HW1, EE 503 (Information theory and coding) spring 2010

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1 Problem 3.2

Consider a source which generates 2 symbols x_1, x_2 with probability $p(x_1) = p$ and $p(x_2) = 1 - p = q$. Now consider a sequence of 2 outputs as a single symbol $X^2 = \{\{x_1x_1\}, \{x_1x_2\}, \{x_2x_1\}, \{x_2x_2\}\}$, Assuming that consecutive outputs from the source are statically independent, show directly that $H(X^2) = 2H(X)$

Solution

First find $H(X)$

$$X = \{x_1, x_2\}$$
$$H(X) = \sum_{i=1}^{M} p_i \log_2 \frac{1}{p_i}$$

Where M in this case is 2. The above becomes

$$\begin{aligned} H(X) &= p \log_2 \frac{1}{p} + q \log_2 \frac{1}{q} \\ &= -p \log p - q \log q \end{aligned}$$

Hence $2H(x)$ becomes

$$\begin{aligned} 2H(X) &= 2(-p \log p - q \log q) \\ &= -p \log p^2 - q \log q^2 \end{aligned} \tag{1}$$

Now, we conside X^2 . Below we write X^2 with the probability of each 2 outputs as single symbol on top of each. Notice that since symbols are statistically independent, then $p(x_1x_2) = p(x_1)p(x_2)$, we obtain

$$X^2 = \left\{ \overbrace{x_1x_1}^{p^2}, \overbrace{x_1x_2}^{pq}, \overbrace{x_2x_1}^{qp}, \overbrace{x_2x_2}^{q^2} \right\}$$

Hence since

$$H(X^2) = \sum_{i=1}^{N} p_i \log_2 \frac{1}{p_i}$$

Where $N=4$ now, the above becomes

$$\begin{aligned} H\left(X^2\right) &= p^2\log_2\frac{1}{p^2} + pq\log_2\frac{1}{pq} + pq\log_2\frac{1}{pq} + q^2\log_2\frac{1}{q^2} \\ &= -p^2\log_2 p^2 - 2pq\log_2 pq - q^2\log_2 q^2 \\ &= -p^2\log_2 p^2 - pq\log_2 (pq)^2 - q^2\log_2 q^2 \\ &= -p^2\log_2 p^2 - pq\log_2\left(p^2q^2\right) - q^2\log_2 q^2 \\ &= -p^2\log_2 p^2 - pq\left(\log_2 p^2 + \log_2 q^2\right) - q^2\log_2 q^2 \\ &= -p^2\log_2 p^2 - \overbrace{pq\log_2 p^2}^{A} - \overbrace{pq\log_2 q^2}^{B} - \overbrace{q^2\log_2 q^2}^{C} \end{aligned} \tag{2}$$

Now we will expand the terms labeled above as A,B,C and simplify, then we will obtain (1) showing the desired results.

$$\begin{aligned} A &= pq\log_2 p^2 \\ &= p(1-p)\log_2 p^2 \\ &= p\log_2 p^2 - p^2\log_2 p^2 \end{aligned}$$

$$\begin{aligned} B &= pq\log_2 q^2 \\ &= p(1-p)\log_2 q^2 \\ &= p\log_2 q^2 - p^2\log_2 q^2 \end{aligned}$$

$$\begin{aligned} C &= q^2\log_2 q^2 \\ &= (1-p)^2\log_2 q^2 \\ &= \left(1+p^2-2p\right)\log_2 q^2 \\ &= \log_2 q^2 + p^2\log_2 q^2 - 2p\log_2 q^2 \end{aligned}$$

Substitute the result we found for A,B,C back into (2) we obtain

$$\begin{aligned} H\left(X^2\right) &= -p^2\log_2 p^2 - \left(p\log_2 p^2 - p^2\log_2 p^2\right) - \left(p\log_2 q^2 - p^2\log_2 q^2\right) - \left(\log_2 q^2 + p^2\log_2 q^2 - 2p\log_2 q^2\right) \\ &= -p\log_2 p^2 + p\log_2 q^2 - \log_2 q^2 \end{aligned}$$

But the above can be written as

$$\begin{aligned} H\left(X^2\right) &= -p\log_2 p^2 + (1-q)\log_2 q^2 - \log_2 q^2 \\ &= -p\log_2 p^2 + \log_2 q^2 - q\log_2 q^2 - \log_2 q^2 \\ &= -p\log_2 p^2 - q\log q^2 \end{aligned} \tag{4}$$

Compare (4) and (1), we see they are the same.
Hence

$$\boxed{H\left(X^2\right) = 2H\left(X\right)}$$

2 Problem 3.3

A source emits sequence of independent symbols from alphabet X of symbols $x_1, \cdots, x_5$ with probabilities $\left\{\frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{3}{16}, \frac{5}{16}\right\}$, find the entropy of the source alphabet

Solution

$$H(X) = \sum_{i=1}^{M} p_i \log_2 \frac{1}{p_i}$$

Where $M = 5$, hence the above becomes

$$\begin{aligned}
H(X) &= p_1 \log_2 \frac{1}{p_1} + p_2 \log_2 \frac{1}{p_2} + p_3 \log_2 \frac{1}{p_3} + p_4 \log_2 \frac{1}{p_4} + p_5 \log_2 \frac{1}{p_5} \\
&= \frac{1}{4} \log_2 4 + \frac{1}{8} \log_2 8 + \frac{1}{8} \log_2 8 + \frac{3}{16} \log_2 \frac{16}{3} + \frac{5}{16} \log_2 \frac{16}{5} \\
&= \frac{1}{4}(2) + \frac{1}{8}(3) + \frac{1}{8}(3) + \frac{3}{16}(\log_2 16 - \log_2 3) + \frac{5}{16}(\log_2 16 - \log_2 5) \\
&= \frac{1}{2} + \frac{3}{8} + \frac{3}{8} + \frac{3}{16}(4 - \log_{10} 3 \log_2 10) + \frac{5}{16}(4 - \log_{10} 5 \log_2 10)
\end{aligned}$$

But $\log_2 10 = 3.32193$, hence the above becomes

$$\begin{aligned}
H(X) &= \frac{1}{2} + \frac{3}{8} + \frac{3}{8} + \frac{3}{4} - \frac{3}{16}(3.32193)\log_{10} 3 + \frac{5}{4} - \frac{5}{16}(3.32193)\log_{10} 5 \\
&= \frac{1}{2} + \frac{3}{8} + \frac{3}{8} + \frac{3}{4} - \frac{3}{16}(3.32193)(0.477121) + \frac{5}{4} - \frac{5}{16}(3.32193)(0.69897) \\
&= 2.2272 \qquad \text{bits/symbol}
\end{aligned}$$

To verify, we know that $H(X)$ must be less than or equal to $\log_2 M$ where $M = 5$ in this case, hence $H(X) \leq \log_2 5$ or $H(X) \leq 2.32193$, therefore, our result above agrees with this upper limit restriction.