

HW 13 Mathematics 503, Mathematical Modeling, CSUF , August 2, 2007

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1 Problem 10 page 268 section 4.5 (Distributions)

problem:

Find fundamental solution associated with operator L defined by $Lu = -x^2u'' - xu' + u, 0 < x < 1$, such that $u(x, \xi) = x$ for $0 < x < \xi$.

answer:

The fundamental solution can be written as

$$u = \begin{cases} x & 0 < x < \xi \\ A(\xi) u_1(x) + B(\xi) u_2(x) & \xi < x < 1 \end{cases}$$

And our goal is to determine $A(\xi), B(\xi)$. In the above u_1, u_2 are the 2 independent solution to the homogenous equation $-x^2u'' - xu' + u = 0$

We start by finding u_1, u_2 . We try solution $u = x^m$ and substitute this into the above homogenous equation, we obtain the characteristic equation $m^2 = 1$, hence $m = \pm 1$ the 2 solution are $\boxed{u_1 = x \text{ and } u_2 = x^{-1}}$. Hence our fundamental solution now looks like

$$u = \begin{cases} x & 0 < x < \xi \\ A(\xi) x + B(\xi) x^{-1} & \xi < x < 1 \end{cases}$$

Now consider the test function ϕ , hence

$$\begin{aligned}
(Lu, \phi) &= (-x^2 u'' - xu' + u, \phi) \\
&= (-x^2 u'', \phi) - (xu', \phi) + (u, \phi) && \text{linearity of distribution} \\
&= (u'', -x^2 \phi) - (u', x\phi) + (u, \phi) && \text{property of distribution} \\
&= \left(u, (-x^2 \phi)'' \right) + (u, (x\phi)') + (u, \phi) && \text{property of distribution} \\
&= \left(u, (-x^2 \phi)'' + (x\phi)' + \phi \right) \\
&= (u, L^* \phi)
\end{aligned}$$

Where $L^* \phi = (-x^2 \phi)'' + (x\phi)' + \phi$

Hence expanding the differentiation in the above and simplifying we obtain

$$\boxed{(Lu, \phi) = (u, -3x\phi' - x^2\phi'')}$$

Now take $Lu = \delta_\xi$, i.e. put a point source as input, then we are looking for $(Lu, \phi) = \phi(\xi)$ from the properties of delta function. In other words, we are looking for

$$(Lu, \phi) = \int_0^1 u(-3x\phi' - x^2\phi'') dx = \phi(\xi)$$

Hence

$$\begin{aligned}
\phi(\xi) &= \int_0^\xi u_1(-3x\phi' - x^2\phi'') dx + \int_\xi^1 u_2(-3x\phi' - x^2\phi'') dx \\
&= \int_0^\xi x(-3x\phi' - x^2\phi'') dx + \int_\xi^1 (A(\xi)x + B(\xi)x^{-1})(-3x\phi' - x^2\phi'') dx \quad (1)
\end{aligned}$$

Looking at the first integral, and perform integration by parts. In these calculations we note that

$$\phi(0) = \phi(1) = \phi'(0) = \phi'(1) = 0$$

Hence

$$\begin{aligned}
\int_0^\xi x(-3x\phi' - x^2\phi'') dx &= \int_0^\xi -3x^2\phi' - x^3\phi'' dx \\
&= \int_0^\xi -3x^2\phi' dx - \int_0^\xi x^3\phi'' dx \\
&= -3\left([x^2\phi]_0^\xi - \int_0^\xi 2x\phi dx\right) - \left([x^3\phi']_0^\xi - \int_0^\xi 3x^2\phi' dx\right) \\
&= -3\left([\xi^2\phi(\xi)] - 2 \int_0^\xi x\phi dx\right) - \left([\xi^3\phi'(\xi)] - 3 \int_0^\xi x^2\phi' dx\right) \\
&= -3\xi^2\phi(\xi) + 6 \int_0^\xi x\phi dx - \xi^3\phi'(\xi) + 3 \int_0^\xi x^2\phi' dx
\end{aligned}$$

Now do integration by part on the last integral above

$$\begin{aligned}
\int_0^\xi x(-3x\phi' - x^2\phi'') dx &= -3\xi^2\phi(\xi) + 6 \int_0^\xi x\phi dx - \xi^3\phi'(\xi) + 3\left([x^2\phi]_0^\xi - \int_0^\xi 2x\phi dx\right) \\
&= -3\xi^2\phi(\xi) + 6 \int_0^\xi x\phi dx - \xi^3\phi'(\xi) + 3\xi^2\phi(\xi) - 6 \int_0^\xi x\phi dx \\
&= -\xi^3\phi'(\xi)
\end{aligned} \tag{2}$$

Now looking at (1) above, we now do integration by part on $\int_\xi^1 (Ax + Bx^{-1})(-3x\phi' - x^2\phi'') dx$

$$\int_\xi^1 (Ax + Bx^{-1})(-3x\phi' - x^2\phi'') dx = \int_\xi^1 Ax(-3x\phi' - x^2\phi'') dx + \int_\xi^1 Bx^{-1}(-3x\phi' - x^2\phi'') dx$$

Consider the first integral above in the RHS, we write

$$\begin{aligned}
\int_{\xi}^1 Ax(-3x\phi' - x^2\phi'') dx &= A \int_{\xi}^1 -3x^2\phi' - x^3\phi'' dx \\
&= A \left(\int_{\xi}^1 -3x^2\phi' dx - \int_{\xi}^1 x^3\phi'' dx \right) \\
&= A \left(3 \left([-x^2\phi]_{\xi}^1 - \int_{\xi}^1 -2x\phi dx \right) - \left([x^3\phi'] - \int_{\xi}^1 3x^2\phi' dx \right) \right) \\
&= A \left(3 \left([-\phi(1) + \xi^2\phi(\xi)] + 2 \int_{\xi}^1 x\phi dx \right) - \left([\phi'(1) - \xi^3\phi'(\xi)] - 3 \int_{\xi}^1 x^2\phi' dx \right) \right) \\
&= A \left(3 \left(\xi^2\phi(\xi) + 2 \int_{\xi}^1 x\phi dx \right) - \left(-\xi^3\phi'(\xi) - 3 \int_{\xi}^1 x^2\phi' dx \right) \right) \\
&= A \left(3\xi^2\phi(\xi) + 6 \int_{\xi}^1 x\phi dx + \xi^3\phi'(\xi) + 3 \int_{\xi}^1 x^2\phi' dx \right)
\end{aligned}$$

Now do integration by part on the last term in the above line

$$\begin{aligned}
\int_{\xi}^1 Ax(-3x\phi' - x^2\phi'') dx &= A \left(3\xi^2\phi(\xi) + 6 \int_{\xi}^1 x\phi dx - \xi^3\phi'(\xi) + 3 \left([x^2\phi]_{\xi}^1 - \int_{\xi}^1 2x\phi dx \right) \right) \\
&= A \left(3\xi^2\phi(\xi) + 6 \int_{\xi}^1 x\phi dx - \xi^3\phi'(\xi) + 3 \left(-\xi^2\phi(\xi) - 2 \int_{\xi}^1 x\phi dx \right) \right) \\
&= A \left(3\xi^2\phi(\xi) + 6 \int_{\xi}^1 x\phi dx - \xi^3\phi'(\xi) - 3\xi^2\phi(\xi) - 6 \int_{\xi}^1 x\phi dx \right) \\
&= A\xi^3\phi'(\xi)
\end{aligned} \tag{3}$$

Now we do integration by parts on $\int_{\xi}^1 Bx^{-1}(-3x\phi' - x^2\phi'') dx$

$$\begin{aligned}
\int_{\xi}^1 Bx^{-1}(-3x\phi' - x^2\phi'') dx &= B \int_{\xi}^1 -3\phi' - x\phi'' dx \\
&= -B \left(\int_{\xi}^1 3\phi' dx + \int_{\xi}^1 x\phi'' dx \right) \\
&= -B \left(3[\phi]_{\xi}^1 + \left([x\phi']_{\xi}^1 - \int_{\xi}^1 \phi' dx \right) \right) \\
&= -B \left(-3\phi(\xi) + \left(-\xi\phi'(\xi) - [\phi]_{\xi}^1 \right) \right) \\
&= -B(-3\phi(\xi) - \xi\phi'(\xi) + \phi(\xi)) \\
&= 2B\phi(\xi) + B\xi\phi'(\xi)
\end{aligned} \tag{5}$$

Hence, from (2),(3),(4),(5), we have

$$\begin{aligned}\phi(\xi) &= \int_0^\xi x(-3x\phi' - x^2\phi'') dx + \int_\xi^1 (A(\xi)x + B(\xi)x^{-1}) (-3x\phi' - x^2\phi'') dx \\ \phi(\xi) &= -\xi^3\phi'(\xi) + A\xi^3\phi'(\xi) + 2B\phi(\xi) + B\xi\phi'(\xi)\end{aligned}$$

or

$$\phi(\xi) = \phi(\xi) (2B) + \phi'(\xi) [-\xi^3 + A\xi^3 + B\xi] \quad (6)$$

By looking at the coefficients on $\phi(\xi)$, and compare, we see that $2B = 1$ or

$$B = \frac{1}{2}$$

We can now use the continuity condition at $x = \xi$ and write

$u_1 = u_2$ at $x = \xi$, hence

$$\begin{aligned}\xi &= Au_1 + Bu_2 \\ &= A\xi + B\xi^{-1} \\ \xi &= A\xi + \frac{1}{2}\xi^{-1} \\ 2\xi^2 &= 2A\xi^2 + 1\end{aligned}$$

Hence

$$A = \frac{2\xi^2 - 1}{2\xi^2}$$

Therefor the fundamental solution is

$$\begin{aligned}u &= \begin{cases} x & 0 < x < \xi \\ Ax + Bx^{-1} & \xi < x < 1 \end{cases} \\ &= \begin{cases} x & 0 < x < \xi \\ \frac{2\xi^2-1}{2\xi^2}x + \frac{1}{2x} & \xi < x < 1 \end{cases}\end{aligned}$$

Here is a plot for few values of ξ


```
In[50]:= u[x_, ξ_] := x /; x < ξ
```

```
u[x_, ξ_] :=  $\frac{2\xi^2 - 1}{2\xi^2} x + \frac{1}{2\xi}$  /; x > ξ
```

```
u[x_, ξ_] := 1 /; x == ξ
```

```
p = Table[Plot[u[x, ξ], {x, 0, 1}, PlotLabel -> "ξ=" <> ToString[ξ], ImageSize
```

