HW 8 Mathematics 503, Mathematical Modeling, CSUF , July 12, 2007

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problem:

Write down the equations that determine the solution of the isoperimetric problem

$$\int_a^b p(x)\,y'^2 + q(x)\,y^2\ dx \to \min$$

Subject to

$$\int_a^b r(x)\,y^2 dx = 1$$

where p, q, r are given functions and $y(a) = y(b) = 0$.

Answer

Since $y(x)$ is fixed at each end, this is not a natural boundary problem. Therefore one can use the auxiliary lagrangian approach, where we write the auxiliary Lagrangian L^* as

$$\boxed{L^* = L + \lambda G}$$

Where $L(x, y, y') = p(x)\,y'^2 + q(x)\,y^2$, and $G = r(x)\,y^2$ and λ is the Lagrangian multiplier. Hence

$$\boxed{L^* = p(x)\,y'^2 + q(x)\,y^2 + \lambda r(x)\,y^2}$$

Hence now we write the solution as the Euler-Lagrange equation, but we use L^* instead of L

$$
\begin{aligned}
L_y^* - \frac{d}{dx} L_{y'}^* &= 0 \\
(2q(x)\, y + 2\lambda r(x)\, y) - \frac{d}{dx}(2p(x)\, y') &= 0 \\
q(x)\, y + \lambda r(x)\, y - (p(x)\, y')' &= 0
\end{aligned}
$$

Therefore the differential equation is

$$
\boxed{(p(x)\, y')' - y(q(x) + \lambda r(x)) = 0}
$$

This is a sturm-Liouville eigenvalue problem. The solution $y(x)$ from the above will contain 3 constants. 2 will be found from boundary conditions, and the third, which is λ is found from plugging in the solution $y(x)$ into the constraint given: $\int\limits_a^b r(x)\, y^2 dx = 1$