HW 6, EECS 203A
Problem 5.16, Digital Image Processing, 2nd edition by Gonzalez, Woods.
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Question
Consider a linear position-invariant image degradation system with impulse response $h(x-\alpha, y-\beta) = e^{-\left[(x-\alpha)^2+(y-\beta)^2\right]}$ Suppose that the input to the system is an image consisting of a line of infinitesimal width located at $x=a$ and modeled by $f(x,y)=\delta(x-a)$ where δ is the impulse. Assuming no noise, find the output image $g(x,y)$

Solution
In general,

$$g(x,y) = h(x,y) \circledast f(x,y) + \eta(x,y)$$

where $\eta(x,y)$ is the noise. Hence since $\eta(x,y)=0$, we have

$$g(x,y) = h(x,y) \circledast f(x,y)$$

Now, I can solve this using spatial domain (convolution), or solve in Fourier transform domain, then inverse transform to get $g(x,y)$. I'll try the direct spatial approach:

$$h(x,y) \circledast f(x,y) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(x,y)\, h(x-m, y-n)$$

where M, N are the dimensions of the image $f(x,y)$
Since $f(x,y) = \delta(x-a)$ then only when $x=a$, that we get a non-zero value for the the output image. At all other values for x, $g(x,y)=0$
Hence

$$\begin{aligned} g(x,y) &= h(x,y) \circledast f(x,y) \\ g(a,y) &= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} 1 \times h(a-m, y-n) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} h(a-m, y-n) \end{aligned}$$

Since $h(x-\alpha, y-\beta) = e^{-\left[(x-\alpha)^2+(y-\beta)^2\right]}$, then Substitute $h(a-m, y-n) = e^{-\left[(a-m)^2+(y-n)^2\right]}$ we get

$$
\begin{aligned}
g(a,y) &= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} e^{-\left[(a-m)^2+(y-n)^2\right]} = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} e^{-(a-m)^2} e^{-(y-n)^2} \\
&= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} e^{-\left(a^2+m^2-2am\right)} e^{-\left(y^2+n^2-2ny\right)} \\
&= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} e^{-a^2} e^{-m^2+2am} e^{-y^2} e^{-n^2+2ny} = e^{-a^2} e^{-y^2} \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} e^{-m^2+2am} e^{-n^2+2ny} \\
&= e^{-\left(a^2+y^2\right)} \sum_{m=0}^{M-1} e^{-\left(m^2-2am\right)} \sum_{n=0}^{N-1} e^{-\left(n^2-2ny\right)}
\end{aligned}
$$

How to evaluate these sums?
Since we are told that the input image is of infinitesimal width, then this means we can consider the sum $\sum_{n=0}^{N-1} e^{-\left(n^2-2ny\right)}$ to have only one point. i.e. $\sum_{n=0}^{0} e^{-\left(n^2-2ny\right)} = 1$
so

$$g(a,y) = e^{-\left(a^2+y^2\right)} \sum_{m=0}^{M-1} e^{-\left(m^2-2am\right)}$$

To continue, best I could do is to look up the tables for integrals, and use the results for $\int_0^k e^{-(m^2-2ma)} dm = \frac{1}{2} e^{a^2} \sqrt{\pi} \left(\operatorname{erf}(k-a) + \operatorname{erf}(a)\right)$ where erf is the error function defined as $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$
Hence, using these, we get

$$g(a,y) = e^{-\left(a^2+y^2\right)} \frac{1}{2} e^{a^2} \sqrt{\pi} \left(\operatorname{erf}(M-1-a) + \operatorname{erf}(a)\right)$$

Since M, a are constants, then $\operatorname{erf}(M-1-a) + \operatorname{erf}(a)$ is a constant, call it k

$$k = \operatorname{erf}(M-1-a) + \operatorname{erf}(a)$$

Then $g(a,y) = e^{-\left(a^2+y^2\right)} \frac{1}{2} e^{a^2} k \quad = \frac{k}{2} e^{-y^2} = A\ e^{-y^2}$ where A new constant$= k/2$

$$g(x,y) = \begin{cases} A\ e^{-y^2} & x = a \\ \\ 0 & x \neq a \end{cases}$$

Notice that g is a function of a as well, since constant A value depends on a via the equation given above for k

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