HW 6, EECS 203A
Problem 5.27, Digital Image Processing, 2nd edition by Gonzalez, Woods.
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Question
A certain Xray imaging geometry produces a blurring degradation that can be modeled as the convolution of the sensed image with the spatial, circularly symmetric function $h(r) = \frac{(r^2-\sigma^2)}{\sigma^4} \; e^{-\frac{r^2}{2\sigma^2}}$ where $r^2 = x^2 + y^2$. Show that the degradation in the frequency domain is given by $H(u,v) = -2\pi\sigma^2 (u^2+v^2) e^{-2\pi^2\sigma^2(u^2+v^2)}$

Solution
In general,

$$g(x,y) = h(x,y) \circledast f(x,y) + \eta(x,y)$$

Since we are told what the model is (no noise involved), then the model is

$$g(x,y) = h(x,y) \circledast f(x,y)$$

Here $f(x,y)$ is the sensed image, and $g(x,y)$ is the degraded, produced image and h is the impulse response.
This means if the input image is an impulse, then the output image will be $h(r)$ or $h(x,y)$ since r is a function of x,y
So a degraded output image can be considered to be $h(x,y)$ convolved with impulse. Since the Fourier transform of an impulse is 1, then in the frequency domain, the fourier transform of a degraded output image is the fourier transform of h times 1.
i.e. in frequency domain, degraded output image transform $= H(u,v) \times 1 = H(u,v)$
But $H(u,v) = \mathcal{F}\left(\frac{(r^2-\sigma^2)}{\sigma 4} \; e^{-\frac{r^2}{2\sigma^2}}\right) = \mathcal{F}\left(\frac{(x^2+y^2-\sigma^2)}{\sigma 4} \; e^{-\frac{(x^2+y^2)}{2\sigma^2}}\right) = \mathcal{F}\left(\left[\frac{x^2}{\sigma 4} + \frac{y^2}{\sigma 4} - \frac{1}{\sigma 2}\right] \; e^{-\frac{(x^2+y^2)}{2\sigma^2}}\right)$
$= \mathcal{F}\left(\frac{x^2}{\sigma 4} e^{-\frac{(x^2+y^2)}{2\sigma^2}} + \frac{y^2}{\sigma 4} e^{-\frac{(x^2+y^2)}{2\sigma^2}} - \frac{1}{\sigma 2} e^{-\frac{(x^2+y^2)}{2\sigma^2}}\right) =$
$= \mathcal{F}\left(\frac{x^2}{\sigma 4} e^{-\frac{(x^2+y^2)}{2\sigma^2}}\right) + \mathcal{F}\left(\frac{y^2}{\sigma 4} e^{-\frac{(x^2+y^2)}{2\sigma^2}}\right) - \mathcal{F}\left(\frac{1}{\sigma 2} e^{-\frac{(x^2+y^2)}{2\sigma^2}}\right)$
$= \frac{1}{\sigma 4}\mathcal{F}\left(x^2 e^{-\frac{(x^2+y^2)}{2\sigma^2}}\right) + \frac{1}{\sigma 4}\mathcal{F}\left(y^2 e^{-\frac{(x^2+y^2)}{2\sigma^2}}\right) - \frac{1}{\sigma 2}\mathcal{F}\left(e^{-\frac{(x^2+y^2)}{2\sigma^2}}\right)$
But

$$\mathcal{F}\left(\; f(x,y) \;\right) = \frac{1}{M \; N} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) \;\; e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

but from tables, we know that $\mathcal{F}\left(A2\pi\sigma^2 e^{-2\pi^2\sigma^2(x^2+y^2)}\right) = Ae^{-\frac{(u^2+v^2)}{2\sigma^2}}$
$\frac{1}{\sigma 4}\mathcal{F}\left(x^2 e^{-\frac{(x^2+y^2)}{2\sigma^2}}\right) \;=\; \sqrt{2\pi}\,(4\pi^2\sigma^2 u^2 - 1)\, e^{-2\sigma^2\pi^2(u^2+v^2)}$

$$\frac{1}{\sigma 4}F\left(y^2 e^{-\frac{\left(x^2+y^2\right)}{2\sigma^2}}\right) = \sqrt{2\pi}\left(4\pi^2\sigma^2 v^2 - 1\right)e^{-2\sigma^2\pi^2\left(u^2+v^2\right)}$$

$$\frac{1}{\sigma 2}F\left(e^{-\frac{\left(x^2+y^2\right)}{2\sigma^2}}\right) = \pi e^{-2\sigma^2\pi^2\left(u^2+v^2\right)}$$

Hence we get

$$\begin{aligned}
H(u,v) &= \sqrt{2\pi}\left(4\pi^2\sigma^2 u^2 - 1\right)e^{-2\sigma^2\pi^2\left(u^2+v^2\right)} + \sqrt{2\pi}\left(4\pi^2\sigma^2 v^2 - 1\right)e^{-2\sigma^2\pi^2\left(u^2+v^2\right)} - \pi e^{-2\sigma^2\pi^2\left(u^2+v^2\right)} \\
&= \sqrt{2\pi}4\pi^2\sigma^2 u^2 e^{-2\sigma^2\pi^2\left(u^2+v^2\right)} - \sqrt{2\pi}e^{-2\sigma^2\pi^2\left(u^2+v^2\right)} + \sqrt{2\pi}4\pi^2\sigma^2 v^2 e^{-2\sigma^2\pi^2\left(u^2+v^2\right)} \\
&\quad -\sqrt{2\pi}e^{-2\sigma^2\pi^2\left(u^2+v^2\right)} - \pi e^{-2\sigma^2\pi^2\left(u^2+v^2\right)} \\
&= \sqrt{2\pi}e^{-2\sigma^2\pi^2\left(u^2+v^2\right)}\left[4\pi^2\sigma^2 u^2 - 4\pi^2\sigma^2 v^2 - 2\right] \\
&= \sqrt{2\pi}e^{-2\sigma^2\pi^2\left(u^2+v^2\right)}\left[4\pi^2\sigma^2\left(u^2 - v^2\right) - 2\right]
\end{aligned}$$

.... do not know what I am doing wrong, can't get the exact expression needed, getting very close...