Initial Goals:

- Understand the mathematics of HYPR and related algorithms (Wright HYPR, First Iteration of I-HYPR).
- Study their mathematical relation to the MLEM algorithm.
- Develop and validate a MATLAB tool to explore the properties of the reconstruction algorithms.
- Explore illustrative examples of where these algorithms work well and where they fail with a time-resolved angiography application in mind.

Data Acquisition:

$$s_t = R_{f_t}[I_t]$$

t : Time

ϕ_t : Angle of projection at time t

s_t : Projection at time t

I_t : True image at time t

R_{ϕ_t} : Radon iransform for angle ϕ_t

Image Reconstruction:

$$J_t = C\left[\frac{1}{N_p}\sum_{i=1}^{N_p}\frac{R^u_{\phi_{t_i}}[s_{t_i}]}{R^u_{\phi_{t_i}}[s_{c_{t_i}}]}\right]$$

C : Composite image

N_p : Number of projections per time frame

S_{t_i} : Projection data at time t_i

$S_{c_{t_i}}$: Projection of composite image at time t_i

R^u_ϕ : Transpose of Radon transform (unfiltered backprojection)

J_t : Reconstructed image

MLEM Algorithm:

Let $g = Hf + \varepsilon$

H : Matrix Projection Operator

f: Discrete Object

g: Projection

ε: Poisson Noise

MLEM maximizes the likelihood that g came from f.

MLEM Algorithm: $f_n^{(k+1)} = \left(\frac{f_n^{(k)}}{s_n} \right) \sum_m \left(\frac{g_m}{\left(Hf^{(k)} \right)_m} H_{mn} \right)$ *where* $s_n = \sum_m H_{mn}$

Using Matrix Notation Unfiltered Backprojection is H^T :

$$\Rightarrow \quad f_n^{(k+1)} = f_n^{(k)} \frac{1}{s_n} \left(H^T \left[\frac{g}{Hf^{(k)}} \right] \right)_n$$

**Notation adopted from Foundations of Image Science, by Barrett and Myers*

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Comparison of MLEM & HYPR

MLEM-1 Algorithm

$$g = Hf + \varepsilon$$

$$f_n^{(1)} = f_n^{(0)} \frac{1}{z_n}\left[H^{\mathrm{T}}\left[\frac{g}{(Hf^{(0)})} \right] \right]_n$$

HYPR

$$s_t = R_{\phi_t}\left[I_t \right]$$

$$J_n = \frac{1}{N_p} C_n \cdot R_\phi^u \left(\frac{s}{R_\phi(C)} \right)_n$$

H ..... R_ϕ – *Radon Transform*

H^T ..... R_ϕ^u – *Unfiltered backprojection*

z_n ..... N_p – *Projections per time frame*

g ..... s – *Original projection*

$f^{(0)}$ ..... C – *Composite image*

For this method to match the original HYPR in the first iteration we need that

$$\frac{R_\phi^u(s)}{R_\phi^u(s_c)} = R_\phi^u\left(\frac{s}{s_c} \right)$$

The ratio of unfiltered backprojections is the unfiltered backprojection of the ratio.

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Relevant Properties of MLEM

- Multiplicative update on each iteration, so if the initial estimate is zero, subsequent estimates remain zero. This property reduces streaking artifacts by using the composite image as the initial guess.

- Enforces non-negativity constraint. If initial estimate is positive and H has non-negative entries, future updates remain non-negative.

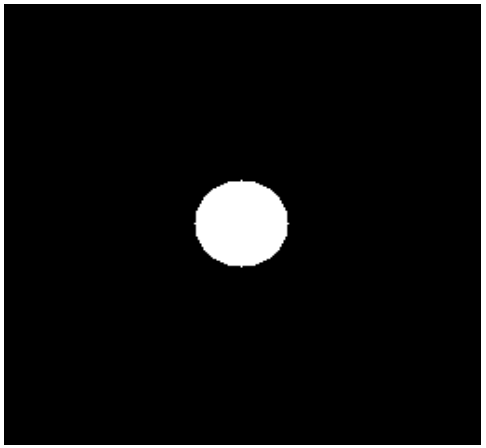
- Non-linear and iterative: while hugely popular in the research community, adoption in clinical nuclear medicine was slow because of unpredictable nature of artifacts. This may be something to discuss with clinical collaborators.

- Noise properties for time resolved MRA very different than in nuclear medicine where the major source of noise is the Poisson noise in the projections.

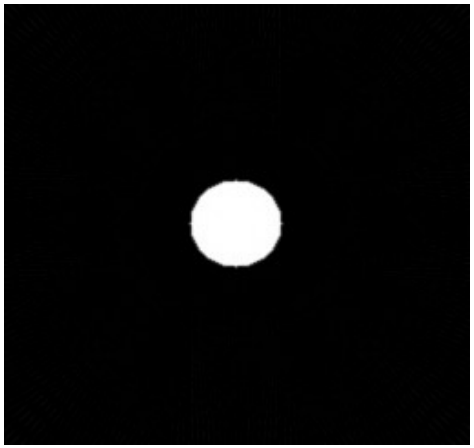
Computational Comparison of HYPR, HYPR-W & MLEM-1

- In the following slide we compare Original HYPR to 1-step MLEM algorithm
- Time-invariant disk used
- 128 projection angles used (bit-reversed ordering)
- Window size: 8 projections
- Also implemented HYPR-W (Huang and Wright)
- For a stationary disk, all methods give a similar result, but the MLEM implementation is slightly better.

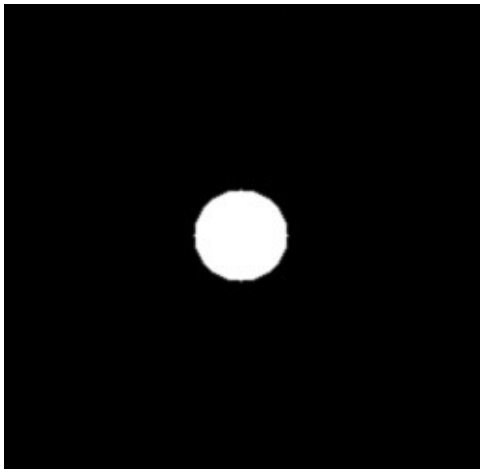
actual image

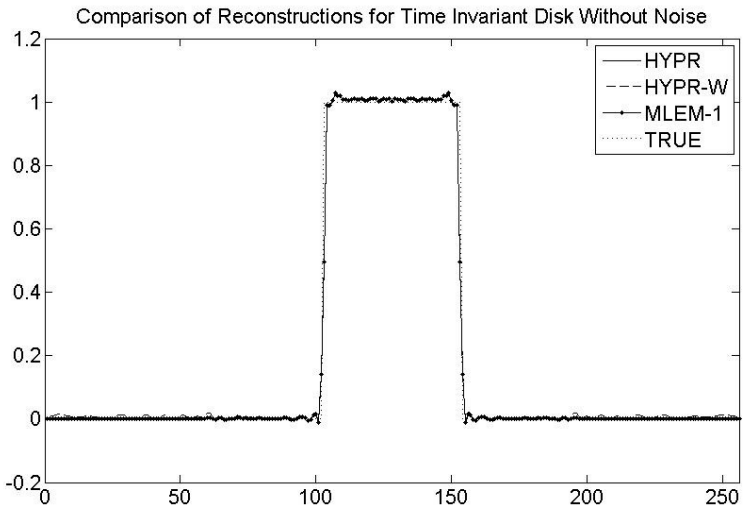


composite image

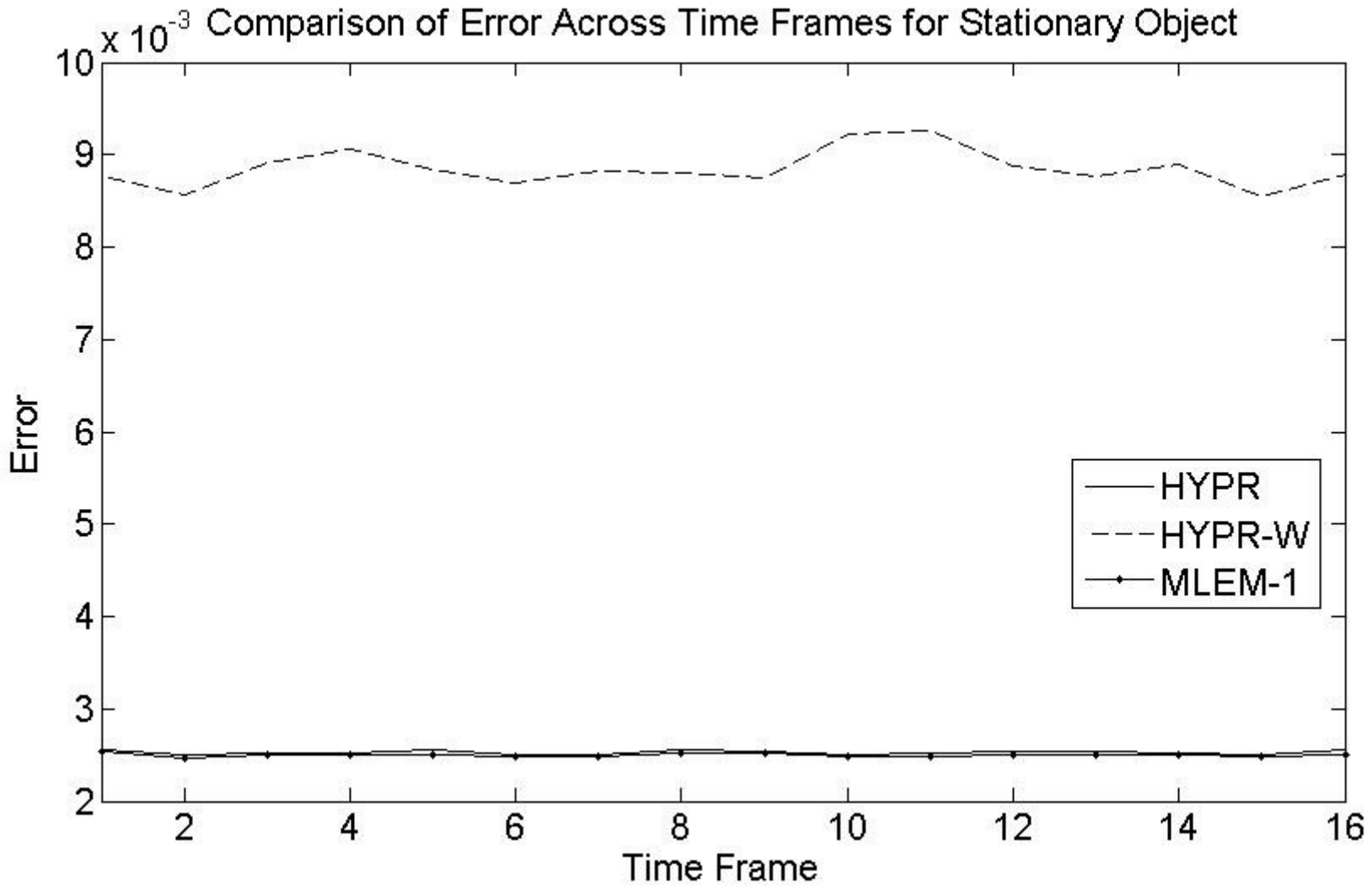


HYPR Reconstruction for Projections 1-8



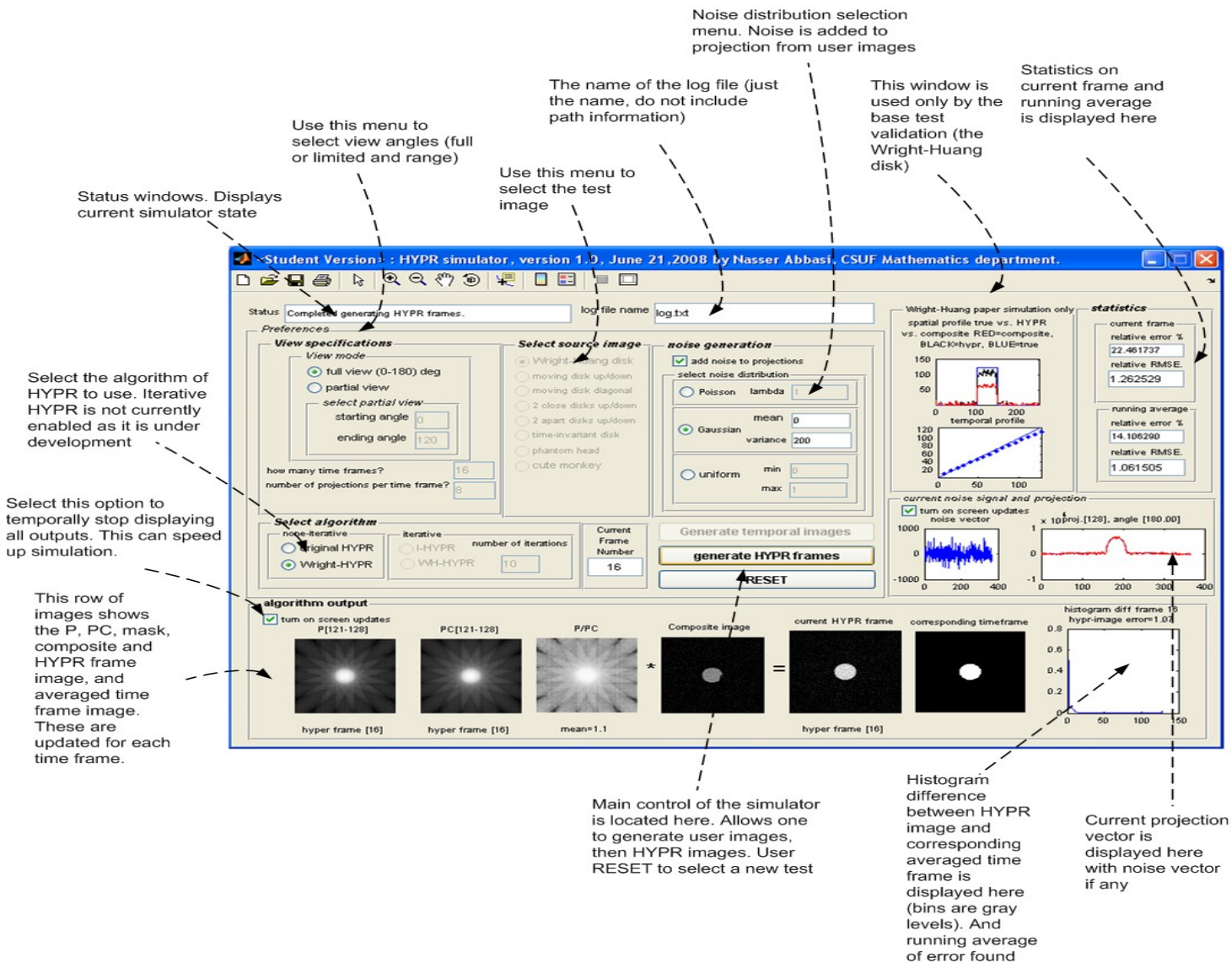


For a stationary disk, with no noise, all methods are similar.

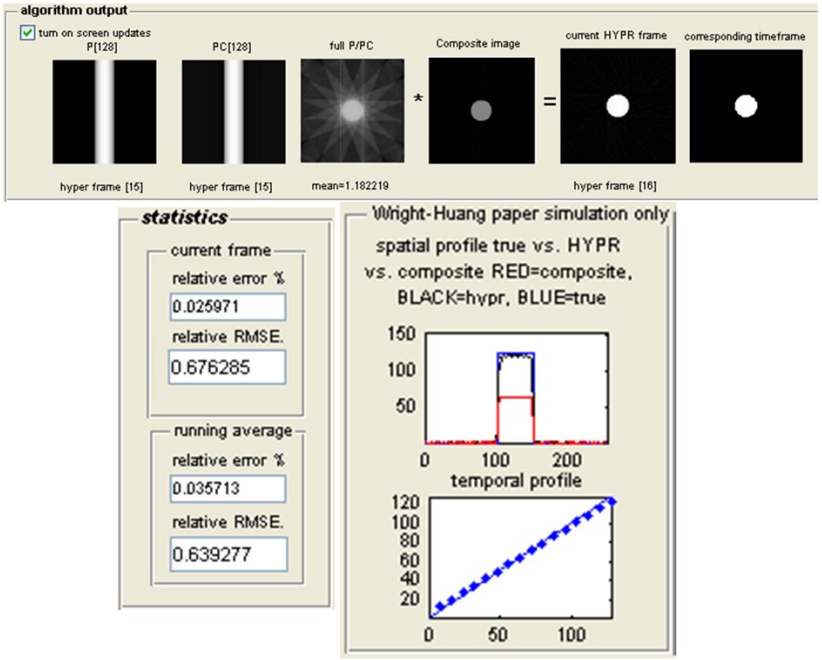


MLEM-1 and HYPR are the same method with different implementation.

MATLAB Computational Workbench



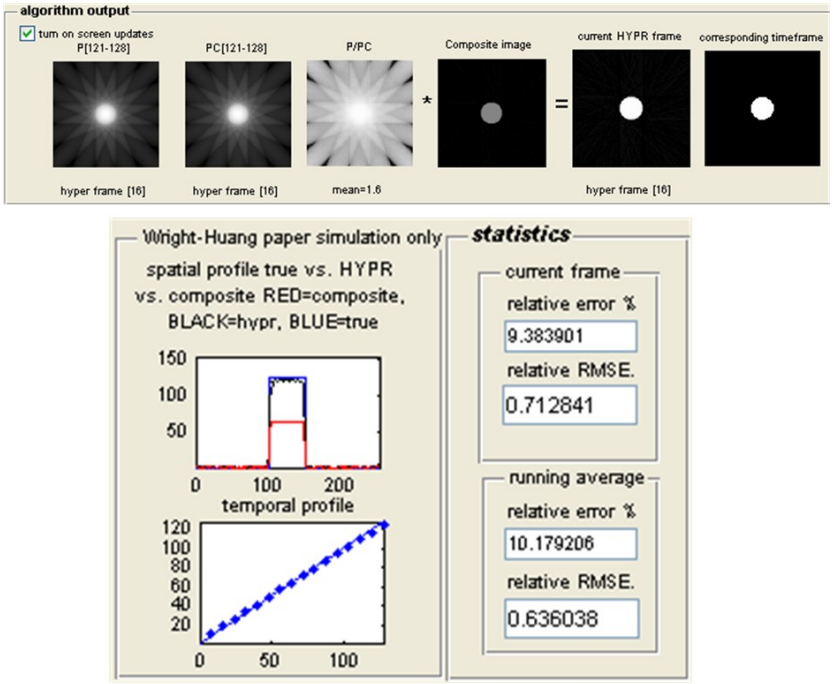
Time Dependent Intensity with Stationary Boundary




```
generated data for simulation [wrightPaperDiskTag]
Completed image generation...
number of time frames = [16]
number of projections per time frame = [8]
[22-Jun-2008 13:56:42]Enter generate HYPR ...

Enter original HYPR ...
No NOISE is being added
frame rmse  relativeError     rmse_averaged     relativeError_averaged  averageMask
1     1.026815    0.124615    1.026815    0.124615    0.810999
2     0.789398    0.080273    0.908107    0.102444    0.834959
3     0.658115    0.049115    0.824776    0.084668    0.856301
4     0.579282    0.057853    0.763403    0.077964    0.888973
5     0.551421    0.015053    0.721006    0.065382    0.905486
6     0.539332    0.016687    0.690727    0.057266    0.934073
7     0.544661    0.011765    0.669861    0.050766    0.957154
8     0.555759    0.021502    0.655598    0.047108    0.988724
9     0.565009    0.019138    0.645533    0.044000    1.005649
10    0.582615    0.021568    0.639241    0.041757    1.030899
11    0.600318    0.024314    0.635702    0.040171    1.053254
12    0.618896    0.016745    0.634302    0.038219    1.085530
13    0.629853    0.033764    0.633959    0.037876    1.101912
14    0.649074    0.025293    0.635039    0.036977    1.132088
15    0.661592    0.027750    0.636809    0.036362    1.153034
16    0.676285    0.025971    0.639277    0.035713    1.182219
Done,  totalHistError =1.33 ...
```

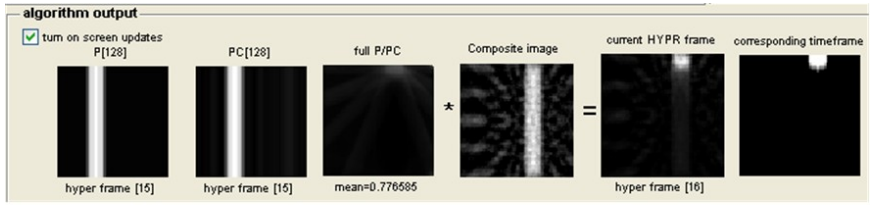
Running the Wright-Huang test case using the original HYPR algorithm




```
Enter Wright-Huang HYPR ...
No NOISE is being added
frame rmse        relErr      MeanRmse    MeanRelErr  MeanMask
1     0.982878    38.035191   0.982878    38.035191   0.396131
2     0.734453    26.583094   0.858665    32.309142   0.483218
3     0.604836    19.085338   0.774055    27.901208   0.557789
4     0.532656    14.054107   0.713705    24.439433   0.639344
5     0.520571    9.217883    0.675079    21.395123   0.715395
6     0.519005    5.920670    0.649066    18.816047   0.798848
7     0.535644    3.097390    0.632863    16.570525   0.877456
8     0.556542    0.955617    0.623323    14.618661   0.958114
9     0.574512    1.083784    0.617900    13.114786   1.035286
10    0.599367    2.818749    0.616046    12.085182   1.114582
11    0.621385    4.294862    0.616532    11.376971   1.193476
12    0.645831    5.560250    0.618973    10.892245   1.277558
13    0.658385    6.633323    0.622005    10.564635   1.353904
14    0.682260    7.683917    0.626309    10.358870   1.434595
15    0.695441    8.459228    0.630918    10.232227   1.508504
16    0.712841    9.383901    0.636038    10.179206   1.596524
Done, totalHistError =1.35 ...
```

Test 1b results

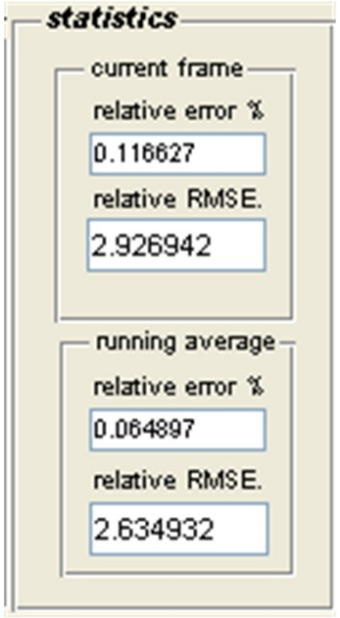
Time Dependent Boundary with Constant Intensity

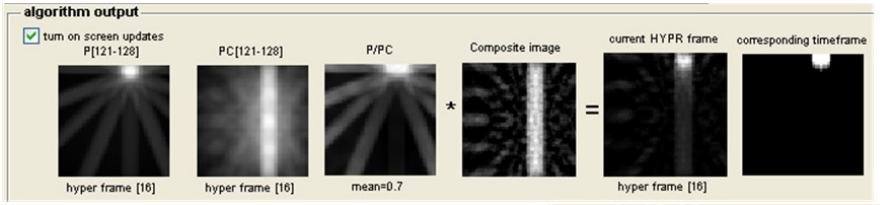



```
generated data for simulation [movingDiskUpDownTag]
Completed image generation...
number of time frames = [16]
number of projections per time frame = [8]
[22-Jun-2008 16:03:26]Enter generate HYPR ...


Enter original HYPR ...
No NOISE is being added
frame rmse       relErr     MeanRmse   MeanRelErr MeanMask
1     2.930777   0.112860   2.930777   0.112860   0.776031
2     2.283642   0.032046   2.607209   0.072453   0.870046
3     2.468333   0.049740   2.560917   0.064882   0.837949
4     2.615779   0.060302   2.574633   0.063737   0.833535
5     2.693877   0.087963   2.598481   0.068582   0.829808
6     2.653286   0.004072   2.607615   0.057831   0.846449
7     2.686605   0.044205   2.618900   0.055884   0.846036
8     2.745714   0.132117   2.634751   0.065413   0.833106
9     2.756394   0.134185   2.648267   0.073055   0.833394
10    2.683672   0.051199   2.651808   0.070869   0.846576
11    2.643038   0.005699   2.651011   0.064944   0.846128
12    2.694688   0.089126   2.654650   0.066960   0.831088
13    2.621380   0.046047   2.652091   0.065351   0.833980
14    2.467277   0.046980   2.638890   0.064039   0.838185
15    2.287501   0.025182   2.615464   0.061448   0.869622
16    2.926942   0.116627   2.634932   0.064897   0.776585
Done,  totalHistError =1.16 ...
```

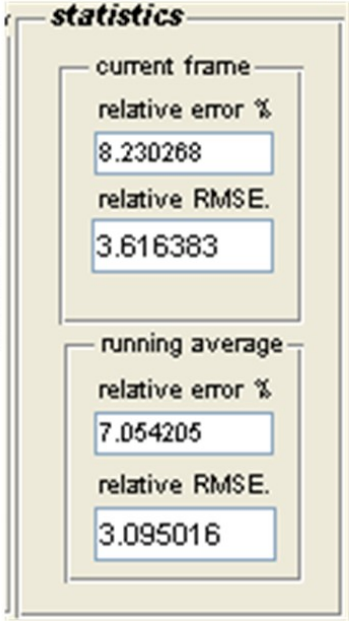
Test 5a



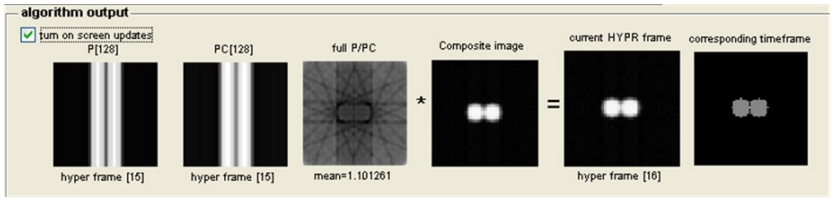


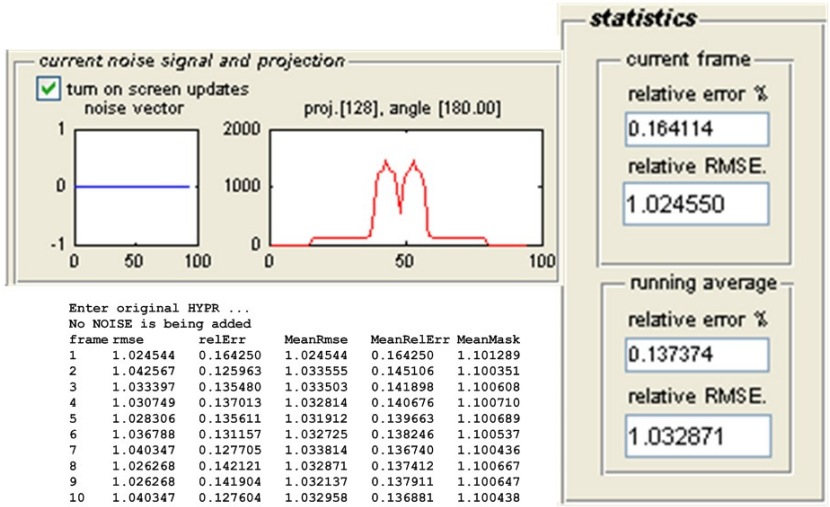

```
Enter Wright-Huang HYPR ...
No NOISE is being added
frame rmse       relErr     MeanRmse   MeanRelErr MeanMask
1     3.621499   8.250962   3.621499   8.250962   0.683032
2     2.683754   2.802784   3.152626   5.526873   0.845110
3     2.913513   3.013436   3.072922   4.689061   0.876864
4     3.029160   6.509401   3.061982   5.144146   0.909015
5     3.119542   8.525960   3.073494   5.820509   0.913877
6     3.110399   7.673412   3.079645   6.129326   0.926160
7     3.120744   8.473927   3.085516   6.464269   0.938696
8     3.161076   11.227506  3.094961   7.059673   0.927934
9     3.169451   11.224866  3.103238   7.522473   0.927989
10    3.118828   8.468160   3.104797   7.617041   0.938733
11    3.101359   7.706259   3.104484   7.625152   0.926111
12    3.120562   8.506358   3.105824   7.698586   0.913934
13    3.035838   6.467908   3.100440   7.603918   0.909125
14    2.913264   3.004449   3.087071   7.275385   0.876923
15    2.684876   2.781627   3.060258   6.975801   0.845124
16    3.616383   8.230268   3.095016   7.054205   0.682970
Done, totalHistError =1.20 ...
```

Test 5b



Two Vessels



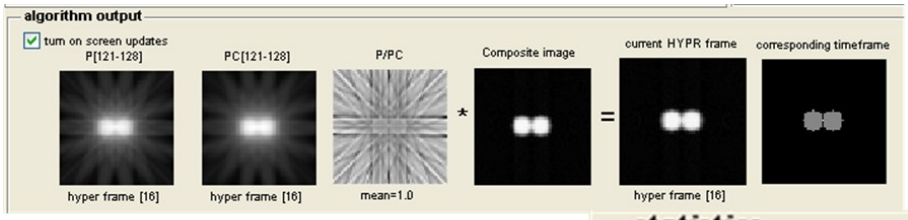


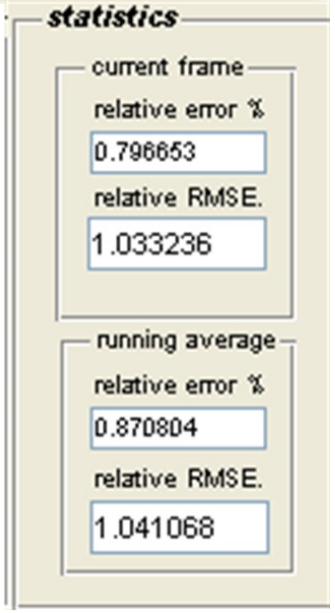
Enter original HYPR ...
No NOISE is being added

frame	rmse	relErr	MeanRmse	MeanRelErr	MeanMask
1	1.024544	0.164250	1.024544	0.164250	1.101289
2	1.042567	0.125963	1.033555	0.145106	1.100351
3	1.033397	0.135480	1.033503	0.141898	1.100608
4	1.030749	0.137013	1.032814	0.140676	1.100710
5	1.028306	0.135611	1.031912	0.139663	1.100689
6	1.036788	0.131157	1.032725	0.138246	1.100537
7	1.040347	0.127705	1.033814	0.136740	1.100436
8	1.026268	0.142121	1.032871	0.137412	1.100667
9	1.026268	0.141904	1.032137	0.137911	1.100647
10	1.040347	0.127604	1.032958	0.136881	1.100438
11	1.036787	0.131156	1.033306	0.136360	1.100511
12	1.028305	0.135576	1.032889	0.136295	1.100685
13	1.030749	0.136989	1.032725	0.136348	1.100705
14	1.033397	0.135413	1.032773	0.136282	1.100645
15	1.042575	0.125936	1.033426	0.135592	1.100335
16	1.024550	0.164114	1.032871	0.137374	1.101261

Done, totalHistError =1.72 ...

Test 3a





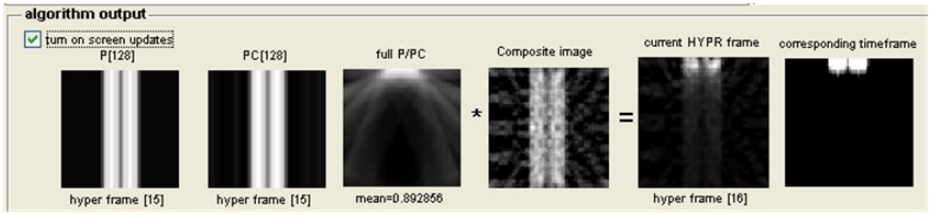
Enter Wright-Huang HYPR ...
No NOISE is being added

frame	rmse	relErr	MeanRmse	MeanRelErr	MeanMask
1	1.033230	0.796456	1.033230	0.796456	1.038977
2	1.048957	0.887767	1.041093	0.842111	1.037790
3	1.041920	0.884762	1.041369	0.856328	1.038249
4	1.039824	0.887151	1.040983	0.864034	1.038365
5	1.037571	0.882855	1.040300	0.867798	1.038507
6	1.044559	0.885161	1.041010	0.870692	1.038137
7	1.047245	0.892601	1.041901	0.873822	1.037962
8	1.035236	0.849602	1.041068	0.870794	1.038600
9	1.035236	0.849525	1.040420	0.868431	1.038602
10	1.047245	0.892414	1.041102	0.870829	1.037970
11	1.044558	0.885209	1.041416	0.872137	1.038135
12	1.037571	0.882776	1.041096	0.873023	1.038515
13	1.039824	0.887281	1.040998	0.874120	1.038361
14	1.041921	0.884700	1.041064	0.874876	1.038251
15	1.048960	0.887956	1.041590	0.875748	1.037768
16	1.033236	0.796653	1.041068	0.870804	1.038961

Done, totalHistError =1.69 ...

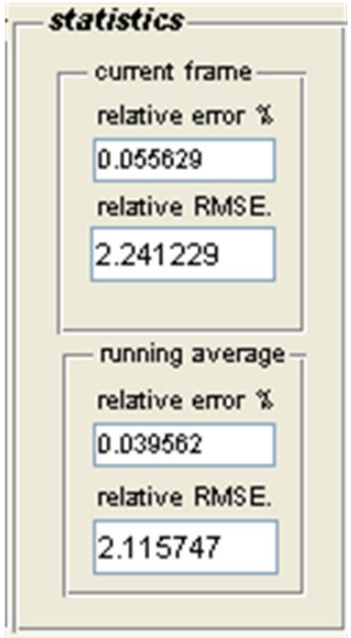
Test 3b

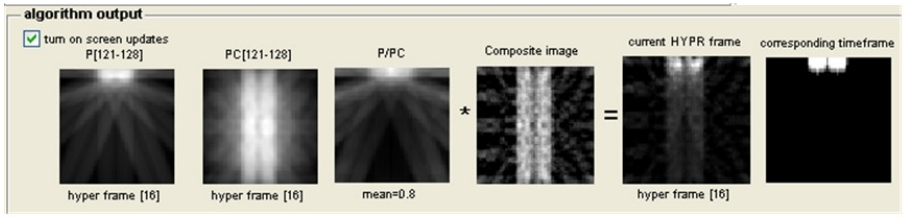
Two Moving Vessels




```
Enter original HYPR ...
No NOISE is being added
frame rmse       relErr     MeanRmse   MeanRelErr MeanMask
1     2.239817   0.032813   2.239817   0.032813   0.893149
2     1.870331   0.037924   2.055074   0.035368   1.007503
3     2.042381   0.041574   2.050843   0.037437   0.964871
4     2.115184   0.049842   2.066928   0.040538   0.949393
5     2.130986   0.048243   2.079740   0.042079   0.944004
6     2.169312   0.009831   2.094668   0.036704   0.950613
7     2.195184   0.017919   2.109028   0.034021   0.948241
8     2.161797   0.065252   2.115624   0.037925   0.945542
9     2.163640   0.065947   2.120959   0.041038   0.945540
10    2.194526   0.015931   2.128316   0.038528   0.947925
11    2.168012   0.014524   2.131924   0.036345   0.950493
12    2.131513   0.049517   2.131890   0.037443   0.944035
13    2.111168   0.048591   2.130296   0.038301   0.949333
14    2.039746   0.039543   2.123828   0.038389   0.964977
15    1.877134   0.039910   2.107382   0.038491   1.007446
16    2.241229   0.055629   2.115747   0.039562   0.892856
Done,  totalHistError =1.49 ...
```

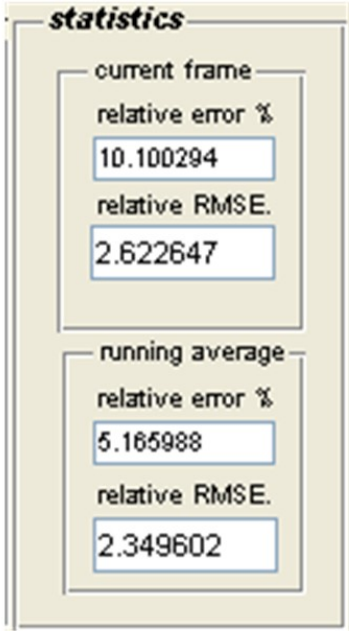
Test 7a






```
Enter Wright-Huang HYPR ...
No NOISE is being added
frame rmse       relErr     MeanRmse   MeanRelErr MeanMask
1     2.621347   10.092964  2.621347   10.092964  0.754539
2     2.129956   4.202134   2.375651   7.147549   0.945006
3     2.243854   0.957437   2.331719   5.084178   0.985303
4     2.308804   3.208890   2.325990   4.615356   1.001461
5     2.334758   4.696902   2.327744   4.631665   1.014916
6     2.377751   5.393613   2.336078   4.758657   1.031932
7     2.396936   5.917429   2.344772   4.924196   1.035065
8     2.383122   6.856131   2.349566   5.165687   1.039493
9     2.384498   6.856367   2.353447   5.353541   1.039498
10    2.395278   5.938221   2.357630   5.412009   1.035004
11    2.377021   5.409679   2.359393   5.411797   1.031946
12    2.335582   4.681225   2.357409   5.350916   1.014995
13    2.305090   3.203002   2.353384   5.185692   1.001508
14    2.241690   0.937821   2.345406   4.882272   0.985397
15    2.135301   4.203699   2.331399   4.837034   0.945082
16    2.622647   10.100294  2.349602   5.165988   0.754457
Done, totalHistError =1.50 ...
```

Test 7b



Preliminary Results

- Identification of the mathematical structure of HYPR
- Verification of HYPR as the first step of MLEM
- Validation of HYPR Computational Workbench
- HYPR surprisingly robust to vessel motion
- Comparison with HYPR-W

Our Intended Direction

- Validation of Computational Tools.
- Characterize the noise amplification and resolution of the HYPR algorithm through simulations and analytical approximations.
- Test on clinically relevant objects (models of occluded arteries, vessels with different time uptake characteristics).
- Comparisons to alternative algorithms (I-HYPR, HYPR-LR, positively constrained least-squares, Level Set Methods).