my study notes, EE 420 Digital Filters
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Contents

1 Finding the spectrum of a sampled signal using a pulse train

Given a continuous signal $x(t)$, and we want to sample it. One way to sample it, is to multiply $x(t)$ it with a pulse train called $p(t)$, where the time between each pulse is T sec (hence sampling frequency is $F_s = \frac{1}{T}$). Let $x_p(t)$ be the generated signal. Notice that $x_p(t)$ will only have values at those time locations where the pulses from the pulse train are located.

The diagram below explains the above. It is from the book "signals and systems" by Oppenheim and Willsky:

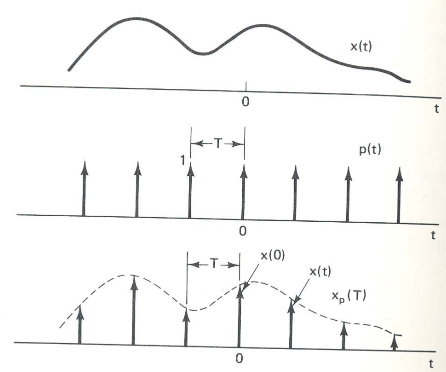


Figure 8.3 Pulse amplitude modulation with an impulse train.

Hence we have

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

And the sampled signal is

$$\begin{aligned} x_p(t) &= x(t)\, p(t) \\ &= x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \end{aligned}$$

The goal now is to find the spectrum of $x_p(t)$ and compare it to spectrum of $x(t)$

Since $x_p(t) = x(t)\, p(t)$, then applying Fourier transform we obtain

$$X_p(\omega) = \frac{1}{2\pi} X(\omega) \circledast P(\omega) \tag{1}$$

Where $X(\omega)$ is Fourier transform of $x(t)$, and $P(\omega)$ is Fourier transform of $p(t)$. This relation is from convolution properties where multiplication in time domain, becomes convolution in frequency domain, with $\frac{1}{2\pi}$ factor added.

Now we need to find $P(\omega)$. But the pulse train is periodic? so we really should be talking about Fourier series coefficients here, not Fourier transform? as Fourier transform is meant for aperiodic signals? Yes, Fourier transform is meant to be used for aperiodic signals, but there is a trick to come up with a Fourier transform for also a periodic signal such as the pulse train here. To apply this trick, first recall that for a periodic signal, it can be approximated by weight sum of complex exponential

$$p(t) = \sum_{n=-\infty}^{\infty} c(n)\, e^{j\omega_0 n t} \tag{2}$$

Where $c(k)$ (the weights) are called the Fourier coefficients, and these can be found using

$$c(n) = \frac{1}{T}\int_{-T/2}^{T/2} p(t)\, e^{j\omega_0 n t} dt$$

Where T is the fundamental period of $p(t)$ and $\omega_0 = \frac{2\pi}{T}$. Hence for the pulse train $p(t)$ we have

$$\begin{aligned} c(n) &= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{j\omega_0 n t} dt \\ &= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, dt \end{aligned}$$

Hence

$$c(n) = \frac{1}{T} \tag{3}$$

The above is all good and well, but we are back the question of how to find Fourier transform $P(\omega)$ for the periodic signal $p(t)$? Ok, here is the trick. We go backward. We ask, what function of $P(\omega)$ has as its inverse Fourier transform an aperiodic signal $\tilde{p}(t)$ whose Fourier series approximation is $p(t)$ as defined in (2) above? i.e. we want to solve for $P(\omega)$ in

$$\begin{aligned} \tilde{p}(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} P(\omega)\, e^{j\omega t} d\omega \\ \sum_{n=-\infty}^{\infty} c(n)\, e^{j\omega_0 n t} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} P(\omega)\, e^{j\omega t} d\omega \end{aligned} \tag{4}$$

Looking at the above long enough, we ask, what should $P(\omega)$ be to make RHS be the same as LHS? Let us try

$$P(\omega) = \sum_{n=-\infty}^{\infty} 2\pi c(n)\, \delta(\omega - n\omega_0) \tag{5}$$

and see if this does it. Plug the above into (4) we have

$$
\begin{aligned}
\sum_{n=-\infty}^{\infty} c(n)\, e^{jn\omega_0 t} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} 2\pi c(n)\, \delta(\omega - n\omega_0) \right) e^{j\omega t} d\omega \\
&= \sum_{n=-\infty}^{\infty} c(n) \left[\int_{-\infty}^{\infty} \delta(\omega - n\omega_0)\, e^{j\omega t} d\omega \right] \\
&= \sum_{n=-\infty}^{\infty} c(n) \left[\int_{-\infty}^{\infty} \delta(\omega - n\omega_0)\, e^{jn\omega_0 t} d\omega \right] \\
&= \sum_{n=-\infty}^{\infty} c(n) \left[e^{jn\omega_0 t} \int_{-\infty}^{\infty} \delta(\omega - n\omega_0)\, d\omega \right] \\
&= \sum_{n=-\infty}^{\infty} c(n)\, e^{jn\omega_0 t}
\end{aligned}
$$

That is it!, hence we conclude that $p(t)$ has Fourier transform given by (5) above. Now that we found $P(\omega)$ we go back to (1) and write

$$
\begin{aligned}
X_p(\omega) &= \frac{1}{2\pi} X(\omega) \circledast P(\omega) \\
&= \frac{1}{2\pi} X(\omega) \circledast \sum_{n=-\infty}^{\infty} 2\pi c(n)\, \delta(\omega - n\omega_0)
\end{aligned}
$$

But we found that $c(n) = \frac{1}{T}$, hence above simplifies to

$$
\begin{aligned}
X_p(\omega) &= \frac{1}{T} X(\omega) \circledast \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0) \\
&= \frac{1}{T} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_0)
\end{aligned}
$$

Hence we see that the spectrum of the sampled signal $x_p(t)$ is a scaled and shifted version of the spectrum of the signal $x(t)$ that was sampled. Also we see that many copies of $X(\omega)$ exist, each one is centered at integer multiples of ω_0 (the radial sampling frequency). The above result immediately leads also to determining what value of ω_0 should be to prevent copies of $X(\omega)$ to overlap with each others. We see that ω_0 must be larger than twice the bandwidth of $X(\omega)$, hence Nyquist theory.

References:

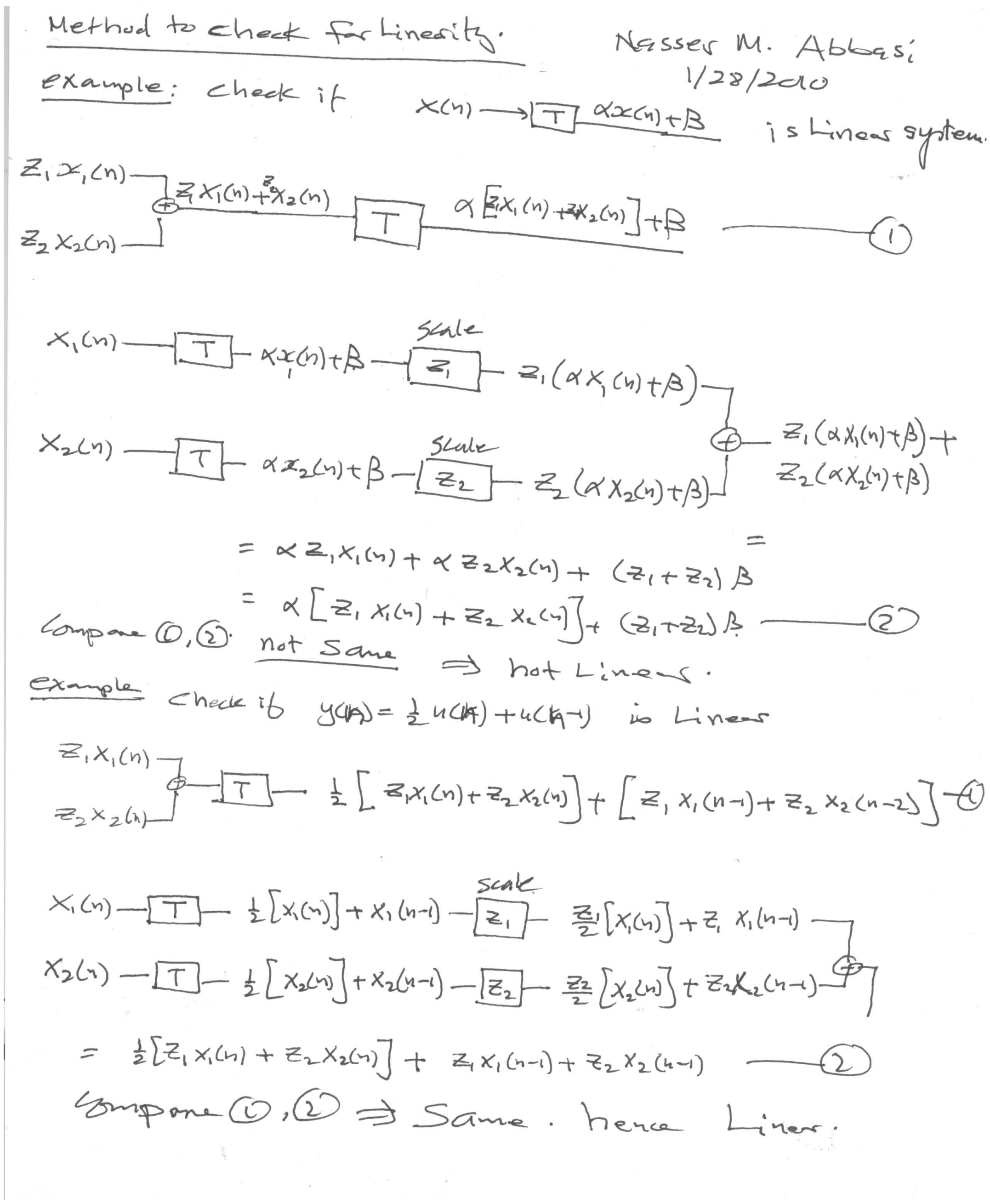
1. Signals and systems. Oppenheim and Willsky

2. Digital signal processing. Proakis and Manolakis

2 How to check system is linear?

Do experiment 1: Feed the system with the sum of 2 scaled signals $x_1(t)$ and $x_2(t)$ and obtain the output.

Do experiment 2: feed the system with $x_1(t)$, and scale the output. feed the system with $x_2(t)$, and scale the output (use same scales that were used in experiment 1). Now, add the above 2 scaled outputs. Check if we get the same result as we did in experiment 1. If the same result, then system is linear, else not linear. Here are 2 examples to illustrate. In these, I used z_1 as scale for the signal $x_1(t)$ and used z_2 as scale for $x_2(t)$. These scales are some numerical values. This has to be valid for any set of scales used.



This below is an algebraic method to check for linearity, which might be easier to use in exams.

1. Let $u_1 \rightarrow y_1$, and $u_2 \rightarrow y_2$, where u_i is input to system, and y_i is the output. Find y_1 and y_2 by using the system definition given

2. Find $u_3 = \alpha u_1 + \beta u_2$, i.e. scaled version of u_1 and u_2
3. Find y_3 when input is u_3 using system definition
4. Let $\tilde{y}_3 = \alpha y_1 + \beta y_2$, i.e. scaled version of the y_1 and y_2 found in step 1 above.
5. Compare $\tilde{y}_3$ and y_3, if they are the same, then system is linear, else not.

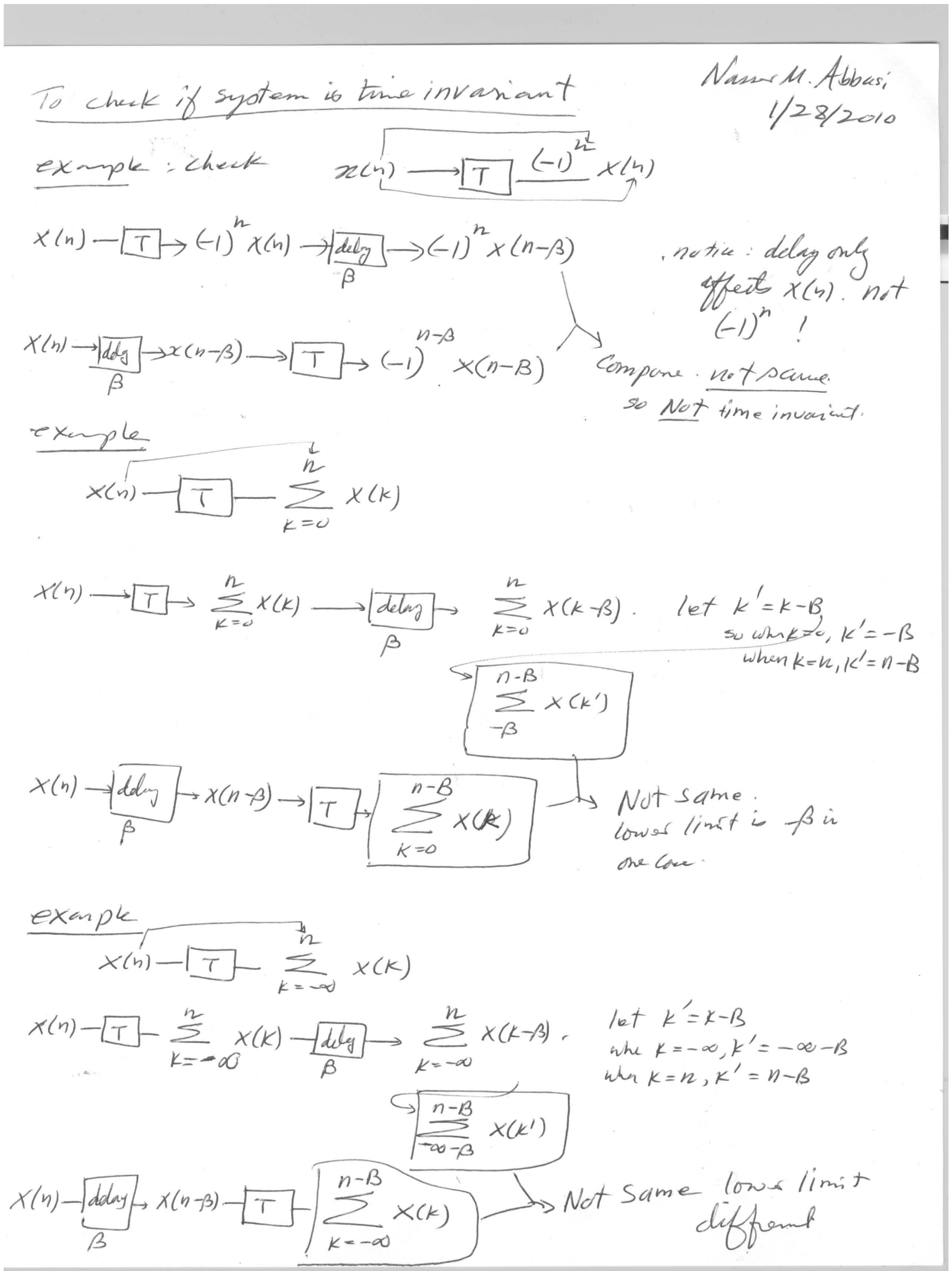
Here is an example, let system be given as $u \to au + b$, and apply the above 5 steps

1. Let $u_1 \to au_1 + b$, and $u_2 \to au_2 + b$. Hence $y_1 = au_2 + b$ and $y_2 = au_2 + b$
2. $u_3 = \alpha u_1 + \beta u_2$
3. $y_3 = au_3 + b = a(\alpha u_1 + \beta u_2) + b$
4. Let $\tilde{y}_3 = \alpha y_1 + \beta y_2$, hence $\tilde{y}_3 = \alpha\,(au_1 + b) + \beta\,(au_2 + b)$, hence $\tilde{y}_3 = a(\alpha u_1 + \beta u_2) + b\,(\alpha + b)$
5. Compare $\tilde{y}_3$ and y_3, we see they are not the same, hence non-linear

3 How to check if system is time invariant?

Experiment 1: Feed the system with the input, delay the output by some amount.

Experiment 2: delay the input by same amount, feed the result to the system. Check the output of experiment 2 is the same as experiment 1. These are some examples



To check if system is time invariant

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1/28/2010

example : check $x(n) \rightarrow \boxed{T} \rightarrow (-1)^n x(n)$

$x(n) - \boxed{T} \rightarrow (-1)^n x(n) \rightarrow \boxed{\text{delay } \beta} \rightarrow (-1)^n x(n-\beta)$

notice : delay only effects $x(n)$. not $(-1)^n$!

$x(n) \rightarrow \boxed{\text{delay } \beta} \rightarrow x(n-\beta) \rightarrow \boxed{T} \rightarrow (-1)^{n-\beta} x(n-\beta)$

Compare. not same. so Not time invariant.

example

$x(n) - \boxed{T} - \sum_{k=0}^{n} x(k)$

$x(n) \rightarrow \boxed{T} \rightarrow \sum_{k=0}^{n} x(k) \rightarrow \boxed{\text{delay } \beta} \rightarrow \sum_{k=0}^{n} x(k-\beta)$. let $k'=k-\beta$

so when $k=0$, $k'=-\beta$
when $k=n$, $k'=n-\beta$

$\sum_{-\beta}^{n-\beta} x(k')$

$x(n) \rightarrow \boxed{\text{delay } \beta} \rightarrow x(n-\beta) \rightarrow \boxed{T} \rightarrow \sum_{k=0}^{n-\beta} x(k)$ → Not same. lower limit is $-\beta$ in one case.

example

$x(n) - \boxed{T} - \sum_{k=-\infty}^{n} x(k)$

$x(n) - \boxed{T} - \sum_{k=-\infty}^{n} x(k) - \boxed{\text{delay } \beta} \rightarrow \sum_{k=-\infty}^{n} x(k-\beta)$. let $k'=k-\beta$

when $k=-\infty$, $k'=-\infty-\beta$
when $k=n$, $k'=n-\beta$

$\sum_{-\infty-\beta}^{n-\beta} x(k')$

$x(n) - \boxed{\text{delay } \beta} \rightarrow x(n-\beta) - \boxed{T} - \sum_{k=-\infty}^{n-\beta} x(k)$ → Not same lower limit different

This below is an algebraic method. Use this below for exams to check if system is time invariant

ex

let system be described by $x(n) \rightarrow \underbrace{(-1)^n x(n)}_{y(n)}$

Then $x(n-\alpha) \longrightarrow (-1)^{n-\alpha} x(n-\alpha)$

but $y(n-\alpha) = (-1)^n x(n-\alpha)$

→ Not same. so Not time invariant

example

$$x(n) \longrightarrow \underbrace{\sum_{k=0}^{n} x(k)}_{y(n)}$$

so $x(n-\alpha) \longrightarrow \sum_{k=0}^{n-\alpha} x(k)$

but $y(n-\alpha) = \sum_{k=0}^{n} x(k-\alpha)$. let $k-\alpha = k'$, so when $k=0$, $k' = -\alpha$; when $k=n$, $k' = n-\alpha$

$= \sum_{k'=-\alpha}^{n-\alpha} x(k')$ → Not same. So not time invariant

example

$$x(n) \longrightarrow \underbrace{\sum_{k=-\infty}^{n} x(k)}_{y(n)}$$

so $x(n-\alpha) \longrightarrow \sum_{k=-\infty}^{n-\alpha} x(k)$ ⟸ The idea is to treat this step as function i.e. change everywhere n appears

but $y(n-\alpha) = \sum_{k=-\infty}^{n} x(k-\alpha)$ let $k' = k-\alpha$, when $k=-\infty$, $k' = -\infty-\alpha$; when $k=n$, $k' = n-\alpha$

$= \sum_{k'=-\infty-\alpha}^{n-\alpha} x(k')$ → Not same. not time inv.

4 How to know if a sampled sequence is periodic?

Given some periodic signal $x(t)$ with fundamental period T, and we sample it with sampling period T_s, what condition we need so that the discrete sequence $x[n]$ generated is also periodic?

The condition is that there exist some multiple of T_s which divides exactly some multiple of T. This is the same thing as saying that frequency of the fundamental harmonic in $x(t)$ over the sampling frequency is a rational number.

5 How to find DTFT of a unit step function which runs over some window?

Added 2/20/2010. It took me a while to derive this. The textbook was not clear. So I am it more clear below.

Give a unit step, over some window, as in

$$u[n] = \begin{cases} 1 & |n| \leq M \\ 0 & o.w. \end{cases}$$

Find its DTFT. There is a good trick to remember

$$X(\omega) = \sum_{n=-M}^{M} e^{-j\omega n}$$

Let $n' = M + n$, then when $n = -M, n' = 0$, and when $n = M, n' = 2M$, hence the above becomes

$$X(\omega) = \sum_{n'=0}^{2M} e^{-j\omega(n'-M)} = \sum_{n'=0}^{2M} e^{-j\omega n'} e^{+j\omega M} = e^{j\omega M} \sum_{n'=0}^{2M} e^{-j\omega n'}$$

but n' is dummy variable, so the above can be written as

$$X(\omega) = e^{j\omega M} \sum_{n=0}^{2M} e^{-j\omega n}$$

Ok, now we use the standard geometric series, the above becomes

$$X(\omega) = e^{j\omega M} \frac{1 - \left(e^{-j\omega}\right)^{2M+1}}{1 - e^{-j\omega}} = e^{j\omega M} \frac{1 - e^{-j\omega(2M+1)}}{1 - e^{-j\omega}}$$

Ok, so far, nothing too exciting has happened, just using standard definitions. Here is the trick, without which I had hard time. The trick is to pull put half the exponential from the numerator and denominator, we obtain

$$X(\omega) = e^{j\omega M} \frac{e^{-\frac{j\omega(2M+1)}{2}} \left[e^{\frac{j\omega(2M+1)}{2}} - e^{-\frac{j\omega(2M+1)}{2}} \right]}{e^{-\frac{j\omega}{2}} \left(e^{\frac{j\omega}{2}} - e^{-\frac{j\omega}{2}} \right)}$$

Now simplify

$$
\begin{aligned}
X(\omega) &= e^{j\omega M-\frac{j\omega(2M+1)}{2}} \frac{\left[e^{\frac{j\omega(2M+1)}{2}} - e^{-\frac{j\omega(2M+1)}{2}}\right]}{e^{-\frac{j\omega}{2}}\left(e^{\frac{j\omega}{2}} - e^{-\frac{j\omega}{2}}\right)} \\
&= e^{\frac{2j\omega M-2j\omega M-j\omega}{2}} \frac{\left[e^{\frac{j\omega(2M+1)}{2}} - e^{-\frac{j\omega(2M+1)}{2}}\right]}{e^{-\frac{j\omega}{2}}\left(e^{\frac{j\omega}{2}} - e^{-\frac{j\omega}{2}}\right)} \\
&= e^{-\frac{j\omega}{2}} \frac{\left[e^{\frac{j\omega(2M+1)}{2}} - e^{-\frac{j\omega(2M+1)}{2}}\right]}{e^{-\frac{j\omega}{2}}\left(e^{\frac{j\omega}{2}} - e^{-\frac{j\omega}{2}}\right)} \\
&= \frac{e^{\frac{j\omega(2M+1)}{2}} - e^{-\frac{j\omega(2M+1)}{2}}}{\left(e^{\frac{j\omega}{2}} - e^{-\frac{j\omega}{2}}\right)}
\end{aligned}
$$

That was fun. Now it is all clear, we have a couple of sin functions sitting on top of each others, we obtain

$$
X(\omega) = \frac{\sin\left(\frac{\omega(2M+1)}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}
$$

6 How to write a window of deltas using unit step function?

Suppose we have

$$
f[n] = \begin{cases} 1 & |n| \leq M \\ 0 & o.w. \end{cases}
$$

To write the above using unit step $u[n]$, do

$$
f[n] = u[n+M] - u[n-M+1]
$$

The thing to notice for is to add 1 for the shifted version on the right side. Else I would not get an even function, and so $X(\omega)$ will turn out to be complex. For example, if $M = 5$, then write

$u[n+5] - u[n-6]$ and NOT $u[n+5] - u[n-5]$

7 How to find the CTFT Fourier transform of a periodic signal?

Given $x(t)$, a periodic signal, such as $\cos\Omega_0 t$, we want to find its $X(\Omega)$, then this is the algorithm

1. Find $c(k)$, the Fourier series coefficients of $x(t)$ from $c(k)=\frac{1}{2T}\int_T x(t)\,e^{-j\Omega t}dt$ where T is the periodic of $x(t)$
2. Now $X(\Omega)=2\pi\sum_{n=-\infty}^{\infty}c(n)\,\delta(\Omega-n\Omega_0)$

For example $\cos\Omega_0 t=\frac{e^{j\Omega_0 t}+e^{-j\Omega_0 t}}{2}$, hence $c(-1)=\frac{1}{2}$ and $c(1)=\frac{1}{2}$, therefore

$$
\begin{aligned}
X(\Omega) &= 2\pi\left(\frac{1}{2}\delta(\Omega+\Omega_0)+\frac{1}{2}\delta(\Omega-\Omega_0)\right)\\
&= \pi\left(\delta(\Omega+\Omega_0)+\delta(\Omega-\Omega_0)\right)
\end{aligned}
$$

Therefore, we see that all periodic functions in time, will have fourier transform which is a pulse train.

8 Review of the sampling theory

by Nasser M. Abbasi, added 2/27/2010

reference: lecture notes by Dr Shiva, EE 420 (digital filters), CSUF and DSP textbook, by Oppenheim, 1975, and textbook DSP by Proakis, 3rd edition.

I made these notes using diagrams, as I like to see things on diagram to help me understand them.

Ω is radial frequency in rad/sec. Used for CTFT

ω is radial frequency in radians. Used for DTFT

$\omega=\Omega T$ where T is sampling frequency.

We start with $f(t)$, the continuous time signal. We multiply it by a train of impulses called $g(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT)$ where T is the sampling period. From this we obtain $\tilde{f}(t)$ the sampled version of $f(t)$. Now we can obtain $f(t)$ by interpolation.

To see how to do the above in frequency domain, we start by taking CTFT of $f(t)$ to obtain $F(\Omega)$ and take the CTFT of $g(t)$ to obtain $G(\Omega)$

Next, we need to map the operation of $f(t)\times g(t)$ that we did in time domain to frequency domain. This becomes a convolution in frequency domain, but with $\frac{1}{2\pi}$ at the front. Since $G(\Omega)=\Omega_s\sum_{n=-\infty}^{\infty}\delta(\Omega-n\Omega_s)$,Hence we obtain

$$
\begin{aligned}
\tilde{G}(\Omega) &= \frac{1}{2\pi}\left[F(\Omega)\circledast G(\Omega)\right]\\
&= \frac{1}{2\pi}\left[F(\Omega)\circledast\Omega_s\sum_{n=-\infty}^{\infty}\delta(\Omega-n\Omega_s)\right]\\
&= \frac{\Omega_s}{2\pi}\sum_{n=-\infty}^{\infty}F(\Omega-n\Omega_s)\\
&= \frac{1}{T}\sum_{n=-\infty}^{\infty}F(\Omega-n\Omega_s)
\end{aligned}
$$

Where we used the convolution property that $F(\Omega)\circledast\sum_{n=-\infty}^{\infty}\delta(\Omega-n\Omega_s)=\sum_{n=-\infty}^{\infty}F(\Omega-n\Omega_s)$, i.e convolving a function with a train of impulses, is the same function evaluated at the location of these impulses. We now see the effect of sampling. The original $F(\Omega)$ is scaled by $\frac{1}{T}$ and is duplicated every Ω_s

Now we multiply $\frac{1}{T}\sum_{n=-\infty}^{\infty} F(\Omega - n\Omega_s)$ by a low pass filter $T\ rect\left(\frac{\Omega}{\Omega_s}\right)$, which pulls out $F(\Omega)$ out of those duplications and adjust the scale back. So now we have the original $F(\Omega)$. But if we wanted to obtain $f(t)$ back, then we need to convert the operation

$$\frac{1}{T}\sum_{n=-\infty}^{\infty} F(\Omega - n\Omega_s) \times T\ rect\left(\frac{\Omega}{\Omega_s}\right) \rightarrow F(\Omega)$$

to the time domain

$$CTFT^{-1}\left(\frac{1}{T}\sum_{n=-\infty}^{\infty} F(\Omega - n\Omega_s)\right) \circledast CTFT^{-1}\left(T\ rect\left(\frac{\Omega}{\Omega_s}\right)\right) \rightarrow f(t)$$

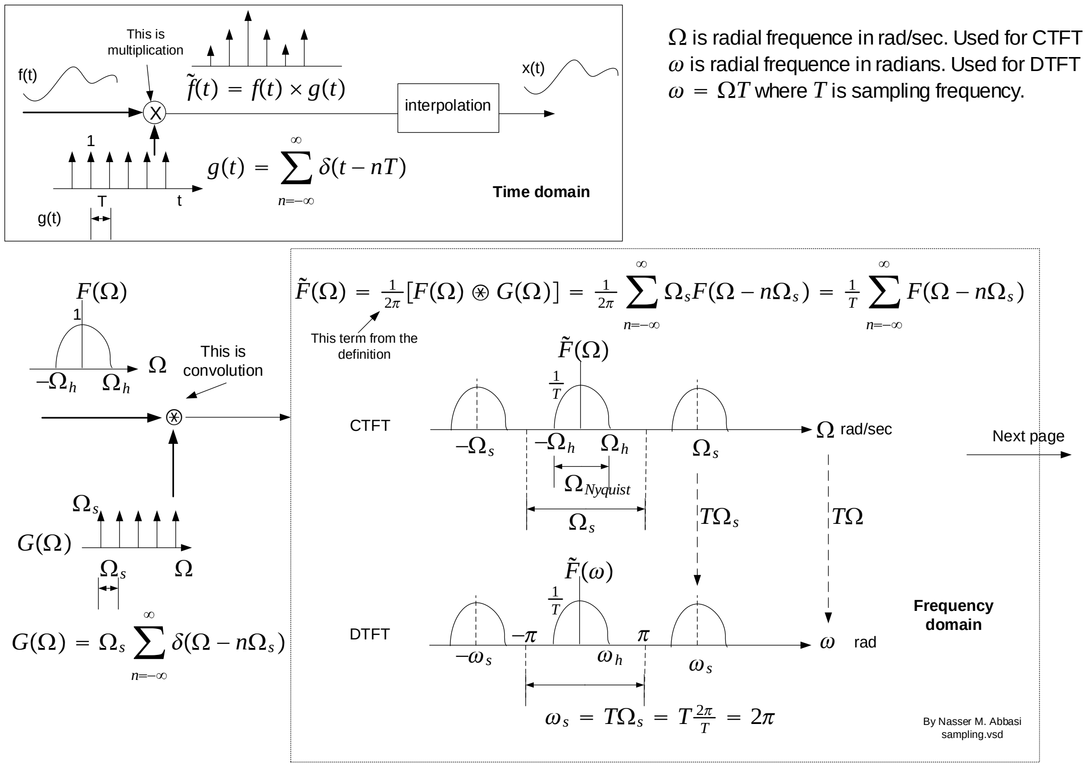
i.e. we apply inverse fourier transform and since multiplication become convolution, we need to convolve a sinc function (which is the inverse transform of the rect) with the inverse fourier transform of $\frac{1}{T}\sum_{n=-\infty}^{\infty} F(\Omega - n\Omega_s)$ which is $\sum_{n=-\infty}^{\infty} f(nT)\,\delta(t - nT)$, hence

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} f(nT)\,\delta(t - nT) \circledast \operatorname{sinc}\left(\frac{\Omega_s}{2} t\right) \\ &= \sum_{n=-\infty}^{\infty} f(nT) \operatorname{sinc}\left(\frac{\Omega_s}{2} t - nT\right) \end{aligned}$$

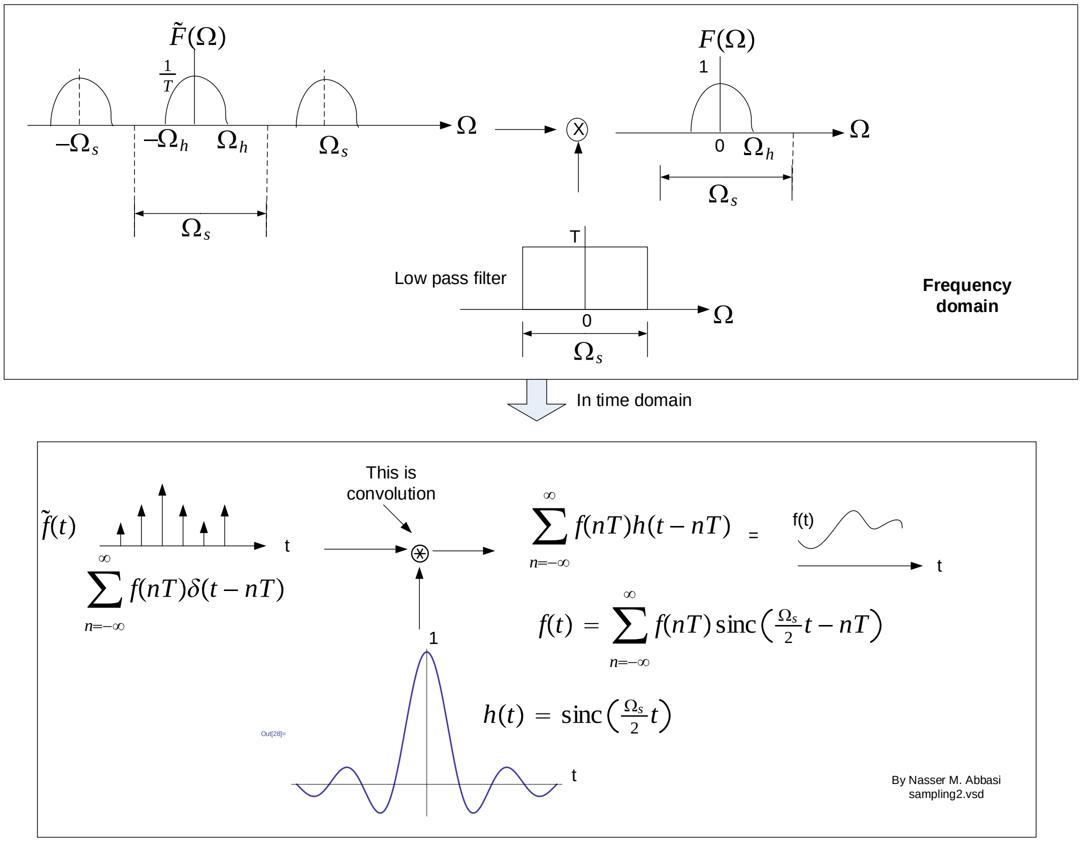
Therefore

$$f(t) = \sum_n f(nT) \operatorname{sinc}\left(\frac{\Omega_s}{2} t - nT\right)$$

To help understand the above, these are some diagrams. This diagram show sampling in time domain. Then shows the same steps, but done in frequency domain.



This diagram shows the operation of obtaining the original continues signal from its samples using convolution with a sinc function, and how it came about.



9 How to find the DTFT from the CTFT given sampling period?

Suppose we have been given the CTFT (cont. time fourier transform) of a signal $x(t)$, then this signal is sampled using sampling period T, and we want now to determine the DTFT of the sequence $x[n]$. We could offcourse apply the definition of DTFT on $x[n]$, but we also obtain the DTFT spectrum directly from the CTFT. (I need to be careful here, we are here talking about magnitude spectrum only, double check on this).

Let $F_a(\Omega)$ be the CTFT and let $F(\omega)$ be the DTFT, then $F(\omega) = \frac{1}{T}F(T\Omega)$ or $F(\omega) = \frac{1}{T}F\left(\frac{\Omega}{f_s}\right)$ where f_s is the sampling frequency in samples per second (or Hz).

Hence if we want to find $F\left(\frac{\pi}{4}\right)$, then this maps to $\Omega = \frac{\omega}{T} = \frac{\pi}{4T}$, and so we read $F\left(\frac{\pi}{4T}\right)$ from the CTFT, and then we multiply the result by $\frac{1}{T}$, this gives $F\left(\frac{\pi}{4}\right)$