1/2:

$$z = b\cos(\omega t)$$

$$\ddot{z} = -b\omega^2\cos(\omega t)$$

$$m\ell^2\ddot{\theta} + c\ell\dot{\theta} + mg\ell\theta = -m\ell\ddot{z}$$

$$\Rightarrow \ddot{\theta} + \frac{c}{m\ell}\dot{\theta} + \frac{g}{\ell}\theta = \frac{b\omega^2}{\ell}\cos(\omega t) \qquad (1)$$

Natural Frequency : $\omega_n^2 = \frac{g}{\ell} \Rightarrow \omega_n = \sqrt{\frac{g}{\ell}}$

Damping ratio : $2\zeta\omega_n = \frac{c}{m\ell} \Rightarrow \zeta = \frac{c}{2m\sqrt{g\ell}}$

3. Homogeneous solution:

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = 0$$

characteristic eqn:

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$$

$$\Rightarrow \lambda = -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}\,i \quad (\zeta^2 < 1)$$

$$\theta_h = e^{-\zeta\omega_n t}\left[A\cos(\omega_n\sqrt{1-\zeta^2}\,t) + B\sin(\omega_n\sqrt{1-\zeta^2}\,t)\right]$$

$$= Ce^{-\zeta\omega_n t}\left[\cos(\omega_n\sqrt{1-\zeta^2}\,t + \varphi)\right]$$

4. particular solution

Assume $\theta_{particular} = C_1 \cos wt + C_2 \sin wt$

$$\Rightarrow \dot{\theta}_p = -C_1 w \sin wt + C_2 w \cos wt$$

$$\Rightarrow \ddot{\theta}_p = -C_1 w^2 \cos wt - C_2 w^2 \sin wt$$

plug them in Eqn (1)

$$-C_1 w^2 \cos wt - C_2 w^2 \sin wt + 2 \zeta w_n [-C_1 w \sin wt + C_2 w \cos wt]$$

$$+ w_n^2 [C_1 \cos wt + C_2 \sin wt] = \frac{b w^2}{\ell} \cos wt$$

Equal SIN terms:

$$-C_2 w^2 - 2 C_1 \zeta w_n w + C_2 w_n^2 = 0$$

$$\Rightarrow C_1 = \frac{w_n^2 - w^2}{2 \zeta w_n w} C_2 \qquad (2)$$

Equal cos terms:

$$-C_1 w^2 + 2 C_2 \zeta w_n w + C_1 w_n^2 = \frac{b w^2}{\ell} \qquad (3)$$

Substitute (2) into (3)

$$-\frac{(w_n^2 - w^2) w^2}{2 \zeta w_n w} C_2 + 2 C_2 \zeta w_n w + \frac{(w_n^2 - w^2) w_n^2}{2 \zeta w_n w} = \frac{b w^2}{\ell}$$

$$\Rightarrow C_2 = \frac{2 \zeta w_n w \left(\frac{b w^2}{\ell}\right)}{(w_n^2 - w^2)^2 + 4 \zeta^2 w_n^2 w^2}$$

$$C_1 = \frac{(w_n^2 - w^2) \left(\frac{b w^2}{\ell}\right)}{(w_n^2 - w^2)^2 + 4 \zeta^2 w_n^2 w^2}$$

$$\theta_p = C_1 \cos\omega t + C_2 \sin\omega t$$

$$= \frac{b\omega^2/\ell}{\sqrt{(\omega_n^2-\omega^2)^2 + 4\zeta^2\omega_n^2\omega^2}} \cos(\omega t - \gamma)$$

$$\cos\gamma = \frac{\omega_n^2 - \omega^2}{\sqrt{(\omega_n^2-\omega^2)^2 + 4\zeta^2\omega_n^2\omega^2}}$$

5. Bode Plots

Amplitude: $\dfrac{b\omega^2/\ell}{\sqrt{(\omega_n^2-\omega^2)^2+4\zeta^2\omega_n^2\omega^2}} = C$

$\omega \to 0 \qquad C \to 0$

$\omega \to \omega_n \qquad C = \dfrac{b/\ell}{\sqrt{\left(\frac{\omega_n}{\omega}-1\right)^2 + 4\zeta^2\left(\frac{\omega_n}{\omega}\right)^2}}$

$= \dfrac{b}{2\zeta\ell}$

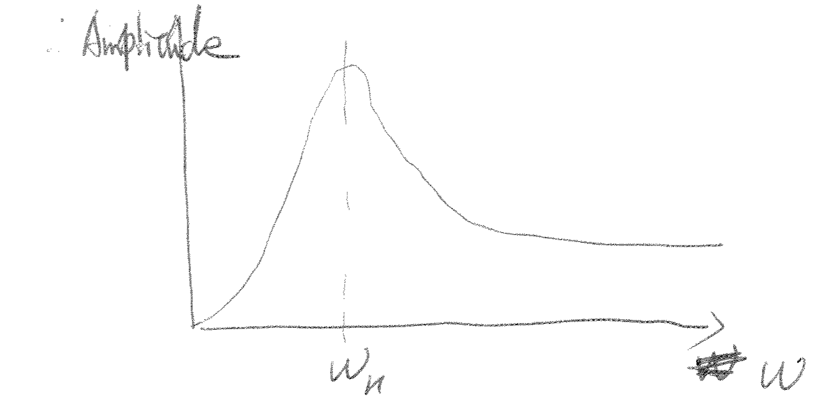
$\omega \to \infty \qquad C = b/\ell$

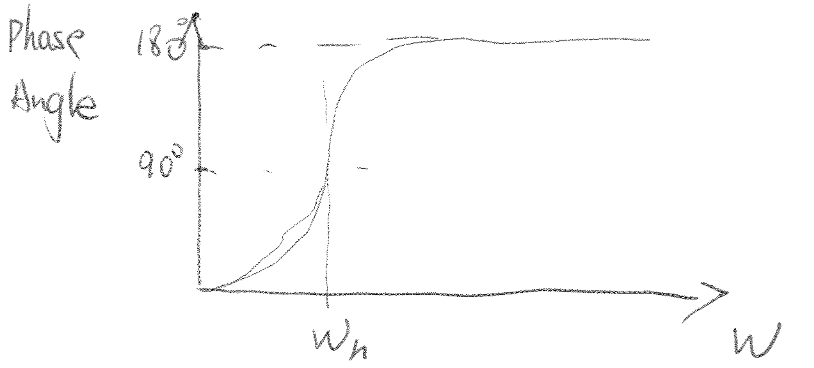
Phase Angle: $\cos\gamma = \dfrac{\omega_n^2 - \omega^2}{\sqrt{(\omega_n^2-\omega^2)^2 + 4\zeta^2\omega_n^2\omega^2}}$

$\omega \to 0 \qquad \cos\gamma = 1 \qquad \gamma = 0$

$\omega \to \omega_n \qquad \cos\gamma = \dfrac{\left(\frac{\omega_n}{\omega}\right)^2 - 1}{\sqrt{\left(\frac{\omega_n}{\omega}-1\right)^2 + 4\zeta^2\left(\frac{\omega_n}{\omega}\right)^2}} = 0 \qquad \gamma = 90^\circ$

$\omega \to \infty \qquad \cos\gamma = -1 \qquad \gamma = 180^\circ$





6. $z = \cos(\omega_1 t) + \cos(\omega_2 t) + \cos(\omega_3 t)$

system is linear, so the particular solution can be superposition of:

$\theta_p(t) = \theta_{p_1} + \theta_{p_2} + \theta_{p_3}$

θ_{p_i} $(i=1,2,3)$ is the particular solution from previous result.

Use the resules in Question 5.

The amplitudes for each terms are

$C_1 \approx 0 \quad \omega_1 \ll \omega_n$

$C_2 \approx \frac{b}{2\zeta \ell} \quad \omega_2 \approx \omega_n$

$C_3 \approx \frac{b}{\ell} \quad \omega_3 \gg \omega_n$

So If the ζ is small enough, the second term (resonant term) dominates.