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1 Problem 1

problem:

Consider a taut string that is held fixed at $x=0$ and $x=L$. Load on string is $f(x)$ which is force per unit length. Tension is $T(x)$. Suppose the deflection of string is $u(x)$ is continuously differentiable function. (a) Argue that when the string is in equilibrium and the deflection is small, the potential energy is

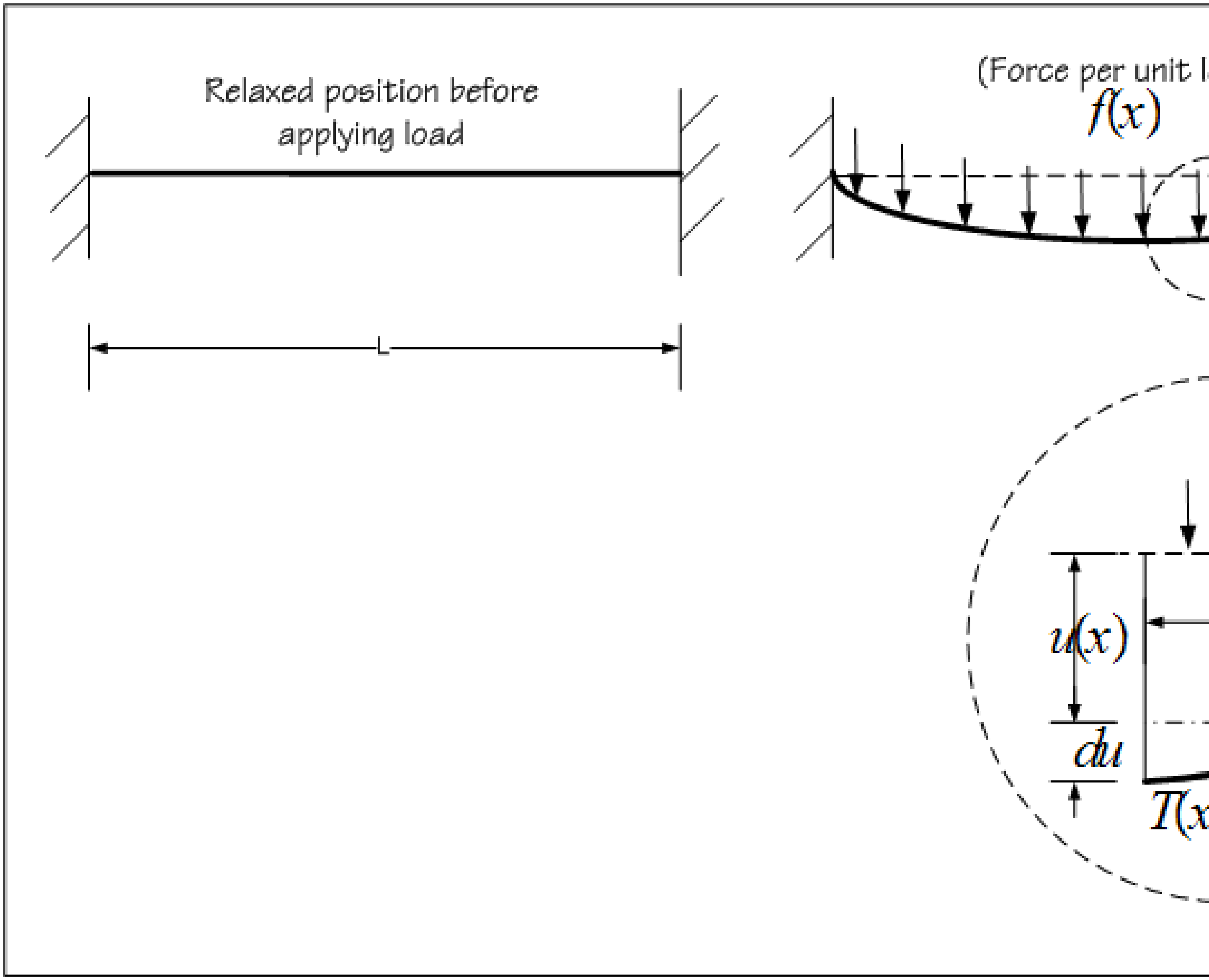
$$J(u)=\int_0^L \left(\frac{1}{2}Tu_x^2-uf\right)dx$$

(b) Use variational argument to characterization of weak solution.

(c) Verify that a weak solution minimizes this energy functional J

Solution:

(a) The potential energy of the string is made up of 2 parts. The first is due to work done by $f(x)$ is moving the string lower by a distance $u(x)$ from its relaxed position, and the second due to the work done by $T(x)$ in stretching the string from length of Δx to length of Δs



Now we calculate the potential energy. Due to $f(x)$ doing work over distance of Δx is

$$PE_1 = -(f(x)\,\Delta x)\,u(x)$$

due to Tension $T(x)$ doing work over distance Δx is

$$\begin{aligned} PE_2 &= -(-T(x)\,(\Delta s - \Delta x)) \\ &= T(x)\,(\Delta s - \Delta x) \end{aligned}$$

But for small deflection $\Delta s^2 = \Delta x^2 + \Delta u^2$, hence $\Delta s = \Delta x\sqrt{1+\left(\frac{\Delta u}{\Delta x}\right)^2}$ hence the above becomes

$$\begin{aligned} PE_2 &= T(x)\left(\Delta x\sqrt{1+\left(\frac{\Delta u}{\Delta x}\right)^2}-\Delta x\right) \\ &= T(x)\left(\sqrt{1+\left(\frac{\Delta u}{\Delta x}\right)^2}-1\right)\Delta x \end{aligned}$$

For small $\frac{\Delta u}{\Delta x}$, $\sqrt{1+\left(\frac{\Delta u}{\Delta x}\right)^2} \approx 1+\frac{1}{2}\left(\frac{\Delta u}{\Delta x}\right)^2$, so the above becomes

$$PE_2 = T(x)\left(\frac{1}{2}\left(\frac{\Delta u}{\Delta x}\right)^2\right)\Delta x$$

Hence total potential energy per unit length is

$$\begin{aligned} PE &= PE_1 + PE_2 \\ &= T(x)\left(\frac{1}{2}\left(\frac{\Delta u}{\Delta x}\right)^2\right)\Delta x - \left(f(x)\,\Delta x\right)u(x) \\ &= \left(\frac{1}{2}T(x)\left(\frac{\Delta u}{\Delta x}\right)^2 - f(x)\,u(x)\right)\Delta x \end{aligned}$$

Hence total PE is found by integrating the above over the total length of the sting

$$\boxed{J(u) = \int_0^L \tfrac{1}{2}T(x)\,u_x^2 - f(x)\,u(x)\quad dx}$$

(b) Assume the string is in equilibrium position. The deflection $u(x)$ at equilibrium will be such that it renders the $J(u)$ minimum. (This is a basic principle in physics, in which physical systems when in equilibrium will be in such configuration such that the total potential energy needed to achieve this configuration is minimal). Hence this is a minimization problem of the above function. Let the set of feasible solution $u(x)$ be $A = \left\{u \in C^1[0,L], \text{ s.t.} u(0) = 0, u(L) = 0\right\}$ and let the set of feasible directions $A_d = \left\{h \in C^1[0,L], \text{ s.t.} h(0) = 0, h(L) = 0\right\}$, then a variation of $J(u)$ in the direction of h is

$$\begin{aligned} J(u+h) &= \int_0^L \frac{1}{2}T\left(\frac{d}{dx}(u+h)\right)^2 - f(u+h)\quad dx \\ &= \int_0^L \frac{1}{2}T(u_x+h_x)^2 - (uf+hf)\quad dx \\ &= \int_0^L \frac{1}{2}T\left(u_x^2+h_x^2+2u_xh_x\right) - uf - hf\quad dx \end{aligned}$$

Hence

$$J(u+h) = \int_0^L \frac{1}{2}T\left(u_x^2\right) - uf\ dx + \int_0^L \frac{1}{2}T\left(h_x^2\right) dx + \int_0^L \frac{1}{2}T(2u_x h_x) - hf\ \ dx$$

$$= J(u) + \overbrace{\int_0^L \frac{1}{2}T\left(h_x^2\right) dx}^{\geq 0} + \int_0^L T(u_x h_x) - hf\ \ dx \tag{1}$$

From the above we see that directional derivative of J in the direction of h is the linear term which is

$$\boxed{J'(u;h) = \int_0^L T(u_x h_x) - hf\ \ dx}$$

The solution $u(x) \in A$ such that $J'(u;h) = 0$ for all $h \in A_d$ is called the weak solution. Notice that a weak solution requires only that u be once differentiable, i.e. $u \in C^1[0,L]$.

(c)A weak solution minimizes $J(u)$. Looking at (1) above, we see that the second term is positive. Hence if $J'(u;h) = 0$, which is the third term in (1), then this implies that a small variation from $J(u)$ in the direction of h would result in larger value of $J(u)$, Hence $J(u)$ is at a minimum.

2 Problem 2

Problem: For the membrane problem, show that a classical solution is a weak solution.

Answer:

From notes, we found that classical solution implied that $0 = \oint\limits_{\partial R} (Pdx + Qdy)$, but $\oint\limits_{\partial R} (Pdx + Qdy) = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$ where $Q = Tu_x h, P = -Tu_y h$

Hence

$$
\begin{aligned}
0 &= \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy \\
&= \int\int_R \left(\frac{\partial (Tu_x h)}{\partial x} - \frac{\partial (-Tu_y h)}{\partial y}\right) dxdy \\
&= \int\int_R T(u_{xx} h + u_x h_x) + T(u_{yy} h + u_y h_y)\ dxdy \\
&= \int\int_R T(u_{xx} + u_{yy})\, h\ \ + T(u_x h_x + u_y h_y)\ dxdy
\end{aligned} \tag{1}
$$

But since a classical solution is assumed, hence

$$(Tu_x)_x + (Tu_y)_y = -f$$

Where the above is the classical solution as derived in the notes. But the above is just $T(u_{xx} + u_{yy}) = -f$, hence if replace the first term in the intrange in (1) by this solution, we have

$$
\begin{aligned}
0 &= \int\int_R -fh\ \ + T(u_x h_x + u_y h_y)\ dxdy \\
&= \int\int_R T(u_x h_x + u_y h_y) - hf\ \ dxdy
\end{aligned}
$$

But this is the weak solution $J'(u;h) = 0$ as shown in the top of page 2 in the membrane notes. Hence a classical solution implied a weak solution.