HW 11 Mathematics 503, Mathematical Modeling, CSUF , July 20, 2007

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Contents

1 Problem 3 page 257 section 4.4 (Green Functions)

problem:

Consider boundary value problem $u'' - 2xu' = f(x)$, $0 < x < 1$, $u(0) = u'(1) = 0$. Find Green function or explain where there isn't one.

answer:

We see that $\boxed{p(x) = -1}$

First, lets see if $\lambda = 0$ or not. Since if $\lambda = 0$ since by theorem 4.19 (page 248) Green function does not exist, and I do not need to try to find it.

Let

$$u'' - 2xu' = \lambda u$$

If $\lambda = 0$ then solve the homogeneous equation $u'' - 2xu' = 0$. Let $\boxed{y(x) = u'(x)}$, hence we obtain $y' - 2xy = 0$, by separation of variables, we then have

$$\frac{y'}{y} = 2x$$
$$\frac{1}{y}dy = 2xdx$$
$$\int \frac{1}{y}dy = 2\int xdx$$

Hence

$$\ln y = x^2 + C$$

Which leads to $y(x) = Ae^{x^2}$. But since $y = u'$, then $\frac{du}{dx} = Ae^{x^2}$ or

$$u(x) = A\int_0^x e^{t^2}dt + B$$

Therefore,

$$u_1(x) = A\int_0^x e^{t^2}dt$$

and

$$u_2(x) = B$$

At $x = 0$ we have $u(0) = 0$, hence $u(0) = A\int_0^0 e^{t^2}dt + B$ or $\boxed{0 = B}$ so now $u(x) = A\int_0^x e^{t^2}dt$. Now lets see if this satisfies the second boundary condition $u'(1) = 0$. First note that

$$\frac{d}{dx}\left(A\int_0^x e^{t^2}dt\right) = Ae^{x^2}$$

hence at $x = 1$ we obtain $0 = A\exp(1)$ which means $\boxed{A = 0}$, but this means trivial solution since both A, B are zero. Hence $\boxed{\lambda \neq 0}$ OK, so now I try to find Green function:

Now we need to find 2 independent solutions as combinations of $A\int_0^x e^{t^2}dt$ and B such that each will satisfies at least one of the boundary conditions.

We need $u(0) = 0$, hence if we take

$$u_1(x) = \int_0^x e^{t^2}dt$$

which will be zero at $x = 0$, and if we take

$$u_2(x) = 1$$

then we see that $u_2'(1) = 0$. Now find the Wronskian

$$W = \det \begin{bmatrix} u_1 & u_2 \\ u_1' & u_2' \end{bmatrix} = \det \begin{bmatrix} \int_0^x e^{t^2} dt & 1 \\ e^{x^2} & 0 \end{bmatrix} = -e^{x^2}$$

Hence using equation 4.46 we obtain, noting that $p = -1$

$$g(x,\xi) = \begin{cases} -\frac{u_1(x)u_2(\xi)}{p\ W(\xi)} & x < \xi \\ -\frac{u_1(\xi)u_2(x)}{p\ W(\xi)} & x > \xi \end{cases} = \begin{cases} -\frac{u_1(x)u_2(\xi)}{(-1)\left(-e^{\xi^2}\right)} & x < \xi \\ -\frac{u_1(\xi)u_2(x)}{(-1)\left(-e^{\xi^2}\right)} & x > \xi \end{cases}$$

$$= \begin{cases} -e^{-\xi^2} \int_0^x e^{t^2} dt & x < \xi \\ -e^{-\xi^2} \int_0^\xi e^{t^2} dt & x > \xi \end{cases}$$

Hence

$$\boxed{g(x,\xi) = -e^{-\xi^2} \left(H(\xi - x) \int_0^x e^{t^2} dt + H(x - \xi) \int_0^\xi e^{t^2} dt \right)}$$

and

$$u(x) = \int_0^x g(x,\xi)\, f(\xi)\, d\xi$$

I used the Green function I derived, and used it to plot the solution (for $f(x) = 1$) and compare the plot with the analytical solution.

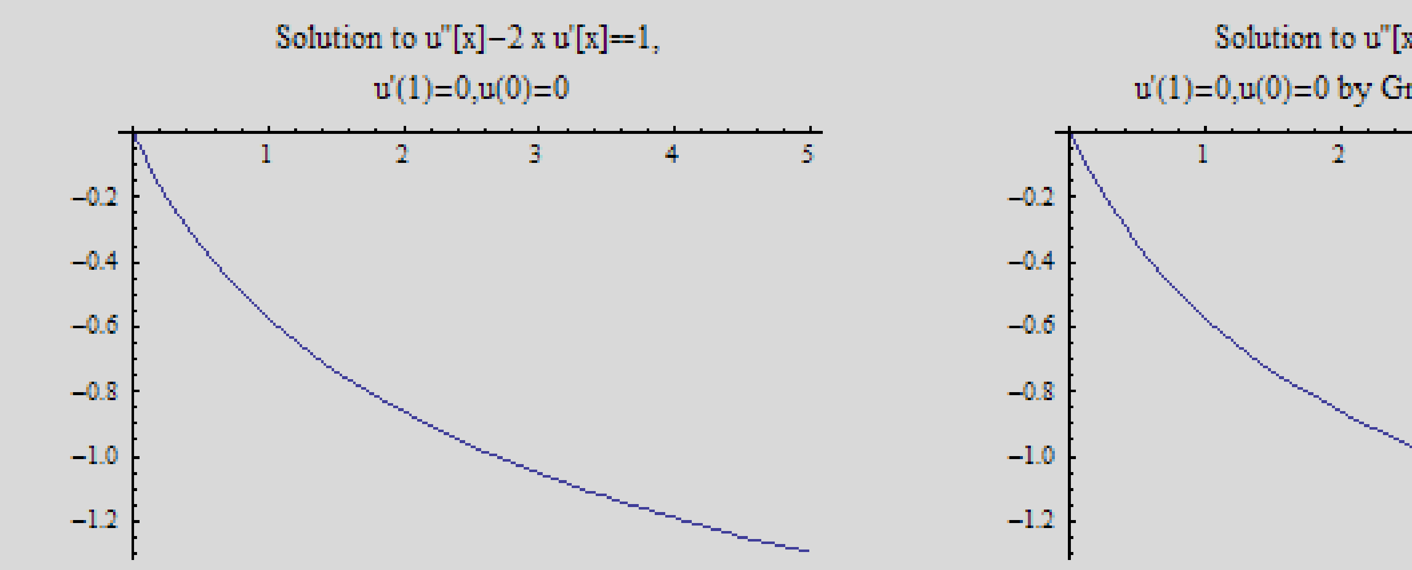
```
Remove["Global`*"]
g[x_, ξ_] := (* 1/Exp[ξ^2] (-UnitStep[ξ-x] N[∫_0^x Exp[t^2] dt] -UnitStep[x-ξ] N[∫_0^ξ Exp[t^2] dt
  -1/Exp[ξ^2] (UnitStep[ξ - x] N[∫_0^x Exp[t^2] dt] + UnitStep[x - ξ] N[∫_0^ξ Exp[t^2] dt])

eq = u''[x] - 2 x u'[x] == 1
s = First@DSolve[{eq, u[0] == 0, u'[5] == 0}, u[x], x]
p = Plot[u[x] /. s, {x, 0, 5}, PlotLabel -> "Solution to u''[x]-2 x u'[x]==1,\n u

mysol[x_] := N[Integrate[g[x, ξ], {ξ, 0, 5}]]
p2 = Plot[mysol[x], {x, 0, 5},
   PlotLabel -> "Solution to u''[x]-2 x u'[x]==1,\n u'(1)=0,u(0)=0 by Green F
GraphicsRow[{p, p2}]
```

$-2\,x\,u'[x] + u''[x] == 1$

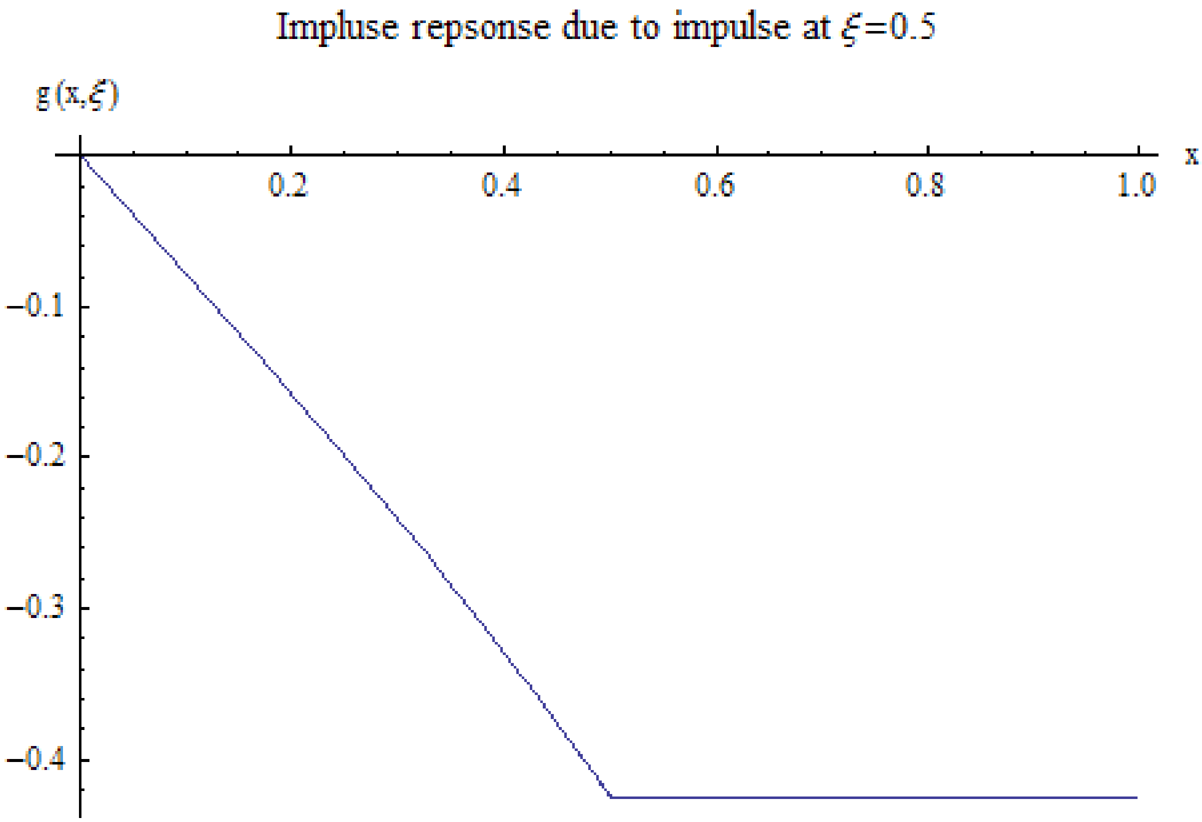
$\left\{u[x] \to \frac{1}{4}\left(-\pi\, \text{Erf}[5]\, \text{Erfi}[x] + 2\, x^2\, \text{HypergeometricPFQ}\left[\{1, 1\}, \left\{\frac{3}{2}, 2\right\}, x^2\right]\right)\right\}$



This is a plot of just the impulse response (green function) due to an impulse at $x = 0.5$

```
In[98]:= g[x_, ξ_] := 1/Exp[ξ^2] (-UnitStep[ξ - x] N[∫_0^x Exp[t^2] dt] - U
         Plot[g[x, .5], {x, 0, 1}, PlotLabel → "Impluse repsonse du
          AxesLabel → {"x", "g(x,ξ)"}, PlotRange → All]
```

Out[99]=



This is another method to solving this problem by using properties of Green function

From above we found $u_1 = \int_0^x e^{t^2} dt$, $u_2 = 1$, but

$$
\begin{aligned}
g(x,\xi) &= A(\xi)\, u_1(x) \;\; 0 < x < \xi \\
&= A(\xi) \int_0^x e^{t^2} dt
\end{aligned}
$$

and

$$
\begin{aligned}
g(x,\xi) &= B(\xi)\, u_2(x) \\
&= B(\xi) \qquad \xi < x < 1
\end{aligned}
$$

At $x = \xi$, due to continuity, we require that

$$
A(\xi) \int_0^{\xi} e^{t^2}\, dt = B(\xi) \tag{1}
$$

and to impose the discontinuity condition on the first derivative we have

$$
\begin{aligned}
g'\left(\xi^+,\xi\right) - g'\left(\xi^-,\xi\right) &= \frac{-1}{p\left(\xi\right)} \\
0 - A(\xi)\, e^{\xi^2} &= 1 \\
A(\xi) &= \frac{-1}{e^{\xi^2}} \qquad (2)
\end{aligned}
$$

From (1) we then obtain that

$$
B(\xi) = \frac{-1}{e^{\xi^2}} \int_0^{\xi} e^{t^2}\, dt
$$

Hence

$$
\begin{aligned}
g(x,\xi) &= A(\xi)\, u_1(x) \\
&= \frac{-1}{e^{\xi^2}} \int_0^{x} e^{t^2}\, dt \qquad 0 < x < \xi
\end{aligned}
$$

and

$$
\begin{aligned}
g(x,\xi) &= B(\xi)\, u_2(x) \\
&= \frac{-1}{e^{\xi^2}} \int_0^{\xi} e^{t^2}\, dt \qquad \xi < x < 1
\end{aligned}
$$

Hence

$$
\boxed{g(x,\xi) = \tfrac{-1}{e^{\xi^2}} \left(H(\xi - x) \int_0^x e^{t^2}\, dt + H(x - \xi) \int_0^{\xi} e^{t^2}\, dt \right)}
$$

Compare this solution to the one found above using the *formula method* we see they are the same.

2 Problem 8, page 258 section 4.5

Problem:

Find the inverse of the differential operator $Lu = -(x^2u')'$ on $1 < x < e$ subject to $u(1) = u(e) = 0$

solution:

This is SLP problem with $p = x^2, q = 0$. First find if $\lambda = 0$ is possible eigenvalue.

$$\lambda u = -\left(x^2u'\right)'$$

Let $\lambda = 0$, hence we have $-(x^2u')' = 0$ or $-(2xu' + x^2u'') = 0$ or

$$u'' + \frac{2}{x}u' = 0$$

Use separation of variables. First let $y = u'$, hence $y' + \frac{2}{x}y = 0$ or $\frac{1}{y}\frac{dy}{dx} = -\frac{2}{x}$ hence

$$\begin{aligned}
\int \frac{1}{y}dy &= -2\int \frac{1}{x}dx \\
\ln y &= -2\ln x + c \\
y &= Ae^{-2\ln x} \\
y &= A\frac{1}{x^2}
\end{aligned}$$

But $y = u'$, hence $du = A\frac{1}{x^2}dx$ or $u = A\int \frac{1}{x^2}dx$

hence $u = -A\frac{1}{x} + B$ or

$$\boxed{u(x) = \frac{A}{x} + B}$$

where the minus sign is absorbed into A. Hence we have 2 independent solutions $\frac{A}{x}$ and B, so we need combination of these 2 solutions to satisfy the BV. At $x = 1$ we have $u = 0$, hence if we take $\boxed{u_1 = \frac{1}{x} - 1}$ then it will satisfy this condition. At $x = e$ we need $u = 0$, hence take $\boxed{u_2 = \frac{1}{x} - \exp(-1)}$

Then

$$W = \det\begin{bmatrix} u_1 & u_2 \\ u_1' & u_2' \end{bmatrix} = \det\begin{bmatrix} \frac{1}{x} - 1 & \frac{1}{x} - \exp(-1) \\ \frac{-1}{x^2} & \frac{-1}{x^2} \end{bmatrix} = -e^{x^2}$$

Hence

$$W = \frac{1-e^{-1}}{x^2}$$

Then green function is

$$g(x,\xi) = \begin{cases} -\frac{u_1(x)u_2(\xi)}{W(\xi)} & x<\xi \\ -\frac{u_1(\xi)u_2(x)}{W(\xi)} & x>\xi \end{cases} = \begin{cases} -\frac{\left(\frac{1}{x}-1\right)\left(\frac{1}{\xi}-e^{-1}\right)}{\xi^2 \frac{1-e^{-1}}{\xi^2}} & x<\xi \\ -\frac{\left(\frac{1}{\xi}-1\right)\left(\frac{1}{x}-e^{-1}\right)}{\xi^2 \frac{1-e^{-1}}{\xi^2}} & x>\xi \end{cases}$$

$$\begin{cases} \left(1-\frac{1}{x}\right)\frac{(1-\xi e^{-1})}{\xi(e^{-1}-1)} & x<\xi \\ \left(\frac{1}{x}-e^{-1}\right)\frac{(1-\xi)}{\xi(e^{-1}-1)} & x>\xi \end{cases}$$

But the inverse L^{-1} is $\int g(x,\xi)\, f(x)\, dx$ where $g(x,\xi)$ is the green function given aboive.

Another way to solve the problem:

From above we found $u_1 = \frac{1}{x}-1$, $u_2 = \frac{1}{x}-e^{-1}$, but

$$\begin{aligned} g(x,\xi) &= A(\xi)\, u_1(x) \\ &= A(\xi)\left(\frac{1}{x}-1\right) \qquad 1<x<\xi \end{aligned}$$

and

$$\begin{aligned} g(x,\xi) &= B(\xi)\, u_2(x) \\ &= B(\xi)\ \left(\frac{1}{x}-e^{-1}\right) \qquad \xi<x<e \end{aligned}$$

At $x=\xi$, due to continuity, we require that

$$A(\xi)\left(\frac{1}{\xi}-1\right) = B(\xi)\ \left(\frac{1}{\xi}-e^{-1}\right) \tag{1}$$

and to impose the discontinuity condition on the first derivative we have

$$
\begin{aligned}
g'(\xi^{+},\xi) - g'(\xi^{-},\xi) &= \frac{-1}{p(\xi)} \\
B(\xi)\ \left(\frac{-1}{\xi^2}\right) - A(\xi)\left(\frac{-1}{\xi^2}\right) &= \frac{-1}{\xi^2} \\
B(\xi)\ - A(\xi) = 1
\end{aligned} \tag{2}
$$

Solve (1) and (2) for $B(\xi)\,, A(\xi)$

From (2) we have $\boxed{B(\xi) = 1 + A(\xi)}$,substitute into (1) we have $A(\xi)\left(\frac{1}{\xi} - 1\right) = (1 + A(\xi))\ \left(\frac{1}{\xi} - e^{-1}\right)$ or

$$
\begin{aligned}
\frac{A(\xi)}{\xi} - A(\xi) &= \frac{1}{\xi} - e^{-1} + \frac{A(\xi)}{\xi} - A(\xi)\, e^{-1} \\
-A(\xi) + A(\xi)\, e^{-1} &= \frac{1}{\xi} - e^{-1} \\
A(\xi)\left(e^{-1} - 1\right) &= \frac{1}{\xi} - e^{-1} \\
A(\xi) &= \frac{1 - \xi e^{-1}}{\xi\left(e^{-1} - 1\right)}
\end{aligned}
$$

Hence

$$
\begin{aligned}
B(\xi) &= 1 + A(\xi) \\
&= 1 + \frac{1 - \xi e^{-1}}{\xi\left(e^{-1} - 1\right)} \\
&= \frac{1 - \xi}{\xi\left(e^{-1} - 1\right)}
\end{aligned}
$$

Then

$$
\begin{aligned}
g(x,\xi) &= A(\xi)\, u_1(x) \\
&= \left(\frac{1 - \xi e^{-1}}{\xi\left(e^{-1} - 1\right)}\right)\left(\frac{1}{x} - 1\right) \qquad 1 < x < \xi
\end{aligned}
$$

$$
\begin{aligned}
g(x,\xi) &= B(\xi)\, u_2(x) \\
&= \left(\frac{1 - \xi}{\xi\left(e^{-1} - 1\right)}\right)\left(\frac{1}{x} - e^{-1}\right) \qquad \xi < x < e
\end{aligned}
$$

Hence

$$\boxed{g(x,\xi) = \frac{1}{e^{\xi^2}}\left(H(\xi - x)\left(\frac{1-\xi e^{-1}}{\xi(e^{-1}-1)}\right)\left(\frac{1}{x}-1\right) + H(x-\xi)\left(\frac{1-\xi}{\xi(e^{-1}-1)}\right)\left(\frac{1}{x}-e^{-1}\right)\right)}$$

Which agree with the *formula method.*

This a plot of Green function for $\xi = 2$

In[36]:=

```
(*Second problem plot of green function *)

r = Exp[-1] // N;
g[x_, ξ_] :=
  1/Exp[ξ^2]
   (UnitStep[ξ - x] ((1 - ξ r)/(ξ (r - 1))) (1/x - 1) + UnitStep[x - ξ]
Plot[g[x, 2], {x, 1, Exp[1]},
 PlotLabel → "Impluse repsonse due to impulse at ξ=2"
 AxesLabel → {"x", "g(x,ξ)"}, PlotRange → All]
```

Out[38]=