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HW#6

EECS 152A, Digital Signal processing

UCI. Fall 2004

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HW 4, EECS 152A DSP.
Problem 4.53 , Digital Signal Processing, 3rd edition, Proakis, anolakis
by Nasser Abbasi
UCI, Fall 2004.

Question
Derive the expression for the resonant frequency of a 2 pole filter with the poles at $p_1 = re^{j\theta}$ and $p_2 = p_1^*$ gien by 4.5.25
Solution

Here, $\omega_0 = \theta$
Hence 4.5.25 is

$$\omega_r = \cos^{-1}\left(\frac{1+r^2}{2r}\cos\theta\right)$$

We know that

$$U_1(\omega) = \sqrt{1+r^2-2r\cos(\theta-\omega)}$$

and

$$U_2(\omega) = \sqrt{1+r^2-2r\cos(\theta+\omega)}$$

Take the product of U_1U_2 and minimize the result and solve for $\omega = \omega_r$

$$\begin{aligned} U_1U_2 &= \sqrt{1+r^2-2r\cos(\theta-\omega)}\sqrt{1+r^2-2r\cos(\theta-\omega)} \\ \frac{d}{d\omega}(U_1U_2) &= U_1\frac{d}{d\omega}(U_2) + U_2\frac{d}{d\omega}(U_1) \end{aligned}$$

But $\frac{d}{d\omega}(U_1) = \frac{1}{2}\frac{1}{\sqrt{1+r^2-2r\cos(\theta-\omega)}}(2r\sin(\theta-\omega)(-1)) = -\frac{r\sin(\theta-\omega)}{\sqrt{1+r^2-2r\cos(\theta-\omega)}}$
and $\frac{d}{d\omega}(U_2) = \frac{1}{2}\frac{1}{\sqrt{1+r^2-2r\cos(\theta+\omega)}}(2r\sin(\theta+\omega)(1)) = \frac{r\sin(\theta+\omega)}{\sqrt{1+r^2-2r\cos(\theta+\omega)}}$
Hence

$$\begin{aligned} \frac{d}{d\omega}(U_1U_2) &= U_1\frac{d}{d\omega}(U_2) + U_2\frac{d}{d\omega}(U_1) \\ &= \frac{\sqrt{1+r^2-2r\cos(\theta-\omega)}r\sin(\theta+\omega)}{\sqrt{1+r^2-2r\cos(\theta+\omega)}} - \frac{\sqrt{1+r^2-2r\cos(\theta+\omega)}r\sin(\theta-\omega)}{\sqrt{1+r^2-2r\cos(\theta-\omega)}} \end{aligned}$$

Take common denomenator

$$\frac{d}{d\omega}(U_1U_2) = \frac{\left(1+r^2-2r\cos(\theta-\omega)\right)\ r\sin(\theta+\omega) - \left(1+r^2-2r\cos(\theta+\omega)\right)\ r\sin(\theta-\omega)}{\sqrt{1+r^2-2r\cos(\theta+\omega)}\sqrt{1+r^2-2r\cos(\theta-\omega)}}$$

This derivative is mimumum when the numerator is zero.
Hence

$$0 = \left(1+r^2-2r\cos(\theta-\omega)\right)\ r\sin(\theta+\omega) - \left(1+r^2-2r\cos(\theta+\omega)\right)\ r\sin(\theta-\omega)$$

But for $r \neq 0$, divide the above by r to simplify, we get

$$0 = \left(1+r^2-2r\cos(\theta-\omega)\right)\ \sin(\theta+\omega) - \left(1+r^2-2r\cos(\theta+\omega)\right)\ \sin(\theta-\omega) \tag{1}$$

Expand (1) and use the following relations to simplify
$\sin(\theta+\omega) = \cos\theta\sin\omega + \sin\theta\cos\omega$
$\sin(\theta-\omega) = \sin\theta\cos\omega - \cos\theta\sin\omega$
$\cos(\theta-\omega) = \cos\theta\cos\omega + \sin\theta\sin\omega$

$\cos(\theta+\omega) = \cos\theta\cos\omega - \sin\theta\sin\omega$
Hence (1) becomes:

$$
\begin{aligned}
0 &= \left(1+r^2-2r\cos(\theta-\omega)\right)\ (\cos\theta\sin\omega+\sin\theta\cos\omega) \\
&\quad -\left(1+r^2-2r\cos(\theta+\omega)\right)\ (\sin\theta\cos\omega-\cos\theta\sin\omega) \\
&= (\cos\theta\sin\omega+\sin\theta\cos\omega) + r^2(\cos\theta\sin\omega+\sin\theta\cos\omega) - 2r\cos(\theta-\omega)(\cos\theta\sin\omega+\sin\theta\cos\omega) \\
&\quad - (\sin\theta\cos\omega-\cos\theta\sin\omega) - r^2\ (\sin\theta\cos\omega-\cos\theta\sin\omega) + 2r\cos(\theta+\omega)(\sin\theta\cos\omega-\cos\theta\sin\omega) \\
&= \cos\theta\sin\omega + \overbrace{\sin\theta\cos\omega} + r^2\cos\theta\sin\omega + \overbrace{r^2\sin\theta\cos\omega} - 2r\cos(\theta-\omega)\cos\theta\sin\omega - 2r\cos(\theta-\omega)\sin\theta\cos\omega \\
&\quad \overbrace{-\sin\theta\cos\omega} + \cos\theta\sin\omega \overbrace{-r^2\ \sin\theta\cos\omega} + r^2\cos\theta\sin\omega + 2r\cos(\theta+\omega)\sin\theta\cos\omega - 2r\cos(\theta+\omega)\cos\theta\sin\omega \\
&= 2\cos\theta\sin\omega + 2r^2\cos\theta\sin\omega - 2r\cos\theta\sin\omega(\cos(\theta-\omega)+\cos(\theta+\omega)) - 2r\sin\theta\cos\omega(\cos(\theta-\omega)-\cos(\theta+\omega)) \\
&= 2\cos\theta\sin\omega + 2r^2\cos\theta\sin\omega - 2r\cos\theta\sin\omega(\cos\theta\cos\omega+\sin\theta\sin\omega+\cos\theta\cos\omega-\sin\theta\sin\omega) \\
&\quad -2r\sin\theta\cos\omega(\cos\theta\cos\omega+\sin\theta\sin\omega-\cos\theta\cos\omega+\sin\theta\sin\omega) \\
&= \\
&= 2\cos\theta\sin\omega + 2r^2\cos\theta\sin\omega - 2r\cos\theta\sin\omega(2\cos\theta\cos\omega) - 2r\sin\theta\cos\omega(2\sin\theta\sin\omega) \\
&= 2\cos\theta\sin\omega + 2r^2\cos\theta\sin\omega - 4r\cos^2\theta\sin\omega\cos\omega - 4r\sin^2\theta\cos\omega\sin\omega) \\
&= 2\cos\theta\sin\omega + 2r^2\cos\theta\sin\omega - 4r\sin\omega\cos\omega\left(\cos^2\theta+\sin^2\theta\right) \\
&= 2\cos\theta\sin\omega + 2r^2\cos\theta\sin\omega - 4r\sin\omega\cos\omega
\end{aligned}
$$

So the solution to $2\cos\theta\sin\omega + 2r^2\cos\theta\sin\omega - 4r\sin\omega\cos\omega = 0$ will give us ω_r

$$
\begin{aligned}
2\cos\theta\sin\omega + 2r^2\cos\theta\sin\omega - 4r\sin\omega\cos\omega &= 0 \\
\sin\omega\left[2\cos\theta + 2r^2\cos\theta - 4r\cos\omega\right] &= 0
\end{aligned}
$$

Hence, we get first solution as $\sin\omega = 0$ or $\boxed{\omega = 0}$
and we get the second solution when

$$
\begin{aligned}
2\cos\theta + 2r^2\cos\theta - 4r\cos\omega &= 0 \\
\left(2+2r^2\right)\cos\theta - 4r\cos\omega &= 0 \\
4r\cos\omega &= \\
\cos\omega &= \frac{(2+2r^2)}{4r}\cos\theta \\
\cos\omega &= \frac{(1+r^2)}{2r}\cos\theta \\
&\boxed{\omega_r = \cos^{-1}\left(\frac{(1+r^2)}{2r}\cos\theta\right)}
\end{aligned}
$$

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HW 6

Problem 4.57

(a) $y(n) = \frac{1}{2M+1} \sum_{K=-M}^{M} x(n-K)$.

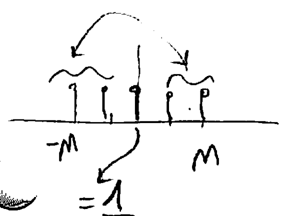
$Y(z) = \frac{1}{2M+1} \sum_{K=-M}^{M} z^{-K} X(z) = \frac{1}{2M+1} X(z) \sum_{K=-M}^{M} z^{-K}$

so $H(z) = \frac{1}{2M+1} \sum_{K=-M}^{M} z^{-K}$

so $\boxed{H(w) = \frac{1}{2M+1} \sum_{-M}^{M} e^{-jwk}}$ but $e^{-jwk} = \cos wk - j\sin wk$

so $H(w) = \frac{1}{2M+1} \sum_{-M}^{M} (\cos kw - j\sin kw)$

$= \frac{1}{2M+1} \underbrace{\sum_{-M}^{M} \cos kw}_{\Downarrow\ 1 + 2\sum_{1}^{M} \cos kw} - j\cancel{\sum_{-M}^{M} \sin kw}$ $= 0$ since sin is an odd function.



$\because \cos 0 = 1$

$1 + 2\sum_{1}^{M} \cos kw$ $\Rightarrow$ since cos is an even function.

So $\boxed{H(w) = \frac{1}{2M+1}\left(1 + 2\sum_{1}^{M} \cos kw\right)}$ ✓ 5.

(b) $y(n) = \frac{1}{4M} x(n+M) + \frac{1}{2M} \sum_{-M+1}^{M-1} x(n-k) + \frac{1}{4M} x(n-M)$

$Y(z) = \frac{1}{4M} z^{M} X(z) + \frac{1}{2M} \sum_{-M+1}^{M-1} z^{k} X(z) + \frac{1}{4M} X(z) z^{-M}$

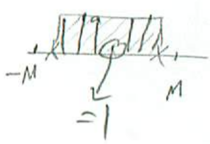
$= \frac{1}{4M} z^{M} X(z) + \frac{1}{2M} X(z) \sum_{-M+1}^{M-1} z^{k} + \frac{1}{4M} X(z) z^{-M}$

$= X(z) \left[\frac{1}{4M} z^{M} + \frac{1}{2M} \sum_{-M+1}^{M-1} z^{k} + \frac{z^{-M}}{4M} \right]$

so $$H(z) = \frac{z^{M}}{4M} + \frac{1}{2M}\sum_{-M+1}^{M-1} z^{k} + \frac{z^{-M}}{4M}$$

so $$H(\omega) = \frac{e^{j\omega M}}{4M} + \frac{1}{2M}\left[\sum_{-M+1}^{M-1} e^{j\omega k}\right] + \frac{e^{-j\omega M}}{4M}$$

$$= \frac{1}{4M}\Big[\underbrace{e^{j\omega M} + e^{-j\omega M}}_{2\cos\omega M}\Big] + \frac{1}{2M}\Big[\underbrace{\sum_{-M+1}^{M-1}\cos\omega k}_{\Downarrow\ \left(1+2\sum_{1}^{M-1}\cos\omega k\right)} + \overset{=0}{\cancel{j\sum_{-M+1}^{M+1}\sin\omega k}}\Big]$$



$$= \frac{1}{4M}\,2\cos\omega M + \frac{1}{2M}\left(1+2\sum_{0}^{M-1}\cos\omega k\right)$$

$$\boxed{H(\omega) = \frac{\cos\omega M}{2M} + \frac{1}{2M} + \frac{1}{M}\sum_{0}^{M-1}\cos\omega k} = \frac{\cos\omega M}{2M} + \frac{1}{2M}\left(1+2\sum_{0}^{M-1}\cos\omega k\right)$$

✓ 5

To find which provides better smoothing:

It is the filter which suppress high ω more.

for a fixed M, we see that $H_b(\omega)$ have one term less in the sum (it only goes up to M-1, but $H_a(\omega)$ goes to M.) so $H_b(\omega)$ will have smaller response.

so $\boxed{H_{(b)} \text{ is better for smoothing}}$

✓ 5

HW 4, EECS 152A DSP.
Problem 4.69 , Digital Signal Processing, 3rd edition, Proakis, anolakis
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Question

Determine the gain b_0 for the digital resonator described by 4.5.28 so that $|H(\omega_0)| = 1$

Solution

From page 342, equation 4.5.28 is

$$H(\omega) = b_0 \frac{1-e^{-j2\omega}}{\left(1-re^{j(\omega_0-\omega)}\right)\left(1-re^{-j(\omega_0+\omega)}\right)} \tag{4.5.28}$$

$$H(\omega) = b_0 \frac{1-(\cos 2\omega - j\sin 2\omega)}{(1-r(\cos(\omega_0-\omega)+j\sin(\omega_0-\omega)))(1-r\cos(\omega_0+\omega)+j\sin(\omega_0+\omega))}$$

Set $\omega = \omega_0$

$$\begin{aligned}
H(\omega) &= b_0 \frac{1-(\cos 2\omega_0 - j\sin 2\omega_0)}{(1-r(\cos(\omega_0-\omega_0)+j\sin(\omega_0-\omega_0)))(1-r\cos(\omega_0+\omega_0)+j\sin(\omega_0+\omega_0))} \\
&= b_0 \frac{1-\cos 2\omega_0 + j\sin 2\omega_0}{(1-r)(1-r\cos 2\omega_0 + j\sin 2\omega_0)} \\
&= b_0 \frac{1-\cos 2\omega_0 + j\sin 2\omega_0}{1-r\cos(2\omega_0)+j\sin 2\omega_0 - r + r^2\cos 2\omega_0 - jr\sin 2\omega_0} \\
&= b_0 \frac{(1-\cos 2\omega_0) + j\sin 2\omega_0}{(1-r\cos 2\omega_0 - r + r^2\cos 2\omega_0) + j(\sin 2\omega_0 - r\sin 2\omega_0)}
\end{aligned}$$

Hence

$$\begin{aligned}
|H(\omega)| &= b_0 \frac{\sqrt{(1-\cos 2\omega_0)^2 + \sin^2 2\omega_0}}{\sqrt{(1-r\cos 2\omega_0 - r + r^2\cos 2\omega_0)^2 + (\sin 2\omega_0 - r\sin 2\omega_0)^2}} \\
&= b_0 \frac{\sqrt{1+\cos^2 2\omega_0 - 2\cos 2\omega_0 + \sin^2 2\omega_0}}{\sqrt{(1+r^2-2r)(1+r^2-2r\cos 2\omega_0)}} \\
&= b_0 \frac{\sqrt{2(1-\cos 2\omega_0)}}{\sqrt{(1+r^2-2r)(1+r^2-2r\cos 2\omega_0)}}
\end{aligned}$$

Set $|H(\omega)| = 1$ and solve for b_0

$$\begin{aligned}
1 &= b_0 \frac{\sqrt{2(1-\cos 2\omega_0)}}{\sqrt{(1+r^2-2r)(1+r^2-2r\cos 2\omega_0)}} \\
b_0 &= \frac{\sqrt{(1+r^2-2r)(1+r^2-2r\cos 2\omega_0)}}{\sqrt{2(1-\cos 2\omega_0)}} \\
&= \frac{\sqrt{(1-r)^2(1+r^2-2r\cos 2\omega_0)}}{\sqrt{2(1-\cos 2\omega_0)}}
\end{aligned}$$

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$$b_0 = (1-r)\frac{\sqrt{1+r^2-2r\cos 2\omega_0}}{\sqrt{2-2\cos 2\omega_0}}$$

HW 6

4.79

(a) $P_1 = 0.8e^{j\frac{2\pi}{9}}$, $P_2 = 0.8e^{-j\frac{2\pi}{9}}$, $P_3 = 0.8e^{j4\frac{\pi}{9}}$, $P_4 = 0.8e^{-j4\frac{\pi}{9}}$

$z_1 = 1$, $z_2 = -1$, $z_3 = e^{3j\frac{\pi}{4}}$, $z_4 = e^{-3j\frac{\pi}{4}}$.

$$H(z) = b_0 \frac{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}{(z-P_1)(z-P_2)(z-P_3)(z-P_4)}$$

$$= b_0 \frac{(z-1)(z+1)(z-e^{3j\frac{\pi}{4}})(z-e^{-j3\frac{\pi}{4}})}{(z-.8e^{j\frac{2\pi}{9}})(z-0.8e^{-j\frac{2\pi}{9}})(z-.8e^{j4\frac{\pi}{9}})(z-.8e^{-j4\frac{\pi}{9}})}$$

$$= \frac{(z^2-1)(e^{jw}-e^{3j\frac{\pi}{4}})(e^{jw}-e^{-j3\frac{\pi}{4}})}{(z^2-0.8z^{-j2\frac{\pi}{9}}-.8ze^{j\frac{2\pi}{9}}+.64)(z^2-0.8ze^{-j4\frac{\pi}{9}}-.8ze^{j\frac{4\pi}{9}}+.64)}$$

$$= \frac{(e^{2jw}-1)(e^{2jw}-e^{jw-j3\frac{\pi}{4}}-e^{3j\frac{\pi}{4}+jw}+1)}{(e^{2jw}-.8(e^{-j\frac{2\pi}{9}+jw}-e^{j\frac{2\pi}{9}+jw})+.64)(e^{2jw}-.8(e^{-j4\frac{\pi}{9}+jw}+e^{jw+j\frac{4\pi}{9}})+.64)}$$

$$= \frac{e^{4jw}-e^{3jw-j3\frac{\pi}{4}}-e^{3jw+3j\frac{\pi}{4}}+\cancel{e^{2jw}}-\cancel{e^{2jw}}+e^{j(w-\frac{3\pi}{4})}+e^{j(w+\frac{3\pi}{4})}-1}{(e^{2jw}-.8(e^{j(-\frac{2\pi}{9}+w)}-e^{j(\frac{2\pi}{9}+w)})+.64)(e^{2jw}-.8(e^{j(w-\frac{4\pi}{9})}+e^{j(w+\frac{4\pi}{9})})+.64)}$$

$$= \frac{e^{4jw}-e^{3jw}(e^{-j3\frac{\pi}{4}}-e^{3j\frac{\pi}{9}})+e^{jw}(e^{-j\frac{3\pi}{4}}+e^{j\frac{3\pi}{4}})-1}{(e^{2jw}-.8e^{jw}(e^{-j\frac{2\pi}{9}}-e^{j\frac{2\pi}{9}})+.64)(e^{2jw}-.8e^{jw}(e^{-j\frac{4\pi}{9}}+e^{j\frac{4\pi}{9}})+.64)}$$

→

In[118]:=

```
(*To TA, I could not simplify this any more, so I solved it
  using Mathematica, please see solution for b0 below *)

(*solution by Nasser Abbasi for HW 6, 152 *)

p1 = 0.8 Exp[I * 2 Pi / 9];
p2 = 0.8 Exp[- I * 2 Pi / 9];
p3 = 0.8 Exp[I * 4 Pi / 9];
p4 = 0.8 Exp[-I * 4 Pi / 9];
z1 = 1;
z2 = -1;
z3 = Exp[3 I Pi / 4];
z4 = Exp[- 3 I Pi / 4];
H[z_] := b0 (z - z1) (z - z2) (z - z3) (z - 4) / ((z - p1) (z - p2) (z - p3) (z - p4))
result = H[Exp[I w]];
Print["Result before substitution for w is=  ", result];

result = result /. w → 5 Pi / 12;
Print["Result before substitution for w is=  ", result];

gain = Solve[Abs[result] == 1, b0];

Print["Gain is = ", gain];
```

```
Result before substitution for w is=
 (b0 (-4 + e^(i w)) (-1 + e^(i w)) (1 + e^(i w)) (-e^(3 i π/4) + e^(i w))) / (((-0.612836 - 0.51423 i) + e^(i w))
   ((-0.612836 + 0.51423 i) + e^(i w)) ((-0.138919 - 0.787846 i) + e^(i w)) ((-0.138919 + 0.787846 i) + e^(i w)))

Result before substitution for w is=  (19.967 - 10.6848 i) b0

Solve::ifun : Inverse functions are being used by Solve, so some
   solutions may not be found; use Reduce for complete solution information. More...

Gain is = {{b0 → -0.0441577}, {b0 → 0.0441577}}
```

So use $\boxed{b_0 = 0.044}$

4

so $H(z) = 0.044 \dfrac{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}{(z-p_1)(z-p_2)(z-p_3)(z-p_4)}$

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HW6
4.84

$$y(n) = 0.9\, y(n-1) + 0.1\, x(n)$$

(a)
$$Y(z) = 0.9\, Y(z)\, z^{-1} + 0.1\, X(z)$$
$$Y(z)(1 - 0.9z^{-1}) = 0.1\, X(z)$$

$$\boxed{H(z) = \frac{0.1}{1-0.9z^{-1}}}$$

$$\boxed{H(w) = \frac{0.1}{1-0.9e^{-jw}}}$$

translation of frequency by $\frac{\pi}{2}$ is equivalent to multiplying the impulse response $h(n)$ by $e^{j\frac{\pi}{2}n}$.

so $h_{bp}(n) = e^{j\frac{\pi}{2}n}\, h_{Lp}(n)$

but $e^{j\frac{\pi}{2}n} = \cos\frac{\pi}{2}n + j\sin\frac{\pi}{2}n$

$n=0 \Rightarrow 1$
$n=1 \Rightarrow J$
$n=2 \Rightarrow -1$
$n=3 \Rightarrow -J$
$n=4 \Rightarrow 1$
$n=5 \Rightarrow J$
$n=6 \Rightarrow -1$
$n=7 \Rightarrow -J$
:

so multiply 2, 6, 10, 14, ... indexes by -1
multiply 1, 5, 9, 13, ... by J
multiply 3, 7, 11, 15, ... by $-J$

so $y(n) = 0.9\, y(n-1) + 0.1\, x(n)$

$K=1$ $\quad$ $K=0$

so $\boxed{y_{bp}(n) = J\, 0.9\, y(n-1) + 0.1\, x(n)}$

5

→

(b) To find impulse response.

let $x(n) = \delta(n)$.

so

$$y_{bp}(n) = J\ 0.9\, y(n-1) + 0.1\, \delta(n).$$

at $n=0$, $y(0) = J\ 0.9\, y(-1) + 0.1$, assume $y(-1) = 0$

$$\boxed{y(0) = 0.1}$$

$n=1$

$y(1) = (0.1)(0.9)J = \boxed{(0.1)(0.9)\,J}$

$y(2) = J\ 0.9\,(0.09J) = \boxed{(-1)(0.1)(0.9)(0.9)}$

$y(3) = (-J)\,(0.1)(0.9)(0.9)(0.9)$

$y(4) = (1)\,(0.1)\,(0.9)(0.9)(0.9)(0.9)$

$y(5) = J\,(0.1)\,(0.9)\,\ldots$

So $h(n) = (0.1)\,(0.9)^n\ \ g(n)$.

where $g(n) = \begin{cases} J & \text{for } n = 1, 5, 9, \ldots \\ -J & \text{for } n = 3, 7, 11, \ldots \end{cases}$

Can also write it as $\boxed{h(n) = (0.1)\,(e^{j\frac{\pi}{2}})^n\ (0.9)^n \text{ for } n \geq 0}$

(c) One problem I see is that the output of the filter is real only for even numbered samples.

$J, -1, -J, 1, J, -1, -J, 1, \ldots$
1 2 3 4 5 6 7 8

√ 5

HW 6
5.1

using property of DFT that for real valued sequence

$$X(N-k) = X^*(k) = X(-k).$$ 5.2.24.

here $N=8$. $X = \{ 0.25, 0.125 - j0.3018, 0, 0.125 - j0.0518, 0, ?, ?, ? \}$

so $X(5) = X(8-3) = X^*(3) = \boxed{0.125 + j0.0518}$

$X(6) = X(8-2) = X^*(2) = \boxed{0}$

$X(7) = X(8-1) = X^*(1) = \boxed{0.125 + j0.3018}$

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HW 6

5.7

$$x_c(n) = x(n) \cos \frac{2\pi k_0 n}{N}$$

$$x_s(n) = x(n) \sin \frac{2\pi k_0 n}{N}.$$

$$x_c(n) = x(n) \left(\frac{e^{j\frac{2\pi k_0 n}{N}} + e^{-j\frac{2\pi k_0 n}{N}}}{2} \right).$$

so $$X_c(k) = \frac{1}{2}\sum_{n=0}^{N-1} x(n) \left(e^{j\frac{2\pi k_0 n}{N}} + e^{-j\frac{2\pi k_0 n}{N}} \right) e^{-j\frac{2\pi k n}{N}}$$

$$= \frac{1}{2}\sum_{n=0}^{N-1} x(n) \left(e^{j\frac{2\pi k_0 n}{N} - j\frac{2\pi k n}{N}} + e^{-j\frac{2\pi k_0 n}{N} - j\frac{2\pi k n}{N}} \right)$$

$$= \frac{1}{2}\sum_{n=0}^{N-1} x(n) \left(e^{-j\frac{2\pi n}{N}(k_0-k)} + e^{-j\frac{2\pi n}{N}(k_0+k)} \right)$$

$$= \frac{1}{2}\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi n}{N}(k_0-k)} + \frac{1}{2}\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi n}{N}(k_0+k)}$$

$$= \frac{1}{2}\sum_{n=0}^{N-1} \underbrace{x(n)\, e^{j\frac{2\pi n}{N}k_0}}\, e^{-j\frac{2\pi k n}{N}} + \frac{1}{2}\sum_{n=0}^{N-1} \underbrace{x(n)\, e^{-j\frac{2\pi n}{N}k_0}}\, e^{-j\frac{2\pi n k}{N}}$$

$$= \frac{1}{2}\, DFT\left(x(n) e^{j\frac{2\pi n}{N}k_0}\right) + \frac{1}{2}\, DFT\left(x(n) e^{-j\frac{2\pi n}{N}k_0}\right)$$

but from Table 5.2, we see that

$$DFT\left(x(n) e^{j\frac{2\pi n}{N}k_0}\right) = X((k-k_0))_N$$

and $$DFT\left(x(n) e^{-j\frac{2\pi n}{N}k_0}\right) = X((k+k_0))_N$$

So $$\boxed{X_c(k) = \frac{1}{2} X((k-k_0))_N + \frac{1}{2} X((k+k_0))_N}$$

$$x_s(n) = x(n)\sin\frac{2\pi k_0 n}{N}$$

$$= x(n)\left(\frac{e^{j\frac{2\pi k_0 n}{N}} - e^{-j\frac{2\pi k_0 n}{N}}}{2j}\right)$$

$$x_s(n) = \frac{1}{2j}\,x(n)\,e^{j\frac{2\pi k_0 n}{N}} - \frac{1}{2j}\,x(n)\,e^{-j\frac{2\pi k_0 n}{N}}$$

$$\mathcal{X}(x_s(n)) = \frac{1}{2j}\left(\sum_{n=0}^{N-1} x(n)\,\underbrace{e^{j\frac{2\pi k_0 n}{N}}\,e^{-j\frac{2\pi k n}{N}}}\right) - \frac{1}{2j}\sum_{n=0}^{N-1} x(n)\,\underbrace{e^{-j\frac{2\pi k_0 n}{N}}\,e^{-j\frac{2\pi k n}{N}}}$$

$$\boxed{\mathcal{X}(x_s(n)) = \frac{1}{2j}\,\mathcal{X}((k-k_0))_N - \frac{1}{2j}\,\mathcal{X}((k+k_0))_N}$$

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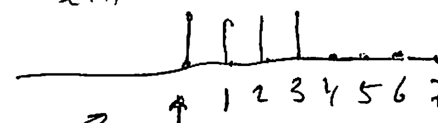
HW 6
5.11

$$x(n) = \begin{cases} 1 & 0 \le n \le 3 \\ 0 & 4 \le n \le 7 \end{cases}$$

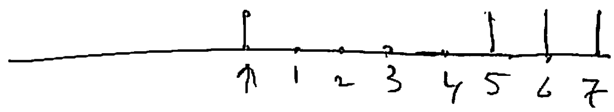
$x(n)$



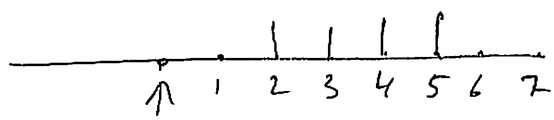
has $X(k)$ as its DFT

compute DFT of

(a) $X_1(n)$



(b) $X_2(n)$



(a) use the circular shift property in time.

we see that $x_1(n)$ is same as $x(n)$, when $x(n)$ is circularly shifted to right by 5 units.

so $x_1(n) = x((n-5))_8$

but circular shift in time means multiply X by $e^{-j\frac{2\pi k}{N}\ell}$ amount of circular shift

so $\boxed{X_1(k) = X(k)\, e^{-j\frac{2\pi k}{8}5}}$

(b) we see that $x_2(n)$ is same as $x(n)$ when $x(n)$ is circularly shifted to right by 2 units. so

$$\boxed{X_2(k) = X(k)\, e^{-j\frac{2\pi k}{8}2}}$$

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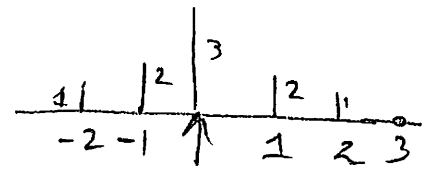
HW6

5.25 (a)

a) Find. Fourier Transform of $X(\omega)$ $x(n)$



b) compute 6 points DFT $V(k)$ of $v(n) = \{3, 2, 1, 0, 1, 2\}$.

(c) any relation between $X(\omega)$ and $V(k)$?

(a) from definition of $X(\omega)$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}.$$

$$= x(-2)e^{-j\omega(-2)} + x(-1)e^{-j\omega(-1)} + x(0)e^{0} + x(1)e^{-j\omega} + x(2)e^{-j\omega 2} + x(3)e^{-j\omega 3}$$

$$= e^{2j\omega} + 2e^{j\omega} + 3 + 2e^{-j\omega} + 1e^{-2j\omega}$$

$$\boxed{X(\omega) = 2\cos 2\omega + 4\cos\omega + 3}$$

(b) $$DFT = \sum_{n=0}^{N-1} v(n)\, e^{-j2\pi\frac{k}{N}n}$$

$$= \sum_{0}^{5} v(n) e^{-j2\pi\frac{k}{N}n} = v(0)e^{0} + v(1)e^{-j2\pi\frac{k}{6}} + v(2)e^{-j\frac{2\pi k 2}{N}} + v(3)e^{-j\frac{2\pi k}{N}3} + v(4)e^{-j\frac{2\pi k 4}{N}} + v(5)e^{-j\frac{2\pi k 5}{N}}$$

$$= 3 + 2e^{-j\frac{\pi k}{3}} + 1e^{-j\frac{4\pi k}{N}} + 0 + 1e^{-j\frac{2\pi}{N}4} + 2e^{-j\frac{2\pi k 5}{N}}$$

$$= 3 + 2e^{-j\frac{\pi k}{3}} + e^{-j\frac{2}{3}\pi k} + e^{-j\frac{4}{3}\pi} + 2e^{-j\frac{5}{3}\pi k} \rightarrow$$

but $e^{-j\frac{\pi}{3}} = e^{-j\frac{5\pi}{3}}$, $e^{-\frac{2}{3}\pi} = e^{-\frac{4}{3}}$ $\Rightarrow \boxed{3 + 4\cos\frac{\pi k}{3} + 2\cos\frac{2}{3}\pi k}$

compare $X(\omega) = 2\cos 2\omega + 4\cos\omega + 3$

with $DFT(v(n)) = 2\cos\frac{2\pi k}{3} + 4\cos\frac{\pi k}{3} + 3$

$\Rightarrow$ when $\boxed{\omega = \frac{\pi k}{3}}$ they are the same.

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