frac{1}{mL} \end{pmatrix} \\ \frac{\partial \mathbf{f}(x,u)}{\partial x} &= 1 \\ \frac{\partial \mathbf{f}(x,u)}{\partial u} &= 0 \end{aligned} \tag{8}$$

Hence, substitute (8) into (4) we obtain the final linearized solution

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \begin{pmatrix} 0 & 1 \\ \frac{g}{L} & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \frac{1}{mL} \end{pmatrix} u(t) \\ \mathbf{y}(t) &= \begin{pmatrix} 1 & 0 \end{pmatrix} x(t) + (0)\ u(t) \end{aligned} \tag{9}$$

The above is the linearized solution. Where

$$
\begin{aligned}
A &= \begin{pmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{pmatrix} \\
B &= \begin{pmatrix} 0 \\ \frac{1}{mL} \end{pmatrix} \\
C &= \begin{pmatrix} 1 & 0 \end{pmatrix} \\
D &= (0)
\end{aligned}
$$

The transfer function is

$$
\begin{aligned}
G(s) &= C\left(sI - A\right)^{-1} B + D \\
&= \begin{pmatrix} 1 & 0 \end{pmatrix} \left(sI - \begin{pmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} 0 \\ \frac{1}{mL} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \end{pmatrix} \left(\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} 0 \\ \frac{1}{mL} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} s & -1 \\ \frac{g}{L} & s \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \frac{1}{mL} \end{pmatrix}
\end{aligned}
$$

but

$$
\begin{pmatrix} s & -1 \\ \frac{g}{L} & s \end{pmatrix}^{-1} = \frac{Adjoint(A)}{Det(A)} = \frac{\begin{pmatrix} s & -\frac{g}{L} \\ 1 & s \end{pmatrix}^{T}}{\begin{vmatrix} s & -1 \\ \frac{g}{L} & s \end{vmatrix}} = \frac{\begin{pmatrix} s & 1 \\ -\frac{g}{L} & s \end{pmatrix}}{s^2 + \frac{g}{L}}
$$

Hence

$$
\begin{aligned}
G(s) &= \frac{1}{s^2+\frac{g}{L}}\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix} s & 1 \\ -\frac{g}{L} & s\end{pmatrix}\begin{pmatrix}0 \\ \frac{1}{mL}\end{pmatrix} \\
&= \frac{1}{s^2+\frac{g}{L}}\begin{pmatrix}s & 1\end{pmatrix}\begin{pmatrix}0 \\ \frac{1}{mL}\end{pmatrix} \\
&= \frac{\frac{1}{mL}}{s^2+\frac{g}{L}} \\
&= \frac{1}{mLs^2+mg} \\
&= \frac{1}{m\left(Ls^2+g\right)}
\end{aligned}
$$