HW1, MAE 270A. Linear systems I. Fall 2005. UCI

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1 Problem 2.2

The impulse response of an ideal lowpass filter is given by

$$g(t) = 2\omega \frac{\sin 2\omega (t - t_0)}{2\omega (t - t_0)}$$

for all t, where ω,t_0 are constants. Is the ideal lowpass filter causal? Is it possible to build the filter in the real world?

Answer

A system is causal when its output measured at time t depends only on its the input at time t and possibly input before that.

Equivalently we can say that a system is causal if its impulse response is always zero when observed at any time before the time of applying the impulse itself.

For the above case, assume that the impulse to the lowpass filter was applied at time t_0 and let us observe the response of the system at some $t < t_0$, The output of the system must be zero for all time before t_0 for it to be a causal system. (Otherwise we are observing a response before applying the input).

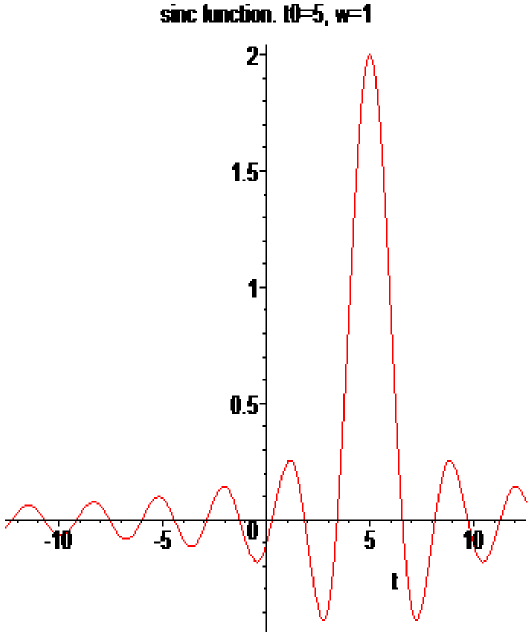
But we can find at least one $t < t_0$ such that $g(t) \neq 0$, hence $\boxed{\text{not causal}}$

Since it is not causal, $\boxed{\text{it is not possible to build in real world}}$. That is why it is called ideal.

For example, assume $t_0 = 5$ and let $t = 4$ then we get

$$g(4) = 2\omega \frac{\sin 2\omega (4 - 5)}{2\omega (4 - 5)} = 2\omega \frac{\sin 2\omega (-1)}{2\omega (-1)} = \sin 2\omega$$

which is not zero unless ω was zero. Since we found at least one time instance before the application of the impulse where the output is not zero, this system is not causal.



2 Problem 2.3

Consider a system whose input $u(t)$ and $y(t)$ are related by

$$y(t) = P_\alpha(u(t)) = \begin{cases} u(t) & t \leq \alpha \\ 0 & t > \alpha \end{cases}$$

where α is a fixed constant. Is this system linear? Time invariant? Causal?
Answer
A system represented by an operator L is linear if

$$L[a\ w(t) + b\ v(t)] = a\ L[\ w(t)] + b\ L[\ v(t)]$$

For all constants a, b and all input $u(t), v(t)$
Applying the definition shown to a combination of input we get

$$
\begin{aligned}
P_\alpha\left(a\ w(t)+b\ v(t)\right) &= \begin{cases} \left(a\ w(t)+b\ v(t)\right) & t\leq\alpha \\ 0 & t>\alpha \end{cases} \\
&= \begin{cases} a\ w(t) & t\leq\alpha \\ 0 & t>\alpha \end{cases} + \begin{cases} b\ v(t) & t\leq\alpha \\ 0 & t>\alpha \end{cases} \\
&= a\begin{cases} w(t) & t\leq\alpha \\ 0 & t>\alpha \end{cases} + b\begin{cases} v(t) & t\leq\alpha \\ 0 & t>\alpha \end{cases} \\
&= aP_\alpha\left(w(t)\right)+b\ P_\alpha\left(v(t)\right)
\end{aligned}
$$

Hence the $\boxed{\text{system is linear}}$

A system is time invariant if the following is true:

Apply an impulse or any input to the system which can be in some initial state at time t and observe the output. Assume the output shows after some $\Delta(t)$ time from the time the input is applied. $\Delta(t)$ can be zero or positive.

Now apply the same input again to the same system when it is in the same initial state as before but with some delay δt. If the same output will result as before and will also appear after the same $\Delta(t)$ from the time the delayed input was applied, then the system is time invariant.

Another way to express this is to say that the by shifting the input by some δt, the output will remain the same and will be shifted by the same δt.

This must be true for any δt.

So we need to ask, is

$$P_\alpha\left(u(t-\tau)\right) \quad = \quad y(t-\tau)$$

for any delay τ ?

This system is clearly not time invariant. Suppose we apply a unit step function as the input. The output will then be of height 1 from $t=0$ up to α and will be zero after that. So the output is 1 for a width of α.

Now if we delay the input by any value, we should also get an output to be of height 1 and of width α (but shifted by the same delay) if this system to be time variant. But if we delay the input by an amount that happens to be α, then the output will be zero. hence $\boxed{\text{system is not time invariant}}$

Is it causal?

Assume that the input was applied at the time t_0 and let us observe the response of the system at some $t<t_0$, hence the output of the system must be zero for all time before t_0 for it to be causal. 2 cases. Assume $t_0\leq\alpha$, then $y(t)=u(t)$, hence if $u(t)$ was zero before t_0 then clearly $y(t)$ will also be zero before t_0 since it is the same signal.

Now assume $t_0>\alpha$, hence $y(t)=0$ by definition.

Hence $\boxed{\text{this is causal system}}$

3 Problem 2.5

Consider a system with input u and output y. 3 experiments are performed on the system using the input u_1, u_2, u_3 for $t \geq 0$. In each case the initial state $x(0)$ time time $t = 0$ is the same. The corresponding output are denoted by y_1, y_2, y_3. which is of the following statements are correct if $x(0) \neq 0$?

1. if $u_3 = u_1 + u_2$ then $y_3 = y_1 + y_2$
2. if $u_3 = 0.5(u_1 + u_2)$ then $y_3 = 0.5(y_1 + y_2)$
3. if $u_3 = u_1 - u_2$ then $y_3 = y_1 - y_2$

which are correct if $x(0) = 0$?
Answer
Assume system is linear.
Let $Y_{j,zir}$ means the output for system j when input is zero
Let $Y_{j,zsr}$ means the output for system j when state $x(0)$ is zero
hence

$$y_j = Y_{j,zir} + Y_{j,zsr}$$

Since it is the same system, and the same initial state, then $Y_{j,zir}$ is the same for all j, call it Z hence

$$Y_{j,zir} = Z$$

hence we write

$$y_j = Z + Y_{j,zir}$$

Where A is the response of the system when input is zero. i.e. it is the response due to the initial state $x(0)$ only.

3.1 part 1

when $u_3 = u_1 + u_2$, since system is linear then

$$Y_{3,zsr} = Y_{1,zsr} + Y_{2,zsr} \tag{1}$$

Now is $y_3 = y_1 + y_2$?

$$\begin{aligned} y_3 &= y_1 + y_2 & ? \\ Z + Y_{3,zsr} &= (Z + Y_{1,zsr}) + (Z + Y_{2,zsr}) & ? \\ Z + Y_{3,zsr} &= 2Z + Y_{1,zsr} + Y_{2,zsr} & ? \end{aligned} \tag{2}$$

substitute (1) into (2) we obtain that $Z = 2Z$. Now when $x(0) \neq 0$ then $Z \neq 0$, hence we can divide by Z and obtain that $1 = 2$ which is not correct. $\boxed{\text{Hence (1) is not correct if } x(0) \neq 0}$

if $x(0) = 0$ then $Z = 0$, then (2)=(1), hence it is correct in this case.

3.2 part 2

When $u_3 = 0.5\,(u_1 + u_2)$, and since system is linear then

$$Y_{3,zsr} = 0.5\,(Y_{1,zsr} + Y_{2,zsr}) \tag{3}$$

is $y_3 = 0.5\,(y_1 + y_2)$?

$$\begin{aligned} y_3 &= 0.5\,(y_1 + y_2) \\ Z + Y_{3,zsr} &= 0.5\,(Z + Y_{1,zsr} + Z + Y_{2,zsr}) \\ Z + Y_{3,zsr} &= Z + 0.5\,(Y_{1,zsr} + Y_{2,zsr}) \end{aligned} \tag{4}$$

subtract Z from each side, we see that (4) is the same as (3) $\boxed{\text{Hence (2) is correct if } x(0) \neq 0}$

It is also correct if $x(0) \neq 0$ since is independent on Z

3.3 part 3

When $u_3 = \; u_1 - u_2$, and since system is linear then

$$Y_{3,zsr} = Y_{1,zsr} - Y_{2,zsr} \tag{5}$$

is $y_3 = \; y_1 - y_2$?

$$\begin{aligned} y_3 &= \; y_1 - y_2 \\ Z + Y_{3,zsr} &= (Z + Y_{1,zsr}) - (Z + Y_{2,zsr}) \\ Z + Y_{3,zsr} &= (Y_{1,zsr} - Y_{2,zsr}) \end{aligned} \tag{6}$$

substitute (5) into equation (6) hence we obtain that $Z = 0$

If $x(0) \neq 0$ then $Z \neq 0$ hence $\boxed{\text{Hence (3) is not correct if } x(0) \neq 0}$ but if $x(0) = 0$ then $Z = 0$ hence it is correct in that case.