HW 9 Mathematics 503, Mathematical Modeling, CSUF , July 16, 2007

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problem:

Consider the problem of minimizing the functional $J(u)=\int_{\Omega} L(\mathbf{x},u,\nabla u)\,d\mathbf{x}$ over all $u\in C^2(\Omega)$ with $u(\mathbf{x})=f(\mathbf{x})$ at boundary Γ where f is a given function. Ω is bounded and well behaved in $\mathbb{R}^2$.

(a) Show that the first variation is (Where L below is meant to be $L(\mathbf{x},u,\nabla u)$) where $\mathbf{x}$ is the vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

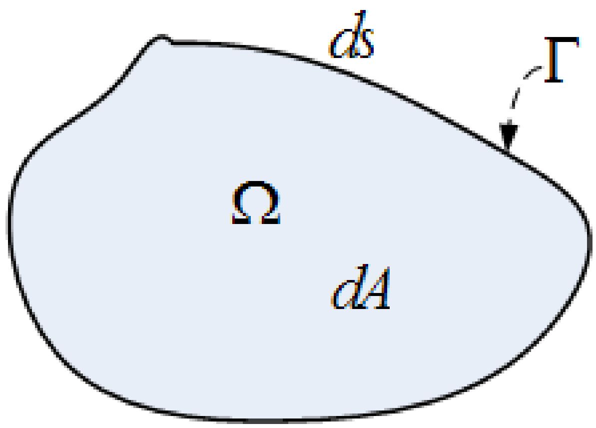
$$
\begin{aligned}
\delta J(u,\mathbf{h}) &= \int_{\Omega} L_u h + L_{\nabla u}\cdot\nabla h \quad dA \\
&= \int_{\Omega} (L_u - \nabla\cdot L_{\nabla u})\, h \;\; dA - \int_{\Gamma} h L_{\nabla u}\cdot\mathbf{n}\; ds
\end{aligned}
$$

Where $L_{\nabla u}$ is the vector $\begin{bmatrix} L_{\left(\frac{\partial u}{\partial x_1}\right)} \\ L_{\left(\frac{\partial u}{\partial x_2}\right)} \end{bmatrix} = \begin{bmatrix} \frac{\partial L}{\partial\left(\frac{\partial u}{\partial x_1}\right)} \\ \frac{\partial L}{\partial\left(\frac{\partial u}{\partial x_2}\right)} \end{bmatrix}$ and $\mathbf{h}\in C^2(\Omega)$ with $h(\mathbf{x})=\mathbf{0}$ at the boundary Γ

(b)Show that the necessary condition for u to minimize J is that u must satisfy the Euler equation $L_u - \nabla\cdot L_{\nabla u} = 0$, $\mathbf{x}\in\Omega$

(c)If u is not fixed on the boundary Γ find the natural boundary conditions.

Answer



(a)

$$\begin{aligned} J(u) &= \int_{\Omega} L(\mathbf{x}, u, \nabla u)\, dA \\ J(u+th) &= \int_{\Omega} L(\mathbf{x}, u+th, \nabla(u+th))\, dA \\ &= \int_{\Omega} L(\mathbf{x}, u+th, \nabla u + t\nabla h)\ dA \end{aligned}$$

Hence

$$\begin{aligned} \frac{dJ(u+th)}{dt} &= \frac{d}{dt}\int_{\Omega} L(\mathbf{x}, u+th, \nabla u + t\nabla h)\ d\mathbf{x} \\ &= \int_{\Omega} \frac{\partial L}{\partial (u+th)} h + \left(\frac{\partial L}{\partial (\nabla u + t\nabla h)} \cdot \nabla h\right)\ d\mathbf{x} \end{aligned}$$

But $\delta J(u, \mathbf{h}) = \lim_{t\to 0} \frac{dJ(u+th)}{dt}$, hence at $t=0$ the above becomes

$$\delta J(u, \mathbf{h}) = \int_{\Omega} \frac{\partial L}{\partial u} h + \left(\frac{\partial L}{\partial (\nabla u)} \cdot \nabla h\right)\ dA \tag{1}$$

But $\nabla u = \begin{bmatrix} \frac{\partial u}{\partial x_1} \\ \frac{\partial u}{\partial x_2} \end{bmatrix}$, hence $\frac{\partial L}{\partial(\nabla u)} = \begin{bmatrix} \frac{\partial L}{\partial\left(\frac{\partial u}{\partial x_1}\right)} \\ \frac{\partial L}{\partial\left(\frac{\partial u}{\partial x_2}\right)} \end{bmatrix}$, and $\nabla h = \begin{bmatrix} \frac{\partial h}{\partial x_1} \\ \frac{\partial h}{\partial x_2} \end{bmatrix}$, therefore

$$
\begin{aligned}
\frac{\partial L}{\partial\left(\nabla u\right)}\cdot\nabla h &= \begin{bmatrix} \frac{\partial L}{\partial\left(\frac{\partial u}{\partial x_1}\right)} \\ \frac{\partial L}{\partial\left(\frac{\partial u}{\partial x_2}\right)} \end{bmatrix}^T \begin{bmatrix} \frac{\partial h}{\partial x_1} \\ \frac{\partial h}{\partial x_2} \end{bmatrix} \\
&= \frac{\partial L}{\partial\left(\frac{\partial u}{\partial x_1}\right)}\frac{\partial h}{\partial x_1} + \frac{\partial L}{\partial\left(\frac{\partial u}{\partial x_2}\right)}\frac{\partial h}{\partial x_2} \\
&= L_{u_{x_1}}h_{x_1} + L_{u_{x_2}}h_{x_2}
\end{aligned}
$$

Hence (1) becomes

$$
\delta J(u,\mathbf{h}) = \int\limits_{\Omega} L_u h + \left(L_{u_{x_1}}h_{x_1} + L_{u_{x_2}}h_{x_2}\right) \ \ dA \tag{2}
$$

Now

$$
\frac{\partial}{\partial x_i}\left(L_{u_{x_i}}h\right) = \frac{\partial L_{u_{x_i}}}{\partial x_i}h + L_{u_{x_i}}h_{x_i}
$$

Hence

$$
L_{u_{x_i}}h_{x_i} = \frac{\partial}{\partial x_i}\left(L_{u_{x_i}}h\right) - \frac{\partial L_{u_{x_i}}}{\partial x_i}h
$$

Hence substitute the above in (2) for $i = 1, 2$ we obtain

$$
\begin{aligned}
\delta J(u,\mathbf{h}) &= \int\limits_{\Omega} L_u h + \left(\frac{\partial}{\partial x_1}\left(L_{u_{x_1}}h\right) - \frac{\partial L_{u_{x_1}}}{\partial x_1}h + \frac{\partial}{\partial x_2}\left(L_{u_{x_2}}h\right) - \frac{\partial L_{u_{x_2}}}{\partial x_2}h\right) \ \ dA \\
&= \int\limits_{\Omega}\left(L_u - \frac{\partial L_{u_{x_1}}}{\partial x_1} - \frac{\partial L_{u_{x_2}}}{\partial x_2}\right) h \ dA + \int\limits_{\Omega}\left(\frac{\partial}{\partial x_1}L_{u_{x_1}} + \frac{\partial}{\partial x_2}L_{u_{x_2}}\right) h \ \ dA
\end{aligned} \tag{3}
$$

Now using Green theorem, where

$$
\int\limits_{\Omega}\left(\frac{\partial Q}{\partial x_1} - \frac{\partial P}{\partial x_2}\right) dx_1 dx_2 = \int_{\Gamma} P dx_1 + Q dx_2
$$

Let $Q \equiv L_{u_{x_1}}h, P \equiv -L_{u_{x_2}}h$, hence Green theorem becomes

$$
\int\limits_{\Omega}\left(\frac{\partial}{\partial x_1}L_{u_{x_1}} + \frac{\partial}{\partial x_2}L_{u_{x_2}}\right) h \ dx_1 dx_2 = \int_{\Gamma}\left(-L_{u_{x_2}}dx_1 + L_{u_{x_1}}dx_2\right) h
$$

Substitute the above into second term in (3) we obtain (noting that $dA = dx_1 dx_2$ since we are in $\mathbb{R}^2$)

$$\delta J(u,\mathbf{h}) = \int_\Omega \left(L_u - \frac{\partial L_{u_{x_1}}}{\partial x_1} - \frac{\partial L_{u_{x_2}}}{\partial x_2}\right) h \ dA + \int_\Gamma \left(L_{u_{x_1}} dx_2 - L_{u_{x_2}} dx_1\right) h \tag{4}$$

But the second integral above can be rewritten as (by dividing and multiplying by ds)

$$\int_\Gamma \left(L_{u_{x_1}} dx_2 - L_{u_{x_2}} dx_1\right) h \equiv \int_\Gamma \left(L_{u_{x_1}} \frac{dx_2}{ds} - L_{u_{x_2}} \frac{dx_1}{ds}\right) h \ \ ds$$

Hence (4) becomes

$$\delta J(u,\mathbf{h}) = \int_\Omega \left(L_u - \frac{\partial L_{u_{x_1}}}{\partial x_1} - \frac{\partial L_{u_{x_2}}}{\partial x_2}\right) h \ dA + \int_\Gamma \left(L_{u_{x_1}} \frac{dx_2}{ds} - L_{u_{x_2}} \frac{dx_1}{ds}\right) h \ \ ds \tag{5}$$

Now Tangent vector at the boundary at point (x_1, x_2) is given by vector $\left(\frac{dx_1}{ds}, \frac{dx_2}{ds}\right)^T$, hence the normal is $\mathbf{n} = \left(\frac{dx_2}{ds}, -\frac{dx_1}{ds}\right)^T$ (since if we take dot product of these 2 vectors we get zero). Now we can rewrite the integrand in the second integral in (5) in terms of this normal vector since

$$\begin{aligned} L_{u_{x_1}} \frac{dx_2}{ds} - L_{u_{x_2}} \frac{dx_1}{ds} &= \begin{bmatrix} L_{u_{x_1}} \\ L_{u_{x_2}} \end{bmatrix}^T \begin{bmatrix} \frac{dx_2}{ds} \\ -\frac{dx_1}{ds} \end{bmatrix} \\ &= \begin{bmatrix} L_{u_{x_1}} \\ L_{u_{x_2}} \end{bmatrix} \cdot \mathbf{n} \\ &= L_{\nabla u} \cdot \mathbf{n} \end{aligned}$$

Substitute the above into the second term of (5) we obtain

$$\boxed{\delta J(u,\mathbf{h}) = \int_\Omega \left(L_u - \frac{\partial L_{u_{x_1}}}{\partial x_1} - \frac{\partial L_{u_{x_2}}}{\partial x_2}\right) h \ dA + \int_\Gamma h(L_{\nabla u} \cdot \mathbf{n}) \ \ ds}$$

Final note on the sign before the second integral above. The book shows it as " $-$ ". I think this is because the normal should be pointing outside? Hence if we make out normal the negative of the normal used here (which I think points inwards), we obtain the result we are asked to show for part (a). (notice, the book has a mistake/typo, it says $\int_\Gamma h(L_{\nabla u} \cdot \mathbf{n}) \ \ dA$

instead of $\int_\Gamma h(L_{\nabla u}\cdot \mathbf{n}) \quad ds$, i.e. the integration is over a line segment, not over a differential area (since obviously this is contour integration).

part (b)

Necessary condition for minimum is that $\delta J(u,\mathbf{h})=0$,. ie.

$$\int\limits_\Omega \left(L_u - \frac{\partial L_{u_{x1}}}{\partial x_1} - \frac{\partial L_{u_{x2}}}{\partial x_2}\right) h \; dA - \int_\Gamma h(L_{\nabla u}\cdot \mathbf{n}) \quad ds = 0$$

Now consider the second integral in the above. Since $h=0$ on Γ, hence we are left to show that

$$\int\limits_\Omega \left(L_u - \frac{\partial L_{u_{x1}}}{\partial x_1} - \frac{\partial L_{u_{x2}}}{\partial x_2}\right) h \; dA = 0$$

But h is arbitrary function, hence by lemma 3.13 again, we argue that for the above to be zero, then

$$\begin{aligned} L_u - \frac{\partial L_{u_{x1}}}{\partial x_1} - \frac{\partial L_{u_{x2}}}{\partial x_2} &= 0 \\ L_u - \nabla\cdot L_{\nabla u} &= 0 \qquad \textit{on } \mathbf{x}\in\Omega \end{aligned}$$

Which is Euler-Lagrange equation.

Part (c)

Here we have free boundary conditions. Hence we can not take $h=0$ everywhere on Γ. Starting with the first variation

$$\delta J(u,\mathbf{h}) = \int\limits_\Omega \left(L_u - \frac{\partial L_{u_{x1}}}{\partial x_1} - \frac{\partial L_{u_{x2}}}{\partial x_2}\right) h \; dA - \int_\Gamma h(L_{\nabla u}\cdot \mathbf{n}) \quad ds = 0$$

Since $h\neq 0$ on Γ then by lemma 3.13 we can argue that $\boxed{L_{\nabla u}\cdot\mathbf{n} = \mathbf{0} \text{ on } \Gamma}$

Hence on $\mathbb{R}^2$, this means $\begin{bmatrix} L_{u_{x1}} \\ L_{u_{x2}}\end{bmatrix}^T \begin{bmatrix} \frac{dx_2}{ds} \\ -\frac{dx_1}{ds}\end{bmatrix} = 0$, i.e.

$$L_{u_{x1}}\frac{dx_2}{ds} - L_{u_{x2}}\frac{dx_1}{ds} = 0$$

Now we need to know the shape of the boundary to evaluate the above at each point. For example, for a circle, $\mathbf{n} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and the above become

$$L_{u_{x_1}} x_1 - L_{u_{x_2}} x_2 = 0$$

And the above equation needs to be satisfied at each point on the boundary after discretization.